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Open Mapping Theorem

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Summary. In this article we formalize one of the most important theorems of linear operator theory the Open Mapping Theorem commonly used in a standard book such as [8] in chapter 2.4.2. It states that a surjective continuous linear operator between Banach spaces is an open map.

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The notation and terminology used here are introduced in the following papers: [13], [14], [3], [9], [2], [7], [1], [4], [5], [10], [6], [12], [11], and [15].

The following proposition is true

(1) For all real numbers x, y such that $0 \leq x < y$ there exists a real number s_0 such that $0 < s_0$ and $x < \frac{y}{1+s_0} < y$.

The scheme *RecExD3* deals with a non empty set $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, an element $\mathcal{C}$ of $\mathcal{A}$, and a 4-ary predicate $\mathcal{P}$, and states that:

There exists a function f from $\mathbb{N}$ into $\mathcal{A}$ such that $f(0) = \mathcal{B}$ and $f(1) = \mathcal{C}$ and for every element n of $\mathbb{N}$ holds $\mathcal{P}[n, f(n), f(n+1), f(n+2)]$

provided the parameters meet the following requirement:

- For every element n of $\mathbb{N}$ and for all elements x, y of $\mathcal{A}$ there exists an element z of $\mathcal{A}$ such that $\mathcal{P}[n, x, y, z]$.

In the sequel X, Y denote real normed spaces.

The following propositions are true:

(2) For every point y_1 of X and for every real number r holds $\mathrm{Ball}(y_1, r) = y_1 + \mathrm{Ball}(0_X, r)$.

(3) For every real number r and for every real number a such that $0 < a$ holds $\mathrm{Ball}(0_X, a \cdot r) = a \cdot \mathrm{Ball}(0_X, r)$.

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(4) For every linear operator T from X into Y and for all subsets B_0, B_1 of X holds $T^\circ(B_0 + B_1) = T^\circ B_0 + T^\circ B_1$.

(5) Let T be a linear operator from X into Y, B_0 be a subset of X, and a be a real number. Then $T^\circ(a \cdot B_0) = a \cdot T^\circ B_0$.

(6) Let T be a linear operator from X into Y, B_0 be a subset of X, and x_1 be a point of X. Then $T^\circ(x_1 + B_0) = T(x_1) + T^\circ B_0$.

(7) For all subsets V, W of X and for all subsets V_1, W_1 of LinearTopSpaceNorm X such that $V = V_1$ and $W = W_1$ holds $V + W = V_1 + W_1$.

(8) Let V be a subset of X, x be a point of X, V_1 be a subset of LinearTopSpaceNorm X, and x_1 be a point of LinearTopSpaceNorm X. If $V = V_1$ and $x = x_1$, then $x + V = x_1 + V_1$.

(9) For every subset V of X and for every real number a and for every subset V_1 of LinearTopSpaceNorm X such that $V = V_1$ holds $a \cdot V = a \cdot V_1$.

(10) For every subset V of TopSpaceNorm X and for every subset V_1 of LinearTopSpaceNorm X such that $V = V_1$ holds $\overline{V} = \overline{V_1}$.

(11) For every point x of X and for every real number r holds $\mathrm{Ball}(0_X, r) = (-1) \cdot \mathrm{Ball}(0_X, r)$.

(12) For every point x of X and for every real number r and for every subset V of LinearTopSpaceNorm X such that $V = \mathrm{Ball}(x, r)$ holds V is convex.

(13) Let x be a point of X, r be a real number, T be a linear operator from X into Y, and V be a subset of LinearTopSpaceNorm Y. If $V = T^\circ \mathrm{Ball}(x, r)$, then V is convex.

(14) For every point x of X and for all real numbers r, s such that $r \leq s$ holds $\mathrm{Ball}(x, r) \subseteq \mathrm{Ball}(x, s)$.

(15) Let X be a real Banach space, Y be a real normed space, T be a bounded linear operator from X into Y, r be a real number, B_2 be a subset of LinearTopSpaceNorm X, and T_1, B_3 be subsets of LinearTopSpaceNorm Y. If $r > 0$ and $B_2 = \mathrm{Ball}(0_X, 1)$ and $B_3 = \mathrm{Ball}(0_Y, r)$ and $T_1 = T^\circ \mathrm{Ball}(0_X, 1)$ and $B_3 \subseteq \overline{T_1}$, then $B_3 \subseteq T_1$.

(16) Let X, Y be real Banach spaces, T be a bounded linear operator from X into Y, and T_2 be a function from LinearTopSpaceNorm X into LinearTopSpaceNorm Y. If $T_2 = T$ and T_2 is onto, then T_2 is open.

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