

BCI-homomorphisms

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Summary. In this article the notion of the power of an element of BCI-algebra and its period in the book [11], sections 1.4 to 1.5 are firstly given. Then the definition of BCI-homomorphism is defined and the fundamental theorem of homomorphism, the first isomorphism theorem and the second isomorphism theorem are proved following the book [9], section 1.6.

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The notation and terminology used in this paper have been introduced in the following articles: [6], [14], [3], [15], [5], [4], [2], [7], [10], [1], [13], [8], and [12].

1. THE POWER OF AN ELEMENT OF BCI-ALGEBRAS

In this paper X is a BCI-algebra and n is an element of $\mathbb{N}$.

Let D be a set, let f be a function from $\mathbb{N}$ into D , and let n be a natural number. Then $f(n)$ is an element of D .

Let G be a non empty BCI structure with 0 . The functor BCI-power G yielding a function from $(\text{the carrier of } G) \times \mathbb{N}$ into the carrier of G is defined as follows:

(Def. 1) For every element x of G holds $(\text{BCI-power } G)(x, 0) = 0_G$ and for every n holds $(\text{BCI-power } G)(x, n + 1) = x \setminus (\text{BCI-power } G)(x, n)^c$.

For simplicity, we adopt the following convention: x, y are elements of X , a, b are elements of $\text{AtomSet } X$, m, n are natural numbers, and i, j are integers.

Let us consider X, i, x . The functor x^i yielding an element of X is defined by:

$$(\text{Def. 2}) \quad x^i = \begin{cases} (\text{BCI-power } X)(x, |i|), & \text{if } 0 \leq i, \\ (\text{BCI-power } X)(x^c, |i|), & \text{otherwise.} \end{cases}$$

Let us consider X, n, x . Then x^n can be characterized by the condition:

$$(\text{Def. 3}) \quad x^n = (\text{BCI-power } X)(x, n).$$

One can prove the following propositions:

- (1) $a \setminus (x \setminus b) = b \setminus (x \setminus a)$.
- (2) $x^{n+1} = x \setminus (x^n)^c$.
- (3) $x^0 = 0_X$.
- (4) $x^1 = x$.
- (5) $x^{-1} = x^c$.
- (6) $x^2 = x \setminus x^c$.
- (7) $(0_X)^n = 0_X$.
- (8) $(a^{-1})^{-1} = a$.
- (9) $x^{-n} = ((x^c)^c)^{-n}$.
- (10) $(a^c)^n = a^{-n}$.
- (11) If $x \in \text{BCK-part } X$ and $n \geq 1$, then $x^n = x$.
- (12) If $x \in \text{BCK-part } X$, then $x^{-n} = 0_X$.
- (13) $a^i \in \text{AtomSet } X$.
- (14) $(a^{n+1})^c = (a^n)^c \setminus a$.
- (15) $(a \setminus b)^n = a^n \setminus b^n$.
- (16) $(a \setminus b)^{-n} = a^{-n} \setminus b^{-n}$.
- (17) $(a^c)^n = (a^n)^c$.
- (18) $(x^c)^n = (x^n)^c$.
- (19) $(a^c)^{-n} = (a^{-n})^c$.
- (20) $x^n \in \text{BranchV}(((x^c)^c)^n)$.
- (21) $(x^n)^c = (((x^c)^c)^n)^c$.
- (22) $a^i \setminus a^j = a^{i-j}$.
- (23) $(a^i)^j = a^{i \cdot j}$.
- (24) $a^{i+j} = a^i \setminus (a^j)^c$.

Let us consider X, x . We say that x is finite-period if and only if:

$$(\text{Def. 4}) \quad \text{There exists an element } n \text{ of } \mathbb{N} \text{ such that } n \neq 0 \text{ and } x^n \in \text{BCK-part } X.$$

One can prove the following proposition

$$(25) \quad \text{If } x \text{ is finite-period, then } (x^c)^c \text{ is finite-period.}$$

Let us consider X, x . Let us assume that x is finite-period. The functor $\text{ord}(x)$ yielding an element of $\mathbb{N}$ is defined as follows:

(Def. 5) $x^{\text{ord}(x)} \in \text{BCK-part } X$ and $\text{ord}(x) \neq 0$ and for every element m of $\mathbb{N}$ such that $x^m \in \text{BCK-part } X$ and $m \neq 0$ holds $\text{ord}(x) \leq m$.

One can prove the following propositions:

- (26) If a is finite-period and $\text{ord}(a) = n$, then $a^n = 0_X$.
- (27) X is a BCK-algebra iff for every x holds x is finite-period and $\text{ord}(x) = 1$.
- (28) If x is finite-period and a is finite-period and $x \in \text{BranchV } a$, then $\text{ord}(x) = \text{ord}(a)$.
- (29) If x is finite-period and $\text{ord}(x) = n$, then $x^m \in \text{BCK-part } X$ iff $n \mid m$.
- (30) If x is finite-period and x^m is finite-period and $\text{ord}(x) = n$ and $m > 0$, then $\text{ord}(x^m) = n \div (m \text{ gcd } n)$.
- (31) If x is finite-period and x^c is finite-period, then $\text{ord}(x) = \text{ord}(x^c)$.
- (32) If $x \setminus y$ is finite-period and $x, y \in \text{BranchV } a$, then $\text{ord}(x \setminus y) = 1$.
- (33) Suppose that $x \setminus y$ is finite-period and $a \setminus b$ is finite-period and x is finite-period and y is finite-period and a is finite-period and b is finite-period and $a \neq b$ and $x \in \text{BranchV } a$ and $y \in \text{BranchV } b$. Then $\text{ord}(a \setminus b) \mid \text{lcm}(\text{ord}(x), \text{ord}(y))$.

2. DEFINITION OF BCI-HOMOMORPHISMS

For simplicity, we follow the rules: X, X', Y, Z, W are BCI-algebras, H' denotes a subalgebra of X' , G denotes a subalgebra of X , A' denotes a non empty subset of X' , I denotes an ideal of X , C_1, K are closed ideals of X , x, y are elements of X , R_1 denotes an I-congruence of X by I , and R_2 denotes an I-congruence of X by K .

One can prove the following proposition

- (34) Let X be a BCI-algebra, Y be a subalgebra of X , x, y be elements of X , and x', y' be elements of Y . If $x = x'$ and $y = y'$, then $x \setminus y = x' \setminus y'$.

Let X, X' be non empty BCI structures with 0 and let f be a function from X into X' . We say that f is multiplicative if and only if:

(Def. 6) For all elements a, b of X holds $f(a \setminus b) = f(a) \setminus f(b)$.

Let X, X' be BCI-algebras. Note that there exists a function from X into X' which is multiplicative.

Let X, X' be BCI-algebras. A BCI-homomorphism from X to X' is a multiplicative function from X into X' .

In the sequel f denotes a BCI-homomorphism from X to X' , g denotes a BCI-homomorphism from X' to X , and h denotes a BCI-homomorphism from X' to Y .

Let us consider X, X', f . We say that f is isotonic if and only if:

(Def. 7) For all x, y such that $x \leq y$ holds $f(x) \leq f(y)$.

Let us consider X . An endomorphism of X is a BCI-homomorphism from X to X .

Let us consider X, X', f . The functor $\text{Ker } f$ is defined by:

(Def. 8) $\text{Ker } f = \{x \in X: f(x) = 0_{X'}\}$.

The following proposition is true

$$(35) \quad f(0_X) = 0_{X'}.$$

Let us consider X, X', f . Observe that $\text{Ker } f$ is non empty.

We now state several propositions:

(36) If $x \leq y$, then $f(x) \leq f(y)$.

(37) f is one-to-one iff $\text{Ker } f = \{0_X\}$.

(38) If f is bijective and $g = f^{-1}$, then g is bijective.

(39) $h \cdot f$ is a BCI-homomorphism from X to Y .

(40) Let f be a BCI-homomorphism from X to Y , g be a BCI-homomorphism from Y to Z , and h be a BCI-homomorphism from Z to W . Then $h \cdot (g \cdot f) = (h \cdot g) \cdot f$.

(41) For every subalgebra Z of X' such that the carrier of $Z = \text{rng } f$ holds f is a BCI-homomorphism from X to Z .

(42) $\text{Ker } f$ is a closed ideal of X .

Let us consider X, X', f . Observe that $\text{Ker } f$ is closed.

Next we state several propositions:

(43) If f is onto, then for every element c of X' there exists x such that $c = f(x)$.

(44) For every element a of X such that a is minimal holds $f(a)$ is minimal.

(45) For every element a of $\text{AtomSet } X$ and for every element b of $\text{AtomSet } X'$ such that $b = f(a)$ holds $f^\circ \text{BranchV } a \subseteq \text{BranchV } b$.

(46) If A' is an ideal of X' , then $f^{-1}(A')$ is an ideal of X .

(47) If A' is a closed ideal of X' , then $f^{-1}(A')$ is a closed ideal of X .

(48) If f is onto, then $f^\circ I$ is an ideal of X' .

(49) If f is onto, then $f^\circ C_1$ is a closed ideal of X' .

Let X, X' be BCI-algebras. We say that X and X' are isomorphic if and only if:

(Def. 9) There exists a BCI-homomorphism from X to X' which is bijective.

Let us consider X , let I be an ideal of X , and let R_1 be an I-congruence of X by I . Note that X/R_1 is strict, B, C, I, and BCI-4.

Let us consider X , let I be an ideal of X , and let R_1 be an I-congruence of X by I . The canonical homomorphism onto cosets of R_1 yielding a BCI-homomorphism from X to X/R_1 is defined as follows:

(Def. 10) For every x holds (the canonical homomorphism onto cosets of R_1)(x) = $[x]_{(R_1)}$.

3. FUNDAMENTAL THEOREM OF HOMOMORPHISMS

The following four propositions are true:

- (50) The canonical homomorphism onto cosets of R_1 is onto.
- (51) Suppose $I = \text{Ker } f$. Then there exists a BCI-homomorphism h from X/R_1 to X' such that $f = h \cdot$ the canonical homomorphism onto cosets of R_1 and h is one-to-one.
- (52) Let given X, X', I, R_1, f . Suppose $I = \text{Ker } f$. Then there exists a BCI-homomorphism h from X/R_1 to X' such that $f = h \cdot$ the canonical homomorphism onto cosets of R_1 and h is one-to-one.
- (53) $\text{Ker}(\text{the canonical homomorphism onto cosets of } R_2) = K$.

4. FIRST ISOMORPHISM THEOREM

One can prove the following propositions:

- (54) If $I = \text{Ker } f$ and the carrier of $H' = \text{rng } f$, then X/R_1 and H' are isomorphic.
- (55) If $I = \text{Ker } f$ and f is onto, then X/R_1 and X' are isomorphic.

5. SECOND ISOMORPHISM THEOREM

Let us consider X, G, K, R_2 . The functor $\text{Union}(G, R_2)$ yielding a non empty subset of X is defined by:

(Def. 11) $\text{Union}(G, R_2) = \bigcup \{[a]_{(R_2)}; a \text{ ranges over elements of } G: [a]_{(R_2)} \in \text{the carrier of } X/R_2\}$.

Let us consider X, G, K, R_2 . The functor $\text{HKOp}(G, R_2)$ yielding a binary operation on $\text{Union}(G, R_2)$ is defined as follows:

(Def. 12) For all elements w_1, w_2 of $\text{Union}(G, R_2)$ and for all elements x, y of X such that $w_1 = x$ and $w_2 = y$ holds $(\text{HKOp}(G, R_2))(w_1, w_2) = x \setminus y$.

Let us consider X, G, K, R_2 . The functor $\text{zeroHK}(G, R_2)$ yields an element of $\text{Union}(G, R_2)$ and is defined as follows:

(Def. 13) $\text{zeroHK}(G, R_2) = 0_X$.

Let us consider X, G, K, R_2 . The functor $\text{HK}(G, R_2)$ yielding a BCI structure with 0 is defined as follows:

(Def. 14) $\text{HK}(G, R_2) = \langle \text{Union}(G, R_2), \text{HKOp}(G, R_2), \text{zeroHK}(G, R_2) \rangle$.

Let us consider X, G, K, R_2 . Observe that $\text{HK}(G, R_2)$ is non empty.

Let us consider X, G, K, R_2 and let w_1, w_2 be elements of $\text{Union}(G, R_2)$.

The functor $w_1 \setminus w_2$ yielding an element of $\text{Union}(G, R_2)$ is defined by:

(Def. 15) $w_1 \setminus w_2 = (\text{HKOp}(G, R_2))(w_1, w_2)$.

We now state the proposition

(56) $\text{HK}(G, R_2)$ is a BCI-algebra.

Let us consider X, G, K, R_2 . Observe that $\text{HK}(G, R_2)$ is strict, B, C, I, and BCI-4.

We now state three propositions:

(57) $\text{HK}(G, R_2)$ is a subalgebra of X .

(58) (The carrier of G) $\cap K$ is a closed ideal of G .

(59) Let K_1 be an ideal of $\text{HK}(G, R_2)$, R_3 be an I-congruence of $\text{HK}(G, R_2)$ by K_1 , I be an ideal of G , and R_1 be an I-congruence of G by I . Suppose $K_1 = K$ and $R_3 = R_2$ and $I = (\text{the carrier of } G) \cap K$. Then G/R_1 and $\text{HK}(G, R_2)/R_3$ are isomorphic.

REFERENCES

- [1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(1):41–46, 1990.
- [2] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(1):175–180, 1990.
- [3] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(1):55–65, 1990.
- [4] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(1):153–164, 1990.
- [5] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(2):357–367, 1990.
- [6] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(1):47–53, 1990.
- [7] Yuzhong Ding. Several classes of BCI-algebras and their properties. *Formalized Mathematics*, 15(1):1–9, 2007.
- [8] Yuzhong Ding and Zhiyong Pang. Congruences and quotient algebras of BCI-algebras. *Formalized Mathematics*, 15(4):175–180, 2007.
- [9] Yisheng Huang. *BCI-algebras*. Science Press, 2006.
- [10] Rafał Kwiatek and Grzegorz Zwara. The divisibility of integers and integer relative primes. *Formalized Mathematics*, 1(5):829–832, 1990.
- [11] Jie Meng and YoungLin Liu. *An Introduction to BCI-algebras*. Shaanxi Scientific and Technological Press, 2001.
- [12] Konrad Raczkowski and Paweł Sadowski. Equivalence relations and classes of abstraction. *Formalized Mathematics*, 1(3):441–444, 1990.
- [13] Michał J. Trybulec. Integers. *Formalized Mathematics*, 1(3):501–505, 1990.
- [14] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(1):67–71, 1990.
- [15] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(1):181–186, 1990.

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