

Basic Properties of Circulant Matrices and Anti-Circular Matrices

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Summary. This article introduces definitions of circulant matrices, line- and column-circulant matrices as well as anti-circular matrices and describes their main properties.

MML identifier: MATRIX16, version: 7.9.03 4.108.1028

The articles [6], [9], [4], [10], [1], [14], [13], [2], [5], [8], [12], [11], [3], and [7] provide the notation and terminology for this paper.

1. SOME PROPERTIES OF CIRCULANT MATRICES

For simplicity, we adopt the following convention: i, j, k, n, l denote elements of $\mathbb{N}$, K denotes a field, a, b, c denote elements of K , p, q denote finite sequences of elements of K , and M_1, M_2, M_3 denote square matrices over K of dimension n .

Next we state two propositions:

- (1) $\mathbf{1}_K \cdot p = p$.
- (2) $(-\mathbf{1}_K) \cdot p = -p$.

Let K be a set, let M be a matrix over K , and let p be a finite sequence. We say that M is line circulant about p if and only if:

(Def. 1) $\text{len } p = \text{width } M$ and for all natural numbers i, j such that $\langle i, j \rangle \in$ the indices of M holds $M_{i,j} = p(((j - i) \bmod \text{len } p) + 1)$.

Let K be a set and let M be a matrix over K . We say that M is line circulant if and only if:

- (Def. 2) There exists a finite sequence p of elements of K such that $\text{len } p = \text{width } M$ and M is line circulant about p .

Let K be a non empty set and let p be a finite sequence of elements of K . We say that p is first-line-of-circulant if and only if:

- (Def. 3) There exists a square matrix over K of dimension $\text{len } p$ which is line circulant about p .

Let K be a set, let M be a matrix over K , and let p be a finite sequence. We say that M is column circulant about p if and only if:

- (Def. 4) $\text{len } p = \text{len } M$ and for all natural numbers i, j such that $\langle i, j \rangle \in$ the indices of M holds $M_{i,j} = p(((i - j) \bmod \text{len } p) + 1)$.

Let K be a set and let M be a matrix over K . We say that M is column circulant if and only if:

- (Def. 5) There exists a finite sequence p of elements of K such that $\text{len } p = \text{len } M$ and M is column circulant about p .

Let K be a non empty set and let p be a finite sequence of elements of K . We say that p is first-column-of-circulant if and only if:

- (Def. 6) There exists a square matrix over K of dimension $\text{len } p$ which is column circulant about p .

Let K be a non empty set and let p be a finite sequence of elements of K . Let us assume that p is first-line-of-circulant. The functor $\text{LCirc } p$ yields a square matrix over K of dimension $\text{len } p$ and is defined by:

- (Def. 7) $\text{LCirc } p$ is line circulant about p .

Let K be a non empty set and let p be a finite sequence of elements of K . Let us assume that p is first-column-of-circulant. The functor $\text{CCirc } p$ yielding a square matrix over K of dimension $\text{len } p$ is defined by:

- (Def. 8) $\text{CCirc } p$ is column circulant about p .

Let K be a field. One can verify that there exists a finite sequence of elements of K which is first-line-of-circulant and first-column-of-circulant.

Let us consider K, n . Observe that $0_K^{n \times n}$ is line circulant and column circulant.

Let us consider K , let us consider n , and let a be an element of K . Observe that $(a)^{n \times n}$ is line circulant and $(a)^{n \times n}$ is column circulant.

Let us consider K . Note that there exists a matrix over K which is line circulant and column circulant.

In the sequel D denotes a non empty set, t denotes a finite sequence of elements of D , and A denotes a square matrix over D of dimension n .

We now state a number of propositions:

- (3) If A is line circulant and $n > 0$, then A^T is column circulant.
- (4) If A is line circulant about t and $n > 0$, then $t = \text{Line}(A, 1)$.

- (5) If A is line circulant and $\langle i, j \rangle \in \text{Seg } n \times \text{Seg } n$ and $k = i + 1$ and $l = j + 1$ and $i < n$ and $j < n$, then $A_{i,j} = A_{k,l}$.
- (6) If M_1 is line circulant, then $a \cdot M_1$ is line circulant.
- (7) If M_1 is line circulant and M_2 is line circulant, then $M_1 + M_2$ is line circulant.
- (8) If M_1 is line circulant and M_2 is line circulant and M_3 is line circulant, then $M_1 + M_2 + M_3$ is line circulant.
- (9) If M_1 is line circulant and M_2 is line circulant, then $a \cdot M_1 + b \cdot M_2$ is line circulant.
- (10) If M_1 is line circulant and M_2 is line circulant and M_3 is line circulant, then $a \cdot M_1 + b \cdot M_2 + c \cdot M_3$ is line circulant.
- (11) If M_1 is line circulant, then $-M_1$ is line circulant.
- (12) If M_1 is line circulant and M_2 is line circulant, then $M_1 - M_2$ is line circulant.
- (13) If M_1 is line circulant and M_2 is line circulant, then $a \cdot M_1 - b \cdot M_2$ is line circulant.
- (14) If M_1 is line circulant and M_2 is line circulant and M_3 is line circulant, then $(a \cdot M_1 + b \cdot M_2) - c \cdot M_3$ is line circulant.
- (15) If M_1 is line circulant and M_2 is line circulant and M_3 is line circulant, then $a \cdot M_1 - b \cdot M_2 - c \cdot M_3$ is line circulant.
- (16) If M_1 is line circulant and M_2 is line circulant and M_3 is line circulant, then $(a \cdot M_1 - b \cdot M_2) + c \cdot M_3$ is line circulant.
- (17) If A is column circulant and $n > 0$, then A^T is line circulant.
- (18) If A is column circulant about t and $n > 0$, then $t = A_{\square,1}$.
- (19) If A is column circulant and $\langle i, j \rangle \in \text{Seg } n \times \text{Seg } n$ and $k = i + 1$ and $l = j + 1$ and $i < n$ and $j < n$, then $A_{i,j} = A_{k,l}$.
- (20) If M_1 is column circulant, then $a \cdot M_1$ is column circulant.
- (21) If M_1 is column circulant and M_2 is column circulant, then $M_1 + M_2$ is column circulant.
- (22) If M_1 is column circulant and M_2 is column circulant and M_3 is column circulant, then $M_1 + M_2 + M_3$ is column circulant.
- (23) If M_1 is column circulant and M_2 is column circulant, then $a \cdot M_1 + b \cdot M_2$ is column circulant.
- (24) Suppose M_1 is column circulant and M_2 is column circulant and M_3 is column circulant. Then $a \cdot M_1 + b \cdot M_2 + c \cdot M_3$ is column circulant.
- (25) If M_1 is column circulant, then $-M_1$ is column circulant.
- (26) If M_1 is column circulant and M_2 is column circulant, then $M_1 - M_2$ is column circulant.

- (27) If M_1 is column circulant and M_2 is column circulant, then $a \cdot M_1 - b \cdot M_2$ is column circulant.
- (28) Suppose M_1 is column circulant and M_2 is column circulant and M_3 is column circulant. Then $(a \cdot M_1 + b \cdot M_2) - c \cdot M_3$ is column circulant.
- (29) Suppose M_1 is column circulant and M_2 is column circulant and M_3 is column circulant. Then $a \cdot M_1 - b \cdot M_2 - c \cdot M_3$ is column circulant.
- (30) Suppose M_1 is column circulant and M_2 is column circulant and M_3 is column circulant. Then $(a \cdot M_1 - b \cdot M_2) + c \cdot M_3$ is column circulant.
- (31) If p is first-line-of-circulant, then $-p$ is first-line-of-circulant.
- (32) If p is first-line-of-circulant, then $\text{LCirc}(-p) = -\text{LCirc } p$.
- (33) Suppose p is first-line-of-circulant and q is first-line-of-circulant and $\text{len } p = \text{len } q$. Then $p + q$ is first-line-of-circulant.
- (34) If $\text{len } p = \text{len } q$ and p is first-line-of-circulant and q is first-line-of-circulant, then $\text{LCirc}(p + q) = \text{LCirc } p + \text{LCirc } q$.
- (35) If p is first-column-of-circulant, then $-p$ is first-column-of-circulant.
- (36) For every finite sequence p of elements of K such that p is first-column-of-circulant holds $\text{CCirc}(-p) = -\text{CCirc } p$.
- (37) Suppose p is first-column-of-circulant and q is first-column-of-circulant and $\text{len } p = \text{len } q$. Then $p + q$ is first-column-of-circulant.
- (38) If $\text{len } p = \text{len } q$ and p is first-column-of-circulant and q is first-column-of-circulant, then $\text{CCirc}(p + q) = \text{CCirc } p + \text{CCirc } q$.
- (39) If $n > 0$, then $I_K^{n \times n}$ is column circulant.
- (40) If $n > 0$, then $I_K^{n \times n}$ is line circulant.
- (41) If p is first-line-of-circulant, then $a \cdot p$ is first-line-of-circulant.
- (42) If p is first-line-of-circulant, then $\text{LCirc}(a \cdot p) = a \cdot \text{LCirc } p$.
- (43) If p is first-line-of-circulant, then $a \cdot \text{LCirc } p + b \cdot \text{LCirc } p = \text{LCirc}((a+b) \cdot p)$.
- (44) If p is first-line-of-circulant and q is first-line-of-circulant and $\text{len } p = \text{len } q$ and $\text{len } p > 0$, then $a \cdot \text{LCirc } p + a \cdot \text{LCirc } q = \text{LCirc}(a \cdot (p + q))$.
- (45) If p is first-line-of-circulant and q is first-line-of-circulant and $\text{len } p = \text{len } q$, then $a \cdot \text{LCirc } p + b \cdot \text{LCirc } q = \text{LCirc}(a \cdot p + b \cdot q)$.
- (46) If p is first-column-of-circulant, then $a \cdot p$ is first-column-of-circulant.
- (47) If p is first-column-of-circulant, then $\text{CCirc}(a \cdot p) = a \cdot \text{CCirc } p$.
- (48) If p is first-column-of-circulant, then $a \cdot \text{CCirc } p + b \cdot \text{CCirc } p = \text{CCirc}((a+b) \cdot p)$.
- (49) Suppose p is first-column-of-circulant and q is first-column-of-circulant and $\text{len } p = \text{len } q$ and $\text{len } p > 0$. Then $a \cdot \text{CCirc } p + a \cdot \text{CCirc } q = \text{CCirc}(a \cdot (p + q))$.

- (50) If p is first-column-of-circulant and q is first-column-of-circulant and $\text{len } p = \text{len } q$, then $a \cdot \text{CCirc } p + b \cdot \text{CCirc } q = \text{CCirc}(a \cdot p + b \cdot q)$.

Let K be a set and let M be a matrix over K . We introduce M is circulant as a synonym of M is line circulant.

2. SOME PROPERTIES OF ANTI-CIRCULAR MATRICES

Let K be a field, let M_1 be a matrix over K , and let p be a finite sequence of elements of K . We say that M_1 is anti-circular about p if and only if the conditions (Def. 9) are satisfied.

- (Def. 9)(i) $\text{len } p = \text{width } M_1$,
 (ii) for all natural numbers i, j such that $\langle i, j \rangle \in$ the indices of M_1 and $i \leq j$ holds $(M_1)_{i,j} = p(((j - i) \bmod \text{len } p) + 1)$, and
 (iii) for all natural numbers i, j such that $\langle i, j \rangle \in$ the indices of M_1 and $i \geq j$ holds $(M_1)_{i,j} = (-p)(((j - i) \bmod \text{len } p) + 1)$.

Let K be a field and let M be a matrix over K . We say that M is anti-circular if and only if:

- (Def. 10) There exists a finite sequence p of elements of K such that $\text{len } p = \text{width } M$ and M is anti-circular about p .

Let K be a field and let p be a finite sequence of elements of K . We say that p is first-line-of-anti-circular if and only if:

- (Def. 11) There exists a square matrix over K of dimension $\text{len } p$ which is anti-circular about p .

Let K be a field and let p be a finite sequence of elements of K . Let us assume that p is first-line-of-anti-circular. The functor $\text{ACirc } p$ yields a square matrix over K of dimension $\text{len } p$ and is defined by:

- (Def. 12) $\text{ACirc } p$ is anti-circular about p .

One can prove the following propositions:

- (51) If M_1 is anti-circular, then $a \cdot M_1$ is anti-circular.
 (52) If M_1 is anti-circular and M_2 is anti-circular, then $M_1 + M_2$ is anti-circular.
 (53) Let K be a Fanoian field, n, i, j be natural numbers, and M_1 be a square matrix over K of dimension n . Suppose $\langle i, j \rangle \in$ the indices of M_1 and $i = j$ and M_1 is anti-circular. Then $(M_1)_{i,j} = 0_K$.
 (54) If M_1 is anti-circular and $\langle i, j \rangle \in \text{Seg } n \times \text{Seg } n$ and $k = i + 1$ and $l = j + 1$ and $i < n$ and $j < n$, then $(M_1)_{k,l} = (M_1)_{i,j}$.
 (55) If M_1 is anti-circular, then $-M_1$ is anti-circular.
 (56) If M_1 is anti-circular and M_2 is anti-circular, then $M_1 - M_2$ is anti-circular.

- (57) If M_1 is anti-circular about p and $n > 0$, then $p = \text{Line}(M_1, 1)$.
 - (58) If p is first-line-of-anti-circular, then $-p$ is first-line-of-anti-circular.
 - (59) If p is first-line-of-anti-circular, then $\text{ACirc}(-p) = -\text{ACirc } p$.
 - (60) Suppose p is first-line-of-anti-circular and q is first-line-of-anti-circular and $\text{len } p = \text{len } q$. Then $p + q$ is first-line-of-anti-circular.
 - (61) If p is first-line-of-anti-circular and q is first-line-of-anti-circular and $\text{len } p = \text{len } q$, then $\text{ACirc}(p + q) = \text{ACirc } p + \text{ACirc } q$.
 - (62) If p is first-line-of-anti-circular, then $a \cdot p$ is first-line-of-anti-circular.
 - (63) If p is first-line-of-anti-circular, then $\text{ACirc}(a \cdot p) = a \cdot \text{ACirc } p$.
 - (64) If p is first-line-of-anti-circular, then $a \cdot \text{ACirc } p + b \cdot \text{ACirc } p = \text{ACirc}((a + b) \cdot p)$.
 - (65) Suppose p is first-line-of-anti-circular and q is first-line-of-anti-circular and $\text{len } p = \text{len } q$ and $\text{len } p > 0$. Then $a \cdot \text{ACirc } p + a \cdot \text{ACirc } q = \text{ACirc}(a \cdot (p + q))$.
 - (66) Suppose p is first-line-of-anti-circular and q is first-line-of-anti-circular and $\text{len } p = \text{len } q$. Then $a \cdot \text{ACirc } p + b \cdot \text{ACirc } q = \text{ACirc}(a \cdot p + b \cdot q)$.
- Let us consider K , n . Observe that $0_K^{n \times n}$ is anti-circular.

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Received August 26, 2008
