FORMALIZED MATHEMATICS
Vol. **16**, No. **4**, Pages 333–338, **2008**
DOI: 10.2478/v10037-008-0041-z

Partial Differentiation of Real Binary Functions

Bing Xie
Qingdao University of Science and Technology
China

Xiquan Liang
Qingdao University of Science and Technology
China

Hongwei Li
Qingdao University of Science and Technology
China

Summary. In this article, we define two single-variable functions SVF1 and SVF2, then discuss partial differentiation of real binary functions by dint of one variable function SVF1 and SVF2. The main properties of partial differentiation are shown [7].

MML identifier: `PDIFF_2`, version: `7.9.03 4.104.1021`

The articles [14], [4], [15], [5], [1], [8], [10], [9], [2], [3], [13], [6], [12], [11], and [7] provide the notation and terminology for this paper.

1. Preliminaries

For simplicity, we adopt the following convention: x, x_0, y, y_0, r are real numbers, z, z_0 are elements of $\mathcal{R}^2$, Z is a subset of $\mathcal{R}^2$, f, f_1, f_2 are partial functions from $\mathcal{R}^2$ to $\mathbb{R}$, R is a rest, and L is a linear function.

Next we state two propositions:

(1) $\operatorname{dom}\operatorname{proj}(1,2) = \mathcal{R}^2$ and $\operatorname{rng}\operatorname{proj}(1,2) = \mathbb{R}$ and for all elements x, y of $\mathbb{R}$ holds $(\operatorname{proj}(1,2))(\langle x, y\rangle) = x$.

(2) $\operatorname{dom}\operatorname{proj}(2,2) = \mathcal{R}^2$ and $\operatorname{rng}\operatorname{proj}(2,2) = \mathbb{R}$ and for all elements x, y of $\mathbb{R}$ holds $(\operatorname{proj}(2,2))(\langle x, y\rangle) = y$.

ISSN 1426–2630(p), 1898-9934(e)

2. Partial Differentiation of Real Binary Functions

Let f be a partial function from $\mathcal{R}^2$ to $\mathbb{R}$ and let z be an element of $\mathcal{R}^2$. The functor $\mathrm{SVF1}(f, z)$ yielding a partial function from $\mathbb{R}$ to $\mathbb{R}$ is defined by:

(Def. 1) $\mathrm{SVF1}(f, z) = f \cdot \mathrm{reproj}(1, z)$.

The functor $\mathrm{SVF2}(f, z)$ yields a partial function from $\mathbb{R}$ to $\mathbb{R}$ and is defined as follows:

(Def. 2) $\mathrm{SVF2}(f, z) = f \cdot \mathrm{reproj}(2, z)$.

Next we state two propositions:

(3) If $z = \langle x, y \rangle$ and f is partially differentiable in z w.r.t. 1 coordinate, then $\mathrm{SVF1}(f, z)$ is differentiable in x.

(4) If $z = \langle x, y \rangle$ and f is partially differentiable in z w.r.t. 2 coordinate, then $\mathrm{SVF2}(f, z)$ is differentiable in y.

Let f be a partial function from $\mathcal{R}^2$ to $\mathbb{R}$ and let z be an element of $\mathcal{R}^2$. We say that f is partial differentiable on 1st coordinate in z if and only if:

(Def. 3) There exist real numbers x_0, y_0 such that $z = \langle x_0, y_0 \rangle$ and $\mathrm{SVF1}(f, z)$ is differentiable in x_0.

We say that f is partial differentiable on 2nd coordinate in z if and only if:

(Def. 4) There exist real numbers x_0, y_0 such that $z = \langle x_0, y_0 \rangle$ and $\mathrm{SVF2}(f, z)$ is differentiable in y_0.

Next we state two propositions:

(5) Suppose $z = \langle x_0, y_0 \rangle$ and f is partial differentiable on 1st coordinate in z. Then there exists a neighbourhood N of x_0 such that $N \subseteq \mathrm{dom}\,\mathrm{SVF1}(f, z)$ and there exist L, R such that for every x such that $x \in N$ holds $(\mathrm{SVF1}(f, z))(x) - (\mathrm{SVF1}(f, z))(x_0) = L(x - x_0) + R(x - x_0)$.

(6) Suppose $z = \langle x_0, y_0 \rangle$ and f is partial differentiable on 2nd coordinate in z. Then there exists a neighbourhood N of y_0 such that $N \subseteq \mathrm{dom}\,\mathrm{SVF2}(f, z)$ and there exist L, R such that for every y such that $y \in N$ holds $(\mathrm{SVF2}(f, z))(y) - (\mathrm{SVF2}(f, z))(y_0) = L(y - y_0) + R(y - y_0)$.

Let f be a partial function from $\mathcal{R}^2$ to $\mathbb{R}$ and let z be an element of $\mathcal{R}^2$. Let us observe that f is partial differentiable on 1st coordinate in z if and only if the condition (Def. 5) is satisfied.

(Def. 5) There exist real numbers x_0, y_0 such that

(i) $z = \langle x_0, y_0 \rangle$, and

(ii) there exists a neighbourhood N of x_0 such that $N \subseteq \mathrm{dom}\,\mathrm{SVF1}(f, z)$ and there exist L, R such that for every x such that $x \in N$ holds $(\mathrm{SVF1}(f, z))(x) - (\mathrm{SVF1}(f, z))(x_0) = L(x - x_0) + R(x - x_0)$.

Let f be a partial function from $\mathcal{R}^2$ to $\mathbb{R}$ and let z be an element of $\mathcal{R}^2$. Let us observe that f is partial differentiable on 2nd coordinate in z if and only if the condition (Def. 6) is satisfied.

(Def. 6) There exist real numbers x_0, y_0 such that

(i) $z = \langle x_0, y_0 \rangle$, and

(ii) there exists a neighbourhood N of y_0 such that $N \subseteq \operatorname{dom} \mathrm{SVF2}(f, z)$ and there exist L, R such that for every y such that $y \in N$ holds $(\mathrm{SVF2}(f, z))(y) - (\mathrm{SVF2}(f, z))(y_0) = L(y - y_0) + R(y - y_0)$.

Next we state two propositions:

(7) Let f be a partial function from $\mathcal{R}^2$ to $\mathbb{R}$ and z be an element of $\mathcal{R}^2$. Then f is partial differentiable on 1st coordinate in z if and only if f is partially differentiable in z w.r.t. 1 coordinate.

(8) Let f be a partial function from $\mathcal{R}^2$ to $\mathbb{R}$ and z be an element of $\mathcal{R}^2$. Then f is partial differentiable on 2nd coordinate in z if and only if f is partially differentiable in z w.r.t. 2 coordinate.

Let f be a partial function from $\mathcal{R}^2$ to $\mathbb{R}$ and let z be an element of $\mathcal{R}^2$. The functor $\mathrm{partdiff1}(f, z)$ yielding a real number is defined by:

(Def. 7) $\mathrm{partdiff1}(f, z) = \mathrm{partdiff}(f, z, 1)$.

The functor $\mathrm{partdiff2}(f, z)$ yielding a real number is defined as follows:

(Def. 8) $\mathrm{partdiff2}(f, z) = \mathrm{partdiff}(f, z, 2)$.

One can prove the following propositions:

(9) Suppose $z = \langle x_0, y_0 \rangle$ and f is partial differentiable on 1st coordinate in z. Then $r = \mathrm{partdiff1}(f, z)$ if and only if there exist real numbers x_0, y_0 such that $z = \langle x_0, y_0 \rangle$ and there exists a neighbourhood N of x_0 such that $N \subseteq \operatorname{dom} \mathrm{SVF1}(f, z)$ and there exist L, R such that $r = L(1)$ and for every x such that $x \in N$ holds $(\mathrm{SVF1}(f, z))(x) - (\mathrm{SVF1}(f, z))(x_0) = L(x - x_0) + R(x - x_0)$.

(10) Suppose $z = \langle x_0, y_0 \rangle$ and f is partial differentiable on 2nd coordinate in z. Then $r = \mathrm{partdiff2}(f, z)$ if and only if there exist real numbers x_0, y_0 such that $z = \langle x_0, y_0 \rangle$ and there exists a neighbourhood N of y_0 such that $N \subseteq \operatorname{dom} \mathrm{SVF2}(f, z)$ and there exist L, R such that $r = L(1)$ and for every y such that $y \in N$ holds $(\mathrm{SVF2}(f, z))(y) - (\mathrm{SVF2}(f, z))(y_0) = L(y - y_0) + R(y - y_0)$.

(11) If $z = \langle x_0, y_0 \rangle$ and f is partial differentiable on 1st coordinate in z, then $\mathrm{partdiff1}(f, z) = (\mathrm{SVF1}(f, z))'(x_0)$.

(12) If $z = \langle x_0, y_0 \rangle$ and f is partial differentiable on 2nd coordinate in z, then $\mathrm{partdiff2}(f, z) = (\mathrm{SVF2}(f, z))'(y_0)$.

Let f be a partial function from $\mathcal{R}^2$ to $\mathbb{R}$ and let Z be a set. We say that f is partial differentiable w.r.t. 1st coordinate on Z if and only if:

(Def. 9) $Z \subseteq \operatorname{dom} f$ and for every element z of $\mathcal{R}^2$ such that $z \in Z$ holds $f \restriction Z$ is partial differentiable on 1st coordinate in z.

We say that f is partial differentiable w.r.t. 2nd coordinate on Z if and only if:

(Def. 10) $Z \subseteq \operatorname{dom} f$ and for every element z of $\mathcal{R}^2$ such that $z \in Z$ holds $f \restriction Z$ is partial differentiable on 2nd coordinate in z.

One can prove the following two propositions:

(13) Suppose f is partial differentiable w.r.t. 1st coordinate on Z. Then $Z \subseteq \operatorname{dom} f$ and for every z such that $z \in Z$ holds f is partial differentiable on 1st coordinate in z.

(14) Suppose f is partial differentiable w.r.t. 2nd coordinate on Z. Then $Z \subseteq \operatorname{dom} f$ and for every z such that $z \in Z$ holds f is partial differentiable on 2nd coordinate in z.

Let f be a partial function from $\mathcal{R}^2$ to $\mathbb{R}$ and let Z be a set. Let us assume that f is partial differentiable w.r.t. 1st coordinate on Z. The functor $f{\restriction}_Z^{1\mathrm{st}}$ yielding a partial function from $\mathcal{R}^2$ to $\mathbb{R}$ is defined as follows:

(Def. 11) $\operatorname{dom}(f{\restriction}_Z^{1\mathrm{st}}) = Z$ and for every element z of $\mathcal{R}^2$ such that $z \in Z$ holds $f{\restriction}_Z^{1\mathrm{st}}(z) = \operatorname{partdiff1}(f, z)$.

Let f be a partial function from $\mathcal{R}^2$ to $\mathbb{R}$ and let Z be a set. Let us assume that f is partial differentiable w.r.t. 2nd coordinate on Z. The functor $f{\restriction}_Z^{2\mathrm{nd}}$ yielding a partial function from $\mathcal{R}^2$ to $\mathbb{R}$ is defined as follows:

(Def. 12) $\operatorname{dom}(f{\restriction}_Z^{2\mathrm{nd}}) = Z$ and for every element z of $\mathcal{R}^2$ such that $z \in Z$ holds $f{\restriction}_Z^{2\mathrm{nd}}(z) = \operatorname{partdiff2}(f, z)$.

3. Main Properties of Partial Differentiation of Real Binary Functions

We now state a number of propositions:

(15) Let z_0 be an element of $\mathcal{R}^2$ and N be a neighbourhood of $(\operatorname{proj}(1,2))(z_0)$. Suppose f is partial differentiable on 1st coordinate in z_0 and $N \subseteq \operatorname{dom} \operatorname{SVF1}(f, z_0)$. Let h be a convergent to 0 sequence of real numbers and c be a constant sequence of real numbers. Suppose $\operatorname{rng} c = \{(\operatorname{proj}(1,2))(z_0)\}$ and $\operatorname{rng}(h+c) \subseteq N$. Then $h^{-1}\,(\operatorname{SVF1}(f, z_0) \cdot (h+c) - \operatorname{SVF1}(f, z_0) \cdot c)$ is convergent and $\operatorname{partdiff1}(f, z_0) = \lim(h^{-1}\,(\operatorname{SVF1}(f, z_0) \cdot (h+c) - \operatorname{SVF1}(f, z_0) \cdot c))$.

(16) Let z_0 be an element of $\mathcal{R}^2$ and N be a neighbourhood of $(\operatorname{proj}(2,2))(z_0)$. Suppose f is partial differentiable on 2nd coordinate in z_0 and $N \subseteq \operatorname{dom} \operatorname{SVF2}(f, z_0)$. Let h be a convergent to 0 sequence of real numbers and c be a constant sequence of real numbers. Suppose $\operatorname{rng} c = \{(\operatorname{proj}(2,2))(z_0)\}$ and $\operatorname{rng}(h+c) \subseteq N$. Then $h^{-1}\,(\operatorname{SVF2}(f, z_0) \cdot (h+c) - \operatorname{SVF2}(f, z_0) \cdot c)$ is convergent and $\operatorname{partdiff2}(f, z_0) = \lim(h^{-1}\,(\operatorname{SVF2}(f, z_0) \cdot (h+c) - \operatorname{SVF2}(f, z_0) \cdot c))$.

(17) Suppose f_1 is partial differentiable on 1st coordinate in z_0 and f_2 is partial differentiable on 1st coordinate in z_0. Then $f_1 + f_2$ is par-

tial differentiable on 1st coordinate in z_0 and $\mathrm{partdiff1}(f_1 + f_2, z_0) = \mathrm{partdiff1}(f_1, z_0) + \mathrm{partdiff1}(f_2, z_0)$.

(18) Suppose f_1 is partial differentiable on 2nd coordinate in z_0 and f_2 is partial differentiable on 2nd coordinate in z_0. Then $f_1 + f_2$ is partial differentiable on 2nd coordinate in z_0 and $\mathrm{partdiff2}(f_1 + f_2, z_0) = \mathrm{partdiff2}(f_1, z_0) + \mathrm{partdiff2}(f_2, z_0)$.

(19) Suppose f_1 is partial differentiable on 1st coordinate in z_0 and f_2 is partial differentiable on 1st coordinate in z_0. Then $f_1 - f_2$ is partial differentiable on 1st coordinate in z_0 and $\mathrm{partdiff1}(f_1 - f_2, z_0) = \mathrm{partdiff1}(f_1, z_0) - \mathrm{partdiff1}(f_2, z_0)$.

(20) Suppose f_1 is partial differentiable on 2nd coordinate in z_0 and f_2 is partial differentiable on 2nd coordinate in z_0. Then $f_1 - f_2$ is partial differentiable on 2nd coordinate in z_0 and $\mathrm{partdiff2}(f_1 - f_2, z_0) = \mathrm{partdiff2}(f_1, z_0) - \mathrm{partdiff2}(f_2, z_0)$.

(21) Suppose f is partial differentiable on 1st coordinate in z_0. Then $r\,f$ is partial differentiable on 1st coordinate in z_0 and $\mathrm{partdiff1}(r\,f, z_0) = r \cdot \mathrm{partdiff1}(f, z_0)$.

(22) Suppose f is partial differentiable on 2nd coordinate in z_0. Then $r\,f$ is partial differentiable on 2nd coordinate in z_0 and $\mathrm{partdiff2}(r\,f, z_0) = r \cdot \mathrm{partdiff2}(f, z_0)$.

(23) Suppose f_1 is partial differentiable on 1st coordinate in z_0 and f_2 is partial differentiable on 1st coordinate in z_0. Then $f_1\,f_2$ is partial differentiable on 1st coordinate in z_0.

(24) Suppose f_1 is partial differentiable on 2nd coordinate in z_0 and f_2 is partial differentiable on 2nd coordinate in z_0. Then $f_1\,f_2$ is partial differentiable on 2nd coordinate in z_0.

(25) Let z_0 be an element of $\mathcal{R}^2$. Suppose f is partial differentiable on 1st coordinate in z_0. Then $\mathrm{SVF1}(f, z_0)$ is continuous in $(\mathrm{proj}(1,2))(z_0)$.

(26) Let z_0 be an element of $\mathcal{R}^2$. Suppose f is partial differentiable on 2nd coordinate in z_0. Then $\mathrm{SVF2}(f, z_0)$ is continuous in $(\mathrm{proj}(2,2))(z_0)$.

(27) If f is partial differentiable on 1st coordinate in z_0, then there exists R such that $R(0) = 0$ and R is continuous in 0.

(28) If f is partial differentiable on 2nd coordinate in z_0, then there exists R such that $R(0) = 0$ and R is continuous in 0.

References

[1] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.

[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[3] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.

[4] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[5] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.
[6] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(**4**):599–603, 1991.
[7] Noboru Endou, Yasunari Shidama, and Keiichi Miyajima. Partial differentiation on normed linear spaces ∇^n. *Formalized Mathematics*, 15(**2**):65–72, 2007.
[8] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[9] Jarosław Kotowicz. Convergent sequences and the limit of sequences. *Formalized Mathematics*, 1(**2**):273–275, 1990.
[10] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.
[11] Konrad Raczkowski and Paweł Sadowski. Real function continuity. *Formalized Mathematics*, 1(**4**):787–791, 1990.
[12] Konrad Raczkowski and Paweł Sadowski. Real function differentiability. *Formalized Mathematics*, 1(**4**):797–801, 1990.
[13] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Formalized Mathematics*, 1(**4**):777–780, 1990.
[14] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[15] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

Received August 5, 2008