

Introduction to Matroids¹

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Summary. The paper includes elements of the theory of matroids [23].
 The formalization is done according to [12].

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The articles [7], [22], [17], [15], [8], [5], [6], [19], [9], [3], [2], [4], [1], [21], [11], [20], [18], [16], [10], [13], and [14] provide the terminology and notation for this paper.

1. DEFINITION BY INDEPENDENT SETS

A subset family structure is a topological structure.

Let M be a subset family structure and let A be a subset of M . We introduce A is independent as a synonym of A is open. We introduce A is dependent as an antonym of A is open.

Let M be a subset family structure. The family of M yielding a family of subsets of M is defined as follows:

(Def. 1) The family of M = the topology of M .

Let M be a subset family structure and let A be a subset of M . Let us observe that A is independent if and only if:

(Def. 2) $A \in$ the family of M .

Let M be a subset family structure. We say that M is subset-closed if and only if:

(Def. 3) The family of M is subset-closed.

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We say that M has exchange property if and only if the condition (Def. 4) is satisfied.

- (Def. 4) Let A, B be finite subsets of M . Suppose $A \in$ the family of M and $B \in$ the family of M and $\text{card } B = \text{card } A + 1$. Then there exists an element e of M such that $e \in B \setminus A$ and $A \cup \{e\} \in$ the family of M .

One can check that there exists a subset family structure which is strict, non empty, non void, finite, and subset-closed and has exchange property.

Let M be a non void subset family structure. One can verify that there exists a subset of M which is independent.

Let M be a subset-closed subset family structure. One can verify that the family of M is subset-closed.

We now state the proposition

- (1) Let M be a non void subset-closed subset family structure, A be an independent subset of M , and B be a set. If $B \subseteq A$, then B is an independent subset of M .

Let M be a non void subset-closed subset family structure. Note that there exists a subset of M which is finite and independent.

A matroid is a non empty non void subset-closed subset family structure with exchange property.

One can prove the following proposition

- (2) For every subset-closed subset family structure M holds M is non void iff $\emptyset \in$ the family of M .

Let M be a non void subset-closed subset family structure. Note that $\emptyset_{\text{the carrier of } M}$ is independent.

The following proposition is true

- (3) Let M be a non void subset family structure. Then M is subset-closed if and only if for all subsets A, B of M such that A is independent and $B \subseteq A$ holds B is independent.

Let M be a non void subset-closed subset family structure, let A be an independent subset of M , and let B be a set. One can check the following observations:

- * $A \cap B$ is independent,
- * $B \cap A$ is independent, and
- * $A \setminus B$ is independent.

Next we state the proposition

- (4) Let M be a non void non empty subset family structure. Then M has exchange property if and only if for all finite subsets A, B of M such that A is independent and B is independent and $\text{card } B = \text{card } A + 1$ there exists an element e of M such that $e \in B \setminus A$ and $A \cup \{e\}$ is independent.

Let A be a set. We introduce A is finite-membered as a synonym of A has finite elements.

Let A be a set. Let us observe that A is finite-membered if and only if:

(Def. 5) For every set B such that $B \in A$ holds B is finite.

Let M be a subset family structure. We say that M is finite-membered if and only if:

(Def. 6) The family of M is finite-membered.

Let M be a subset family structure. We say that M is finite-degree if and only if the conditions (Def. 7) are satisfied.

(Def. 7)(i) M is finite-membered, and

(ii) there exists a natural number n such that for every finite subset A of M such that A is independent holds $\text{card } A \leq n$.

Let us note that every subset family structure which is finite-degree is also finite-membered and every subset family structure which is finite is also finite-degree.

2. EXAMPLES

Let us note that there exists a set which is mutually-disjoint and non empty and has non empty elements.

The following propositions are true:

(5) For all finite sets A, B such that $\text{card } A < \text{card } B$ there exists a set x such that $x \in B \setminus A$.

(6) For every mutually-disjoint non empty set P with non empty elements holds every choice function of P is one-to-one.

Let us mention that every discrete subset family structure is non void and subset-closed and has exchange property.

Next we state the proposition

(7) Every non empty discrete topological structure is a matroid.

Let P be a set. The functor $\text{ProdMatroid } P$ yields a strict subset family structure and is defined by the conditions (Def. 8).

(Def. 8)(i) The carrier of $\text{ProdMatroid } P = \bigcup P$, and

(ii) the family of $\text{ProdMatroid } P = \{A \subseteq \bigcup P : \bigwedge_{D \in \text{set}} (D \in P \Rightarrow \bigvee_{d \in \text{set}} A \cap D \subseteq \{d\})\}$.

Let P be a non empty set with non empty elements. One can verify that $\text{ProdMatroid } P$ is non empty.

Next we state the proposition

(8) Let P be a set and A be a subset of $\text{ProdMatroid } P$. Then A is independent if and only if for every element D of P there exists an element d of D such that $A \cap D \subseteq \{d\}$.

Let P be a set. One can verify that $\text{ProdMatroid } P$ is non void and subset-closed.

Next we state two propositions:

- (9) Let P be a mutually-disjoint set and x be a subset of $\text{ProdMatroid } P$. Then there exists a function f from x into P such that for every set a such that $a \in x$ holds $a \in f(a)$.
- (10) Let P be a mutually-disjoint set, x be a subset of $\text{ProdMatroid } P$, and f be a function from x into P . Suppose that for every set a such that $a \in x$ holds $a \in f(a)$. Then x is independent if and only if f is one-to-one.

Let P be a mutually-disjoint set. Observe that $\text{ProdMatroid } P$ has exchange property.

Let X be a finite set and let P be a subset of 2^X . One can check that $\text{ProdMatroid } P$ is finite.

Let X be a set. Observe that every partition of X is mutually-disjoint.

One can check that there exists a matroid which is finite and strict.

Let M be a finite-membered non void subset family structure. Observe that every independent subset of M is finite.

Let F be a field and let V be a vector space over F . The matroid of linearly independent subsets of V is a strict subset family structure and is defined by the conditions (Def. 9).

- (Def. 9)(i) The carrier of the matroid of linearly independent subsets of V = the carrier of V , and
- (ii) the family of the matroid of linearly independent subsets of $V = \{A \subseteq V : A \text{ is linearly independent}\}$.

Let F be a field and let V be a vector space over F . Note that the matroid of linearly independent subsets of V is non empty, non void, and subset-closed.

Let F be a field and let V be a vector space over F . Observe that there exists a subset of V which is linearly independent and empty.

The following three propositions are true:

- (11) Let F be a field, V be a vector space over F , and A be a subset of the matroid of linearly independent subsets of V . Then A is independent if and only if A is a linearly independent subset of V .
- (12) Let F be a field, V be a vector space over F , and A, B be finite subsets of V . Suppose $B \subseteq A$. Let v be a vector of V . Suppose $v \in \text{Lin}(A)$ and $v \notin \text{Lin}(B)$. Then there exists a vector w of V such that $w \in A \setminus B$ and $w \in \text{Lin}((A \setminus \{w\}) \cup \{v\})$.
- (13) Let F be a field, V be a vector space over F , and A be a subset of V . Suppose A is linearly independent. Let a be an element of V . If $a \notin$ the carrier of $\text{Lin}(A)$, then $A \cup \{a\}$ is linearly independent.

Let F be a field and let V be a vector space over F . Observe that the matroid of linearly independent subsets of V has exchange property.

Let F be a field and let V be a finite dimensional vector space over F . Note that the matroid of linearly independent subsets of V is finite-membered.

3. MAXIMAL INDEPENDENT SUBSETS, RANKS, AND BASIS

Let M be a subset family structure and let A, C be subsets of M . We say that A is maximal independent in C if and only if:

- (Def. 10) A is independent and $A \subseteq C$ and for every subset B of M such that B is independent and $B \subseteq C$ and $A \subseteq B$ holds $A = B$.

The following propositions are true:

- (14) Let M be a non void finite-degree subset family structure and C, A be subsets of M . Suppose $A \subseteq C$ and A is independent. Then there exists an independent subset B of M such that $A \subseteq B$ and B is maximal independent in C .
- (15) Let M be a non void finite-degree subset-closed subset family structure and C be a subset of M . Then there exists an independent subset of M which is maximal independent in C .
- (16) Let M be a non empty non void subset-closed finite-degree subset family structure. Then M is a matroid if and only if for every subset C of M and for all independent subsets A, B of M such that A is maximal independent in C and B is maximal independent in C holds $\text{card } A = \text{card } B$.

Let M be a finite-degree matroid and let C be a subset of M . The functor $\text{Rnk } C$ yields a natural number and is defined by:

- (Def. 11) $\text{Rnk } C = \bigcup \{\text{card } A; A \text{ ranges over independent subsets of } M: A \subseteq C\}$.

One can prove the following propositions:

- (17) Let M be a finite-degree matroid, C be a subset of M , and A be an independent subset of M . If $A \subseteq C$, then $\text{card } A \leq \text{Rnk } C$.
- (18) Let M be a finite-degree matroid and C be a subset of M . Then there exists an independent subset A of M such that $A \subseteq C$ and $\text{card } A = \text{Rnk } C$.
- (19) Let M be a finite-degree matroid, C be a subset of M , and A be an independent subset of M . Then A is maximal independent in C if and only if $A \subseteq C$ and $\text{card } A = \text{Rnk } C$.
- (20) For every finite-degree matroid M and for every finite subset C of M holds $\text{Rnk } C \leq \text{card } C$.
- (21) Let M be a finite-degree matroid and C be a finite subset of M . Then C is independent if and only if $\text{card } C = \text{Rnk } C$.

Let M be a finite-degree matroid. The functor $\text{Rnk } M$ yielding a natural number is defined by:

(Def. 12) $\text{Rnk } M = \text{Rnk}(\Omega_M)$.

Let M be a non void finite-degree subset family structure. An independent subset of M is said to be a basis of M if:

(Def. 13) It is maximal independent in Ω_M .

One can prove the following propositions:

(22) For every finite-degree matroid M and for all bases B_1, B_2 of M holds $\text{card } B_1 = \text{card } B_2$.

(23) For every finite-degree matroid M and for every independent subset A of M there exists a basis B of M such that $A \subseteq B$.

We follow the rules: M is a finite-degree matroid, A, B, C are subsets of M , and e, f are elements of M .

Next we state four propositions:

(24) If $A \subseteq B$, then $\text{Rnk } A \leq \text{Rnk } B$.

(25) $\text{Rnk}(A \cup B) + \text{Rnk}(A \cap B) \leq \text{Rnk } A + \text{Rnk } B$.

(26) $\text{Rnk } A \leq \text{Rnk}(A \cup B)$ and $\text{Rnk}(A \cup \{e\}) \leq \text{Rnk } A + 1$.

(27) If $\text{Rnk}(A \cup \{e\}) = \text{Rnk}(A \cup \{f\})$ and $\text{Rnk}(A \cup \{f\}) = \text{Rnk } A$, then $\text{Rnk}(A \cup \{e, f\}) = \text{Rnk } A$.

4. DEPENDENCE ON A SET, SPANS, AND CYCLES

Let M be a finite-degree matroid, let e be an element of M , and let A be a subset of M . We say that e is dependent on A if and only if:

(Def. 14) $\text{Rnk}(A \cup \{e\}) = \text{Rnk } A$.

We now state two propositions:

(28) If $e \in A$, then e is dependent on A .

(29) If $A \subseteq B$ and e is dependent on A , then e is dependent on B .

Let M be a finite-degree matroid and let A be a subset of M . The functor $\text{Span } A$ yielding a subset of M is defined as follows:

(Def. 15) $\text{Span } A = \{e \in M : e \text{ is dependent on } A\}$.

Next we state several propositions:

(30) $e \in \text{Span } A$ iff $\text{Rnk}(A \cup \{e\}) = \text{Rnk } A$.

(31) $A \subseteq \text{Span } A$.

(32) If $A \subseteq B$, then $\text{Span } A \subseteq \text{Span } B$.

(33) $\text{Rnk Span } A = \text{Rnk } A$.

(34) If e is dependent on $\text{Span } A$, then e is dependent on A .

(35) $\text{Span Span } A = \text{Span } A$.

(36) If $f \notin \text{Span } A$ and $f \in \text{Span}(A \cup \{e\})$, then $e \in \text{Span}(A \cup \{f\})$.

Let M be a subset family structure and let A be a subset of M . We say that A is cycle if and only if:

- (Def. 16) A is dependent and for every element e of M such that $e \in A$ holds $A \setminus \{e\}$ is independent.

Next we state the proposition

- (37) If A is cycle, then A is non empty and finite.

Let us consider M . Note that every subset of M which is cycle is also non empty and finite.

One can prove the following propositions:

- (38) A is cycle iff A is non empty and for every e such that $e \in A$ holds $A \setminus \{e\}$ is maximal independent in A .
- (39) If A is cycle, then $\text{Rnk } A + 1 = \overline{\overline{A}}$.
- (40) If A is cycle and $e \in A$, then e is dependent on $A \setminus \{e\}$.
- (41) If A is cycle and B is cycle and $A \subseteq B$, then $A = B$.
- (42) If for every B such that $B \subseteq A$ holds B is not cycle, then A is independent.
- (43) If A is cycle and B is cycle and $A \neq B$ and $e \in A \cap B$, then there exists C such that C is cycle and $C \subseteq (A \cup B) \setminus \{e\}$.
- (44) If A is independent and B is cycle and C is cycle and $B \subseteq A \cup \{e\}$ and $C \subseteq A \cup \{e\}$, then $B = C$.

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