

Integral of Complex-Valued Measurable Function

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Summary. In this article, we formalized the notion of the integral of a complex-valued function considered as a sum of its real and imaginary parts. Then we defined the measurability and integrability in this context, and proved the linearity and several other basic properties of complex-valued measurable functions. The set of properties showed in this paper is based on [15], where the case of real-valued measurable functions is considered.

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The notation and terminology used here are introduced in the following papers: [17], [1], [11], [18], [6], [19], [7], [2], [12], [14], [16], [5], [4], [3], [9], [10], [13], [8], and [15].

1. DEFINITIONS FOR COMPLEX-VALUED FUNCTIONS

One can prove the following proposition

- (1) For all real numbers a, b holds $\overline{\mathbb{R}}(a) + \overline{\mathbb{R}}(b) = a + b$ and $-\overline{\mathbb{R}}(a) = -a$ and $\overline{\mathbb{R}}(a) - \overline{\mathbb{R}}(b) = a - b$ and $\overline{\mathbb{R}}(a) \cdot \overline{\mathbb{R}}(b) = a \cdot b$.

Let X be a non empty set and let f be a partial function from X to $\mathbb{C}$. The functor $\Re(f)$ yields a partial function from X to $\mathbb{R}$ and is defined as follows:

- (Def. 1) $\text{dom } \Re(f) = \text{dom } f$ and for every element x of X such that $x \in \text{dom } \Re(f)$ holds $\Re(f)(x) = \Re(f(x))$.

The functor $\Im(f)$ yields a partial function from X to $\mathbb{R}$ and is defined as follows:

- (Def. 2) $\text{dom } \Im(f) = \text{dom } f$ and for every element x of X such that $x \in \text{dom } \Im(f)$ holds $\Im(f)(x) = \Im(f(x))$.

2. THE MEASURABILITY OF COMPLEX-VALUED FUNCTIONS

For simplicity, we use the following convention: X is a non empty set, Y is a set, S is a σ -field of subsets of X , M is a σ -measure on S , f, g are partial functions from X to $\mathbb{C}$, r is a real number, c is a complex number, and E, A, B are elements of S .

Let X be a non empty set, let S be a σ -field of subsets of X , let f be a partial function from X to $\mathbb{C}$, and let E be an element of S . We say that f is measurable on E if and only if:

- (Def. 3) $\Re(f)$ is measurable on E and $\Im(f)$ is measurable on E .

One can prove the following propositions:

- (2) $r \Re(f) = \Re(r f)$ and $r \Im(f) = \Im(r f)$.
- (3) $\Re(c f) = \Re(c) \Re(f) - \Im(c) \Im(f)$ and $\Im(c f) = \Im(c) \Re(f) + \Re(c) \Im(f)$.
- (4) $-\Im(f) = \Re(i f)$ and $\Re(f) = \Im(i f)$.
- (5) $\Re(f + g) = \Re(f) + \Re(g)$ and $\Im(f + g) = \Im(f) + \Im(g)$.
- (6) $\Re(f - g) = \Re(f) - \Re(g)$ and $\Im(f - g) = \Im(f) - \Im(g)$.
- (7) $\Re(f)|_A = \Re(f|_A)$ and $\Im(f)|_A = \Im(f|_A)$.
- (8) $f = \Re(f) + i \Im(f)$.
- (9) If $B \subseteq A$ and f is measurable on A , then f is measurable on B .
- (10) If f is measurable on A and f is measurable on B , then f is measurable on $A \cup B$.
- (11) If f is measurable on A and g is measurable on A , then $f+g$ is measurable on A .
- (12) If f is measurable on A and g is measurable on A and $A \subseteq \text{dom } g$, then $f - g$ is measurable on A .
- (13) If $Y \subseteq \text{dom}(f+g)$, then $\text{dom}(f|_Y + g|_Y) = Y$ and $(f+g)|_Y = f|_Y + g|_Y$.
- (14) If f is measurable on B and $A = \text{dom } f \cap B$, then $f|_B$ is measurable on A .
- (15) If $\text{dom } f, \text{dom } g \in S$, then $\text{dom}(f + g) \in S$.
- (16) If $\text{dom } f = A$, then f is measurable on B iff f is measurable on $A \cap B$.
- (17) If f is measurable on A and $A \subseteq \text{dom } f$, then $c f$ is measurable on A .
- (18) Given an element A of S such that $\text{dom } f = A$. Let c be a complex number and B be an element of S . If f is measurable on B , then $c f$ is measurable on B .

3. THE INTEGRAL OF A COMPLEX-VALUED MEASURABLE FUNCTION

Let X be a non empty set, let S be a σ -field of subsets of X , let M be a σ -measure on S , and let f be a partial function from X to $\mathbb{C}$. We say that f is integrable on M if and only if:

(Def. 4) $\Re(f)$ is integrable on M and $\Im(f)$ is integrable on M .

Let X be a non empty set, let S be a σ -field of subsets of X , let M be a σ -measure on S , and let f be a partial function from X to $\mathbb{C}$. Let us assume that f is integrable on M . The functor $\int f dM$ yielding a complex number is defined by:

(Def. 5) There exist real numbers R, I such that $R = \int \Re(f) dM$ and $I = \int \Im(f) dM$ and $\int f dM = R + I \cdot i$.

We now state several propositions:

- (19) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , f be a partial function from X to $\overline{\mathbb{R}}$, and A be an element of S . Suppose there exists an element E of S such that $E = \text{dom } f$ and f is measurable on E and $M(A) = 0$. Then $f|_A$ is integrable on M .
- (20) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , f be a partial function from X to $\mathbb{R}$, and E, A be elements of S . Suppose there exists an element E of S such that $E = \text{dom } f$ and f is measurable on E and $M(A) = 0$. Then $f|_A$ is integrable on M .
- (21) Suppose there exists an element E of S such that $E = \text{dom } f$ and f is measurable on E and $M(A) = 0$. Then $f|_A$ is integrable on M and $\int f|_A dM = 0$.
- (22) If $E = \text{dom } f$ and f is integrable on M and $M(A) = 0$, then $\int f|(E \setminus A) dM = \int f dM$.
- (23) If f is integrable on M , then $f|_A$ is integrable on M .
- (24) If f is integrable on M and A misses B , then $\int f|(A \cup B) dM = \int f|_A dM + \int f|_B dM$.
- (25) If f is integrable on M and $B = \text{dom } f \setminus A$, then $f|_A$ is integrable on M and $\int f dM = \int f|_A dM + \int f|_B dM$.

Let k be a real number, let X be a non empty set, and let f be a partial function from X to $\mathbb{R}$. The functor f^k yields a partial function from X to $\mathbb{R}$ and is defined as follows:

(Def. 6) $\text{dom}(f^k) = \text{dom } f$ and for every element x of X such that $x \in \text{dom}(f^k)$ holds $f^k(x) = f(x)^k$.

Let us consider X . Observe that there exists a partial function from X to $\mathbb{R}$ which is non-negative.

Let k be a non negative real number, let us consider X , and let f be a non-negative partial function from X to $\mathbb{R}$. Observe that f^k is non-negative.

We now state a number of propositions:

- (26) Let k be a real number, given X, S, E , and f be a partial function from X to $\mathbb{R}$. If f is non-negative and $0 \leq k$, then f^k is non-negative.
- (27) Let x be a set, given X, S, E , and f be a partial function from X to $\mathbb{R}$. If f is non-negative, then $f(x)^{\frac{1}{2}} = \sqrt{f(x)}$.
- (28) For every partial function f from X to $\mathbb{R}$ and for every real number a such that $A \subseteq \text{dom } f$ holds $A \cap \text{LE-dom}(f, a) = A \setminus A \cap \text{GTE-dom}(f, a)$.
- (29) Let k be a real number, given X, S, E , and f be a partial function from X to $\mathbb{R}$. Suppose f is non-negative and $0 \leq k$ and $E \subseteq \text{dom } f$ and f is measurable on E . Then f^k is measurable on E .
- (30) If f is measurable on A and $A \subseteq \text{dom } f$, then $|f|$ is measurable on A .
- (31) Given an element A of S such that $A = \text{dom } f$ and f is measurable on A . Then f is integrable on M if and only if $|f|$ is integrable on M .
- (32) If f is integrable on M and g is integrable on M , then $\text{dom}(f + g) \in S$.
- (33) If f is integrable on M and g is integrable on M , then $f + g$ is integrable on M .
- (34) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , and f, g be partial functions from X to $\mathbb{R}$. Suppose f is integrable on M and g is integrable on M . Then $f - g$ is integrable on M .
- (35) If f is integrable on M and g is integrable on M , then $f - g$ is integrable on M .
- (36) Suppose f is integrable on M and g is integrable on M . Then there exists an element E of S such that $E = \text{dom } f \cap \text{dom } g$ and $\int f + g \, dM = \int f \upharpoonright E \, dM + \int g \upharpoonright E \, dM$.
- (37) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , and f, g be partial functions from X to $\mathbb{R}$. Suppose f is integrable on M and g is integrable on M . Then there exists an element E of S such that $E = \text{dom } f \cap \text{dom } g$ and $\int f - g \, dM = \int f \upharpoonright E \, dM + \int (-g) \upharpoonright E \, dM$.
- (38) If f is integrable on M , then $r f$ is integrable on M and $\int r f \, dM = r \cdot \int f \, dM$.
- (39) If f is integrable on M , then $i f$ is integrable on M and $\int i f \, dM = i \cdot \int f \, dM$.
- (40) If f is integrable on M , then $c f$ is integrable on M and $\int c f \, dM = c \cdot \int f \, dM$.
- (41) For every partial function f from X to $\mathbb{R}$ and for all Y, r holds $(r f) \upharpoonright Y = r (f \upharpoonright Y)$.
- (42) Let f, g be partial functions from X to $\mathbb{R}$. Suppose that

- (i) there exists an element A of S such that $A = \text{dom } f \cap \text{dom } g$ and f is measurable on A and g is measurable on A ,
- (ii) f is integrable on M ,
- (iii) g is integrable on M , and
- (iv) $g - f$ is non-negative.

Then there exists an element E of S such that $E = \text{dom } f \cap \text{dom } g$ and $\int f \upharpoonright E \, dM \leq \int g \upharpoonright E \, dM$.

- (43) Suppose there exists an element A of S such that $A = \text{dom } f$ and f is measurable on A and f is integrable on M . Then $|\int f \, dM| \leq \int |f| \, dM$.

Let X be a non empty set, let S be a σ -field of subsets of X , let M be a σ -measure on S , let f be a partial function from X to $\mathbb{C}$, and let B be an element of S . The functor $\int_B f \, dM$ yields a complex number and is defined by:

$$(\text{Def. 7}) \quad \int_B f \, dM = \int f \upharpoonright B \, dM.$$

Next we state two propositions:

- (44) Suppose f is integrable on M and g is integrable on M and $B \subseteq \text{dom}(f + g)$. Then $f + g$ is integrable on M and $\int_B f + g \, dM = \int_B f \, dM + \int_B g \, dM$.
- (45) If f is integrable on M and f is measurable on B , then $\int_B c f \, dM = c \cdot \int_B f \, dM$.

4. SEVERAL PROPERTIES OF REAL-VALUED MEASURABLE FUNCTIONS

In the sequel f denotes a partial function from X to $\mathbb{R}$ and a denotes a real number.

One can prove the following four propositions:

- (46) If $A \subseteq \text{dom } f$, then $A \cap \text{GTE-dom}(f, a) = A \setminus A \cap \text{LE-dom}(f, a)$.
- (47) If $A \subseteq \text{dom } f$, then $A \cap \text{GT-dom}(f, a) = A \setminus A \cap \text{LEQ-dom}(f, a)$.
- (48) If $A \subseteq \text{dom } f$, then $A \cap \text{LEQ-dom}(f, a) = A \setminus A \cap \text{GT-dom}(f, a)$.
- (49) $A \cap \text{EQ-dom}(f, a) = A \cap \text{GTE-dom}(f, a) \cap \text{LEQ-dom}(f, a)$.

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