

Jordan Matrix Decomposition

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Summary. In this paper I present the Jordan Matrix Decomposition Theorem which states that an arbitrary square matrix M over an algebraically closed field can be decomposed into the form

$$M = SJS^{-1}$$

where S is an invertible matrix and J is a matrix in a Jordan canonical form, i.e. a special type of block diagonal matrix in which each block consists of Jordan blocks (see [13]).

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The terminology and notation used here are introduced in the following articles: [11], [2], [3], [12], [34], [7], [10], [8], [4], [28], [33], [30], [18], [6], [14], [15], [36], [23], [37], [35], [9], [29], [32], [31], [5], [19], [24], [22], [17], [1], [21], [20], [16], [25], [27], and [26].

1. JORDAN BLOCKS

We follow the rules: i, j, m, n, k denote natural numbers, K denotes a field, and a, λ denote elements of K .

Let us consider K, λ, n . The Jordan block of λ and n yields a matrix over K and is defined by the conditions (Def. 1).

- (Def. 1)(i) $\text{len}(\text{the Jordan block of } \lambda \text{ and } n) = n$,
 (ii) $\text{width}(\text{the Jordan block of } \lambda \text{ and } n) = n$, and
 (iii) for all i, j such that $\langle i, j \rangle \in$ the indices of the Jordan block of λ and n holds if $i = j$, then $(\text{the Jordan block of } \lambda \text{ and } n)_{i,j} = \lambda$ and if $i + 1 = j$, then $(\text{the Jordan block of } \lambda \text{ and } n)_{i,j} = \mathbf{1}_K$ and if $i \neq j$ and $i + 1 \neq j$, then $(\text{the Jordan block of } \lambda \text{ and } n)_{i,j} = 0_K$.

Let us consider K , λ , n . Then the Jordan block of λ and n is an upper triangular matrix over K of dimension n .

The following propositions are true:

- (1) The diagonal of the Jordan block of λ and $n = n \mapsto \lambda$.
- (2) $\text{Det}(\text{the Jordan block of } \lambda \text{ and } n) = \text{power}_K(\lambda, n)$.
- (3) The Jordan block of λ and n is invertible iff $n = 0$ or $\lambda \neq 0_K$.
- (4) If $i \in \text{Seg } n$ and $i \neq n$, then $\text{Line}(\text{the Jordan block of } \lambda \text{ and } n, i) = \lambda \cdot \text{Line}(I_K^{n \times n}, i) + \text{Line}(I_K^{n \times n}, i + 1)$.
- (5) $\text{Line}(\text{the Jordan block of } \lambda \text{ and } n, n) = \lambda \cdot \text{Line}(I_K^{n \times n}, n)$.
- (6) Let F be an element of $(\text{the carrier of } K)^n$ such that $i \in \text{Seg } n$. Then
 - (i) if $i \neq n$, then $\text{Line}(\text{the Jordan block of } \lambda \text{ and } n, i) \cdot F = \lambda \cdot F_i + F_{i+1}$,
and
 - (ii) if $i = n$, then $\text{Line}(\text{the Jordan block of } \lambda \text{ and } n, i) \cdot F = \lambda \cdot F_i$.
- (7) Let F be an element of $(\text{the carrier of } K)^n$ such that $i \in \text{Seg } n$. Then
 - (i) if $i = 1$, then $(\text{the Jordan block of } \lambda \text{ and } n)_{\square, i} \cdot F = \lambda \cdot F_i$, and
 - (ii) if $i \neq 1$, then $(\text{the Jordan block of } \lambda \text{ and } n)_{\square, i} \cdot F = \lambda \cdot F_i + F_{i-1}$.
- (8) Suppose $\lambda \neq 0_K$. Then there exists a square matrix M over K of dimension n such that
 - (i) $(\text{the Jordan block of } \lambda \text{ and } n)^\sim = M$, and
 - (ii) for all i, j such that $\langle i, j \rangle \in$ the indices of M holds if $i > j$, then $M_{i,j} = 0_K$ and if $i \leq j$, then $M_{i,j} = -\text{power}_K(-\lambda^{-1}, (j - i) + 1)$.
- (9) $(\text{The Jordan block of } \lambda \text{ and } n) + a \cdot I_K^{n \times n} = \text{the Jordan block of } \lambda + a \text{ and } n$.

2. FINITE SEQUENCES OF JORDAN BLOCKS

Let us consider K and let G be a finite sequence of elements of $((\text{the carrier of } K)^*)^*$. We say that G is Jordan-block-yielding if and only if:

- (Def. 2) For every i such that $i \in \text{dom } G$ there exist λ, n such that $G(i) = \text{the Jordan block of } \lambda \text{ and } n$.

Let us consider K . Observe that there exists a finite sequence of elements of $((\text{the carrier of } K)^*)^*$ which is Jordan-block-yielding.

Let us consider K . One can verify that every finite sequence of elements of $((\text{the carrier of } K)^*)^*$ which is Jordan-block-yielding is also square-matrix-yielding.

Let us consider K . A finite sequence of Jordan blocks of K is a Jordan-block-yielding finite sequence of elements of $((\text{the carrier of } K)^*)^*$.

Let us consider K, λ . A finite sequence of Jordan blocks of K is said to be a finite sequence of Jordan blocks of λ and K if:

(Def. 3) For every i such that $i \in \text{dom } \lambda$ there exists n such that $\text{it}(i) =$ the Jordan block of λ and n .

Next we state two propositions:

- (10) $\emptyset$ is a finite sequence of Jordan blocks of λ and K .
- (11) $\langle \text{the Jordan block of } \lambda \text{ and } n \rangle$ is a finite sequence of Jordan blocks of λ and K .

Let us consider K, λ . Observe that there exists a finite sequence of Jordan blocks of λ and K which is non-empty.

Let us consider K . Note that there exists a finite sequence of Jordan blocks of K which is non-empty.

Next we state the proposition

- (12) Let J be a finite sequence of Jordan blocks of λ and K . Then $J \oplus \text{len } J \mapsto a \bullet I_K^{\text{Len } J \times \text{Len } J}$ is a finite sequence of Jordan blocks of $\lambda + a$ and K .

Let us consider K and let J_1, J_2 be finite sequences of Jordan blocks of K . Then $J_1 \cap J_2$ is a finite sequence of Jordan blocks of K .

Let us consider K , let J be a finite sequence of Jordan blocks of K , and let us consider n . Then $J \upharpoonright n$ is a finite sequence of Jordan blocks of K . Then $J|_n$ is a finite sequence of Jordan blocks of K .

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3. NILPOTENT TRANSFORMATIONS

Let K be a double loop structure, let V be a non empty vector space structure over K , and let f be a function from V into V . We say that f is nilpotent if and only if:

(Def. 4) There exists n such that for every vector v of V holds $f^n(v) = 0_V$.

We now state the proposition

- (13) Let K be a double loop structure, V be a non empty vector space structure over K , and f be a function from V into V . Then f is nilpotent if and only if there exists n such that $f^n = \text{ZeroMap}(V, V)$.

Let K be a double loop structure and let V be a non empty vector space structure over K . Observe that there exists a function from V into V which is nilpotent.

Let R be a ring and let V be a left module over R . Observe that there exists a function from V into V which is nilpotent and linear.

Next we state the proposition

- (14) Let V be a vector space over K and f be a linear transformation from V to V . Then $f|_{\ker f^n}$ is a nilpotent linear transformation from $\ker f^n$ to $\ker f^n$.

Let K be a double loop structure, let V be a non empty vector space structure over K , and let f be a nilpotent function from V into V . The degree of nilpotence of f yielding a natural number is defined by the conditions (Def. 5).

- (Def. 5)(i) $f^{\text{the degree of nilpotence of } f} = \text{ZeroMap}(V, V)$, and
(ii) for every k such that $f^k = \text{ZeroMap}(V, V)$ holds the degree of nilpotence of $f \leq k$.

Let K be a double loop structure, let V be a non empty vector space structure over K , and let f be a nilpotent function from V into V . We introduce $\deg f$ as a synonym of the degree of nilpotence of f .

One can prove the following propositions:

- (15) Let K be a double loop structure, V be a non empty vector space structure over K , and f be a nilpotent function from V into V . Then $\deg f = 0$ if and only if $\Omega_V = \{0_V\}$.
- (16) Let K be a double loop structure, V be a non empty vector space structure over K , and f be a nilpotent function from V into V . Then there exists a vector v of V such that for every i such that $i < \deg f$ holds $f^i(v) \neq 0_V$.
- (17) Let K be a field, V be a vector space over K , W be a subspace of V , and f be a nilpotent function from V into V . Suppose $f|_W$ is a function from W into W . Then $f|_W$ is a nilpotent function from W into W .
- (18) Let K be a field, V be a vector space over K , W be a subspace of V , f be a nilpotent linear transformation from V to V , and f_1 be a nilpotent function from $\text{im}(f^n)$ into $\text{im}(f^n)$. If $f_1 = f|_{\text{im}(f^n)}$ and $n \leq \deg f$, then $\deg f_1 + n = \deg f$.

For simplicity, we adopt the following convention: V_1, V_2 denote finite dimensional vector spaces over K , W_1, W_2 denote subspaces of V_1, U_1, U_2 denote subspaces of V_2 , b_1 denotes an ordered basis of V_1 , B_1 denotes a finite sequence of elements of V_1 , b_2 denotes an ordered basis of V_2 , B_2 denotes a finite sequence of elements of V_2 , b_3 denotes an ordered basis of W_1 , b_4 denotes an ordered basis of W_2 , B_3 denotes a finite sequence of elements of U_1 , and B_4 denotes a finite sequence of elements of U_2 .

Next we state a number of propositions:

- (19) Let M be a matrix over K of dimension $\text{len } b_1 \times \text{len } B_2$, M_1 be a matrix over K of dimension $\text{len } b_3 \times \text{len } B_3$, and M_2 be a matrix over K of dimension $\text{len } b_4 \times \text{len } B_4$ such that $b_1 = b_3 \hat{\ } b_4$ and $B_2 = B_3 \hat{\ } B_4$ and $M = \text{the } 0_K\text{-block diagonal of } \langle M_1, M_2 \rangle$ and width $M_1 = \text{len } B_3$ and width $M_2 = \text{len } B_4$. Then

- (i) if $i \in \text{dom } b_3$, then $(\text{Mx2Tran}(M, b_1, B_2))((b_1)_i) = (\text{Mx2Tran}(M_1, b_3, B_3))((b_3)_i)$, and
- (ii) if $i \in \text{dom } b_4$, then $(\text{Mx2Tran}(M, b_1, B_2))((b_1)_{i+\text{len } b_3}) = (\text{Mx2Tran}(M_2, b_4, B_4))((b_4)_i)$.
- (20) Let M be a matrix over K of dimension $\text{len } b_1 \times \text{len } B_2$ and F be a finite sequence of matrices over K . Suppose $M =$ the 0_K -block diagonal of F . Let given i, m . Suppose $i \in \text{dom } b_1$ and $m = \min(\text{Len } F, i)$. Then $(\text{Mx2Tran}(M, b_1, B_2))((b_1)_i) = \sum \text{lmlt}(\text{Line}(F(m), i -' \sum \text{Len}(F \upharpoonright (m -' 1))), (B_2 \upharpoonright \sum \text{Width}(F \upharpoonright m)) \downharpoonright \sum \text{Width}(F \upharpoonright (m -' 1)))$ and $\text{len}((B_2 \upharpoonright \sum \text{Width}(F \upharpoonright m)) \downharpoonright \sum \text{Width}(F \upharpoonright (m -' 1))) = \text{width } F(m)$.
- (21) If $\text{len } B_1 \in \text{dom } B_1$, then $\sum \text{lmlt}(\text{Line}(\text{the Jordan block of } \lambda \text{ and } \text{len } B_1, \text{len } B_1), B_1) = \lambda \cdot (B_1)_{\text{len } B_1}$.
- (22) If $i \in \text{dom } B_1$ and $i \neq \text{len } B_1$, then $\sum \text{lmlt}(\text{Line}(\text{the Jordan block of } \lambda \text{ and } \text{len } B_1, i), B_1) = \lambda \cdot (B_1)_i + (B_1)_{i+1}$.
- (23) Let M be a matrix over K of dimension $\text{len } b_1 \times \text{len } B_2$. Suppose $M =$ the Jordan block of λ and n . Let given i such that $i \in \text{dom } b_1$. Then
 - (i) if $i = \text{len } b_1$, then $(\text{Mx2Tran}(M, b_1, B_2))((b_1)_i) = \lambda \cdot (B_2)_i$, and
 - (ii) if $i \neq \text{len } b_1$, then $(\text{Mx2Tran}(M, b_1, B_2))((b_1)_i) = \lambda \cdot (B_2)_i + (B_2)_{i+1}$.
- (24) Let J be a finite sequence of Jordan blocks of λ and K and M be a matrix over K of dimension $\text{len } b_1 \times \text{len } B_2$. Suppose $M =$ the 0_K -block diagonal of J . Let given i, m such that $i \in \text{dom } b_1$ and $m = \min(\text{Len } J, i)$. Then
 - (i) if $i = \sum \text{Len}(J \upharpoonright m)$, then $(\text{Mx2Tran}(M, b_1, B_2))((b_1)_i) = \lambda \cdot (B_2)_i$, and
 - (ii) if $i \neq \sum \text{Len}(J \upharpoonright m)$, then $(\text{Mx2Tran}(M, b_1, B_2))((b_1)_i) = \lambda \cdot (B_2)_i + (B_2)_{i+1}$.
- (25) Let J be a finite sequence of Jordan blocks of 0_K and K and M be a matrix over K of dimension $\text{len } b_1 \times \text{len } b_1$. Suppose $M =$ the 0_K -block diagonal of J . Let given m . If for every i such that $i \in \text{dom } J$ holds $\text{len } J(i) \leq m$, then $(\text{Mx2Tran}(M, b_1, b_1))^m = \text{ZeroMap}(V_1, V_1)$.
- (26) Let J be a finite sequence of Jordan blocks of λ and K and M be a matrix over K of dimension $\text{len } b_1 \times \text{len } b_1$. Suppose $M =$ the 0_K -block diagonal of J . Then $\text{Mx2Tran}(M, b_1, b_1)$ is nilpotent if and only if $\text{len } b_1 = 0$ or $\lambda = 0_K$.
- (27) Let J be a finite sequence of Jordan blocks of 0_K and K and M be a matrix over K of dimension $\text{len } b_1 \times \text{len } b_1$. Suppose $M =$ the 0_K -block diagonal of J and $\text{len } b_1 > 0$. Let F be a nilpotent function from V_1 into V_1 . Suppose $F = \text{Mx2Tran}(M, b_1, b_1)$. Then there exists i such that $i \in \text{dom } J$ and $\text{len } J(i) = \deg F$ and for every i such that $i \in \text{dom } J$ holds $\text{len } J(i) \leq \deg F$.
- (28) Let given $V_1, V_2, b_1, b_2, \lambda$. Suppose $\text{len } b_1 = \text{len } b_2$. Let F be a linear

transformation from V_1 to V_2 . Suppose that for every i such that $i \in \text{dom } b_1$ holds $F((b_1)_i) = \lambda \cdot (b_2)_i$ or $i+1 \in \text{dom } b_1$ and $F((b_1)_i) = \lambda \cdot (b_2)_i + (b_2)_{i+1}$. Then there exists a non-empty finite sequence J of Jordan blocks of λ and K such that $\text{AutMt}(F, b_1, b_2) = \text{the } 0_K\text{-block diagonal of } J$.

- (29) Let V_1 be a finite dimensional vector space over K and F be a nilpotent linear transformation from V_1 to V_1 . Then there exists a non-empty finite sequence J of Jordan blocks of 0_K and K and there exists an ordered basis b_1 of V_1 such that $\text{AutMt}(F, b_1, b_1) = \text{the } 0_K\text{-block diagonal of } J$.
- (30) Let V be a vector space over K , F be a linear transformation from V to V , V_1 be a finite dimensional subspace of V , and F_1 be a linear transformation from V_1 to V_1 . Suppose $V_1 = \ker(F + (-\lambda) \cdot \text{id}_V)^n$ and $F|V_1 = F_1$. Then there exists a non-empty finite sequence J of Jordan blocks of λ and K and there exists an ordered basis b_1 of V_1 such that $\text{AutMt}(F_1, b_1, b_1) = \text{the } 0_K\text{-block diagonal of } J$.

4. THE MAIN THEOREM

The following two propositions are true:

- (31) Let K be an algebraic-closed field, V be a non trivial finite dimensional vector space over K , and F be a linear transformation from V to V . Then there exists a non-empty finite sequence J of Jordan blocks of K and there exists an ordered basis b_1 of V such that
 - (i) $\text{AutMt}(F, b_1, b_1) = \text{the } 0_K\text{-block diagonal of } J$, and
 - (ii) for every scalar λ of K holds λ is an eigenvalue of F iff there exists i such that $i \in \text{dom } J$ and $J(i) = \text{the Jordan block of } \lambda$ and $\text{len } J(i)$.
- (32) Let K be an algebraic-closed field and M be a square matrix over K of dimension n . Then there exists a non-empty finite sequence J of Jordan blocks of K and there exists a square matrix P over K of dimension n such that $\sum \text{Len } J = n$ and P is invertible and $M = P \cdot \text{the } 0_K\text{-block diagonal of } J \cdot P^\sim$.

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