

Eigenvalues of a Linear Transformation

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Summary. The article presents well known facts about eigenvalues of linear transformation of a vector space (see [13]). I formalize main dependencies between eigenvalues and the diagram of the matrix of a linear transformation over a finite-dimensional vector space. Finally, I formalize the subspace $\bigcup_{i=0}^{\infty} \text{Ker}(f - \lambda I)^i$ called a generalized eigenspace for the eigenvalue λ and show its basic properties.

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The articles [11], [33], [2], [3], [12], [34], [8], [10], [9], [5], [31], [27], [15], [7], [14], [32], [35], [25], [30], [29], [28], [26], [6], [22], [16], [23], [20], [1], [19], [4], [21], [17], [18], and [24] provide the notation and terminology for this paper.

1. PRELIMINARIES

We adopt the following convention: i, j, m, n denote natural numbers, K denotes a field, and a denotes an element of K .

Next we state several propositions:

- (1) Let A, B be matrices over K , n_1 be an element of $\mathbb{N}^n$, and m_1 be an element of $\mathbb{N}^m$. If $\text{rng } n_1 \times \text{rng } m_1 \subseteq \text{the indices of } A$, then $\text{Segm}(A + B, n_1, m_1) = \text{Segm}(A, n_1, m_1) + \text{Segm}(B, n_1, m_1)$.
- (2) For every without zero finite subset P of $\mathbb{N}$ such that $P \subseteq \text{Seg } n$ holds $\text{Segm}(I_K^{n \times n}, P, P) = I_K^{\text{card } P \times \text{card } P}$.
- (3) Let A, B be matrices over K and P, Q be without zero finite subsets of $\mathbb{N}$. If $P \times Q \subseteq \text{the indices of } A$, then $\text{Segm}(A + B, P, Q) = \text{Segm}(A, P, Q) + \text{Segm}(B, P, Q)$.

- (4) For all square matrices A, B over K of dimension n such that $i, j \in \text{Seg } n$ holds $\text{Delete}(A + B, i, j) = \text{Delete}(A, i, j) + \text{Delete}(B, i, j)$.
- (5) For every square matrix A over K of dimension n such that $i, j \in \text{Seg } n$ holds $\text{Delete}(a \cdot A, i, j) = a \cdot \text{Delete}(A, i, j)$.
- (6) If $i \in \text{Seg } n$, then $\text{Delete}(I_K^{n \times n}, i, i) = I_K^{(n-1) \times (n-1)}$.
- (7) Let A, B be square matrices over K of dimension n . Then there exists a polynomial P of K such that $\text{len } P \leq n + 1$ and for every element x of K holds $\text{eval}(P, x) = \text{Det}(A + x \cdot B)$.
- (8) Let A be a square matrix over K of dimension n . Then there exists a polynomial P of K such that $\text{len } P = n + 1$ and for every element x of K holds $\text{eval}(P, x) = \text{Det}(A + x \cdot I_K^{n \times n})$.

Let us consider K . Observe that there exists a vector space over K which is non trivial and finite dimensional.

2. MAPS WITH EIGENVALUES

Let R be a non empty double loop structure, let V be a non empty vector space structure over R , and let I_1 be a function from V into V . We say that I_1 has eigenvalues if and only if:

- (Def. 1) There exists a vector v of V and there exists a scalar a of R such that $v \neq 0_V$ and $I_1(v) = a \cdot v$.

For simplicity, we follow the rules: V denotes a non trivial vector space over K , V_1, V_2 denote vector spaces over K , f denotes a linear transformation from V_1 to V_1 , v, w denote vectors of V , v_1 denotes a vector of V_1 , and L denotes a scalar of K .

Let us consider K, V . One can verify that there exists a linear transformation from V to V which has eigenvalues.

Let R be a non empty double loop structure, let V be a non empty vector space structure over R , and let f be a function from V into V . Let us assume that f has eigenvalues. An element of R is called an eigenvalue of f if:

- (Def. 2) There exists a vector v of V such that $v \neq 0_V$ and $f(v) = \text{it} \cdot v$.

Let R be a non empty double loop structure, let V be a non empty vector space structure over R , let f be a function from V into V , and let L be a scalar of R . Let us assume that f has eigenvalues and L is an eigenvalue of f . A vector of V is called an eigenvector of f and L if:

- (Def. 3) $f(\text{it}) = L \cdot \text{it}$.

We now state several propositions:

- (9) Let given a . Suppose $a \neq 0_K$. Let f be a function from V into V with eigenvalues and L be an eigenvalue of f . Then
 - (i) $a \cdot f$ has eigenvalues,

- (ii) $a \cdot L$ is an eigenvalue of $a \cdot f$, and
- (iii) w is an eigenvector of f and L iff w is an eigenvector of $a \cdot f$ and $a \cdot L$.
- (10) Let f_1, f_2 be functions from V into V with eigenvalues and L_1, L_2 be scalars of K . Suppose that
 - (i) L_1 is an eigenvalue of f_1 ,
 - (ii) L_2 is an eigenvalue of f_2 , and
 - (iii) there exists v such that v is an eigenvector of f_1 and L_1 and an eigenvector of f_2 and L_2 and $v \neq 0_V$.

Then

- (iv) $f_1 + f_2$ has eigenvalues,
- (v) $L_1 + L_2$ is an eigenvalue of $f_1 + f_2$, and
- (vi) for every w such that w is an eigenvector of f_1 and L_1 and an eigenvector of f_2 and L_2 holds w is an eigenvector of $f_1 + f_2$ and $L_1 + L_2$.
- (11) id_V has eigenvalues and $\mathbf{1}_K$ is an eigenvalue of id_V and every v is an eigenvector of id_V and $\mathbf{1}_K$.
- (12) For every eigenvalue L of id_V holds $L = \mathbf{1}_K$.
- (13) If $\ker f$ is non trivial, then f has eigenvalues and 0_K is an eigenvalue of f .
- (14) f has eigenvalues and L is an eigenvalue of f iff $\ker f + (-L) \cdot \text{id}_{(V_1)}$ is non trivial.
- (15) Let V_1 be a finite dimensional vector space over K , b_1, b'_1 be ordered bases of V_1 , and f be a linear transformation from V_1 to V_1 . Then f has eigenvalues and L is an eigenvalue of f if and only if $\text{Det AutEqMt}(f + (-L) \cdot \text{id}_{(V_1)}, b_1, b'_1) = 0_K$.
- (16) Let K be an algebraic-closed field and V_1 be a non trivial finite dimensional vector space over K . Then every linear transformation from V_1 to V_1 has eigenvalues.
- (17) Let given f, L . Suppose f has eigenvalues and L is an eigenvalue of f . Then v_1 is an eigenvector of f and L if and only if $v_1 \in \ker f + (-L) \cdot \text{id}_{(V_1)}$.

Let S be a 1-sorted structure, let F be a function from S into S , and let n be a natural number. The functor F^n yields a function from S into S and is defined as follows:

- (Def. 4) For every element F' of the semigroup of functions onto the carrier of S such that $F' = F$ holds $F^n = \prod (n \mapsto F')$.

In the sequel S denotes a 1-sorted structure and F denotes a function from S into S .

Next we state several propositions:

- (18) $F^0 = \text{id}_S$.
- (19) $F^1 = F$.
- (20) $F^{i+j} = F^i \cdot F^j$.

- (21) For all elements s_1, s_2 of S and for all n, m such that $F^m(s_1) = s_2$ and $F^n(s_2) = s_2$ holds $F^{m+i \cdot n}(s_1) = s_2$.
- (22) Let K be an add-associative right zeroed right complementable Abelian associative well unital distributive non empty double loop structure, V_1 be an Abelian add-associative right zeroed right complementable vector space-like non empty vector space structure over K , W be a subspace of V_1 , f be a function from V_1 into V_1 , and f_3 be a function from W into W . If $f_3 = f|_W$, then $f^n|_W = f_3^n$.

Let us consider K, V_1 , let f be a linear transformation from V_1 to V_1 , and let n be a natural number. Then f^n is a linear transformation from V_1 to V_1 .

We now state the proposition

- (23) If $f^i(v_1) = 0_{(V_1)}$, then $f^{i+j}(v_1) = 0_{(V_1)}$.

3. GENERALIZED EIGENSPACE OF A LINEAR TRANSFORMATION

Let us consider K, V_1, f . The functor $\text{UnionKers } f$ yielding a strict subspace of V_1 is defined by:

- (Def. 5) The carrier of $\text{UnionKers } f = \{v; v \text{ ranges over vectors of } V_1: \bigvee_n f^n(v) = 0_{(V_1)}\}$.

We now state a number of propositions:

- (24) $v_1 \in \text{UnionKers } f$ iff there exists n such that $f^n(v_1) = 0_{(V_1)}$.
- (25) $\ker f^i$ is a subspace of $\text{UnionKers } f$.
- (26) $\ker f^i$ is a subspace of $\ker f^{i+j}$.
- (27) Let V be a finite dimensional vector space over K and f be a linear transformation from V to V . Then there exists n such that $\text{UnionKers } f = \ker f^n$.
- (28) $f|_{\ker f^n}$ is a linear transformation from $\ker f^n$ to $\ker f^n$.
- (29) $f|_{\ker (f + L \cdot \text{id}_{(V_1)})^n}$ is a linear transformation from $\ker (f + L \cdot \text{id}_{(V_1)})^n$ to $\ker (f + L \cdot \text{id}_{(V_1)})^n$.
- (30) $f|_{\text{UnionKers } f}$ is a linear transformation from $\text{UnionKers } f$ to $\text{UnionKers } f$.
- (31) $f|_{\text{UnionKers}(f + L \cdot \text{id}_{(V_1)})}$ is a linear transformation from $\text{UnionKers}(f + L \cdot \text{id}_{(V_1)})$ to $\text{UnionKers}(f + L \cdot \text{id}_{(V_1)})$.
- (32) $f|_{\text{im}(f^n)}$ is a linear transformation from $\text{im}(f^n)$ to $\text{im}(f^n)$.
- (33) $f|_{\text{im}((f + L \cdot \text{id}_{(V_1)})^n)}$ is a linear transformation from $\text{im}((f + L \cdot \text{id}_{(V_1)})^n)$ to $\text{im}((f + L \cdot \text{id}_{(V_1)})^n)$.
- (34) If $\text{UnionKers } f = \ker f^n$, then $\ker f^n \cap \text{im}(f^n) = \mathbf{0}_{(V_1)}$.

- (35) Let V be a finite dimensional vector space over K , f be a linear transformation from V to V , and given n . If $\text{UnionKers } f = \ker f^n$, then V is the direct sum of $\ker f^n$ and $\text{im}(f^n)$.
- (36) For every linear complement I of $\text{UnionKers } f$ holds $f|I$ is one-to-one.
- (37) Let I be a linear complement of $\text{UnionKers}(f + (-L) \cdot \text{id}_{(V_1)})$ and f_4 be a linear transformation from I to I . If $f_4 = f|I$, then for every vector v of I such that $f_4(v) = L \cdot v$ holds $v = 0_{(V_1)}$.
- (38) Suppose $n \geq 1$. Then there exists a linear transformation h from V_1 to V_1 such that $(f + L \cdot \text{id}_{(V_1)})^n = f \cdot h + (L \cdot \text{id}_{(V_1)})^n$ and for every i holds $f^i \cdot h = h \cdot f^i$.
- (39) Let L_1, L_2 be scalars of K . Suppose f has eigenvalues and $L_1 \neq L_2$ and L_1 is an eigenvalue of f and L_2 is an eigenvalue of f . Let I be a linear complement of $\text{UnionKers}(f + (-L_1) \cdot \text{id}_{(V_1)})$ and f_4 be a linear transformation from I to I . Suppose $f_4 = f|I$. Then f_4 has eigenvalues and L_1 is not an eigenvalue of f_4 and L_2 is an eigenvalue of f_4 and $\text{UnionKers}(f + (-L_2) \cdot \text{id}_{(V_1)})$ is a subspace of I .
- (40) Let U be a finite subset of V_1 . Suppose U is linearly independent. Let u be a vector of V_1 . Suppose $u \in U$. Let L be a linear combination of $U \setminus \{u\}$. Then $\overline{U} = \overline{(U \setminus \{u\}) \cup \{u + \sum L\}}$ and $(U \setminus \{u\}) \cup \{u + \sum L\}$ is linearly independent.
- (41) Let A be a subset of V_1 , L be a linear combination of V_2 , and f be a linear transformation from V_1 to V_2 . Suppose the support of $L \subseteq f^\circ A$. Then there exists a linear combination M of A such that $f(\sum M) = \sum L$.
- (42) Let f be a linear transformation from V_1 to V_2 , A be a subset of V_1 , and B be a subset of V_2 . If $f^\circ A = B$, then $f^\circ(\text{the carrier of } \text{Lin}(A)) = \text{the carrier of } \text{Lin}(B)$.
- (43) Let L be a linear combination of V_1 , F be a finite sequence of elements of V_1 , and f be a linear transformation from V_1 to V_2 . Suppose $f|((\text{the support of } L) \cap \text{rng } F)$ is one-to-one and $\text{rng } F \subseteq \text{the support of } L$. Then there exists a linear combination L_3 of V_2 such that
 - (i) the support of $L_3 = f^\circ((\text{the support of } L) \cap \text{rng } F)$,
 - (ii) $f \cdot (L F) = L_3 (f \cdot F)$, and
 - (iii) for every v_1 such that $v_1 \in (\text{the support of } L) \cap \text{rng } F$ holds $L(v_1) = L_3(f(v_1))$.
- (44) Let A, B be subsets of V_1 and L be a linear combination of V_1 . Suppose the support of $L \subseteq A \cup B$ and $\sum L = 0_{(V_1)}$. Let f be a linear function from V_1 into V_2 . Suppose $f|B$ is one-to-one and $f^\circ B$ is a linearly independent subset of V_2 and $f^\circ A \subseteq \{0_{(V_2)}\}$. Then the support of $L \subseteq A$.

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