

Linear Map of Matrices

Karol Pąk
 Institute of Computer Science
 University of Białystok
 Poland

Summary. The paper is concerned with a generalization of concepts introduced in [13], i.e. introduced are matrices of linear transformations over a finite-dimensional vector space. Introduced are linear transformations over a finite-dimensional vector space depending on a given matrix of the transformation. Finally, I prove that the rank of linear transformations over a finite-dimensional vector space is the same as the rank of the matrix of that transformation.

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The notation and terminology used here are introduced in the following papers: [24], [2], [3], [9], [25], [6], [8], [7], [4], [23], [19], [12], [10], [27], [28], [26], [22], [20], [18], [29], [5], [15], [13], [17], [11], [14], [21], [1], and [16].

1. PRELIMINARIES

We adopt the following rules: i, j, m, n are natural numbers, K is a field, and a is an element of K .

One can prove the following propositions:

- (1) Let V be a vector space over K , W_1, W_2, W_{12} be subspaces of V , and U_1, U_2 be subspaces of W_{12} . If $U_1 = W_1$ and $U_2 = W_2$, then $W_1 \cap W_2 = U_1 \cap U_2$ and $W_1 + W_2 = U_1 + U_2$.
- (2) Let V be a vector space over K and W_1, W_2 be subspaces of V . Suppose $W_1 \cap W_2 = \mathbf{0}_V$. Let B_1 be a linearly independent subset of W_1 and B_2 be a linearly independent subset of W_2 . Then $B_1 \cup B_2$ is a linearly independent subset of $W_1 + W_2$.

- (3) Let V be a vector space over K and W_1, W_2 be subspaces of V . Suppose $W_1 \cap W_2 = \mathbf{0}_V$. Let B_1 be a basis of W_1 and B_2 be a basis of W_2 . Then $B_1 \cup B_2$ is a basis of $W_1 + W_2$.
- (4) For every finite dimensional vector space V over K holds every ordered basis of Ω_V is an ordered basis of V .
- (5) Let V_1 be a vector space over K and A be a finite subset of V_1 . If $\dim(\text{Lin}(A)) = \text{card } A$, then A is linearly independent.
- (6) For every vector space V over K and for every finite subset A of V holds $\dim(\text{Lin}(A)) \leq \text{card } A$.

2. MORE ON THE PRODUCT OF FINITE SEQUENCE OF SCALARS AND VECTORS

For simplicity, we follow the rules: V_1, V_2, V_3 are finite dimensional vector spaces over K , f is a function from V_1 into V_2 , b_1, b'_1 are ordered bases of V_1 , B_1 is a finite sequence of elements of V_1 , b_2 is an ordered basis of V_2 , B_2 is a finite sequence of elements of V_2 , B_3 is a finite sequence of elements of V_3 , v_1, w_1 are elements of V_1 , R, R_1, R_2 are finite sequences of elements of V_1 , and p, p_1, p_2 are finite sequences of elements of K .

We now state a number of propositions:

- (7) $\text{lmlt}(p_1 + p_2, R) = \text{lmlt}(p_1, R) + \text{lmlt}(p_2, R)$.
- (8) $\text{lmlt}(p, R_1 + R_2) = \text{lmlt}(p, R_1) + \text{lmlt}(p, R_2)$.
- (9) If $\text{len } p_1 = \text{len } R_1$ and $\text{len } p_2 = \text{len } R_2$, then $\text{lmlt}(p_1 \frown p_2, R_1 \frown R_2) = (\text{lmlt}(p_1, R_1)) \frown \text{lmlt}(p_2, R_2)$.
- (10) If $\text{len } R_1 = \text{len } R_2$, then $\sum(R_1 + R_2) = (\sum R_1) + \sum R_2$.
- (11) $\sum \text{lmlt}(\text{len } R \mapsto a, R) = a \cdot \sum R$.
- (12) $\sum \text{lmlt}(p, \text{len } p \mapsto v_1) = (\sum p) \cdot v_1$.
- (13) $\sum \text{lmlt}(a \cdot p, R) = a \cdot \sum \text{lmlt}(p, R)$.
- (14) Let B_1 be a finite sequence of elements of V_1 , W_1 be a subspace of V_1 , and B_2 be a finite sequence of elements of W_1 . If $B_1 = B_2$, then $\text{lmlt}(p, B_1) = \text{lmlt}(p, B_2)$.
- (15) Let B_1 be a finite sequence of elements of V_1 , W_1 be a subspace of V_1 , and B_2 be a finite sequence of elements of W_1 . If $B_1 = B_2$, then $\sum B_1 = \sum B_2$.
- (16) If $i \in \text{dom } R$, then $\sum \text{lmlt}(\text{Line}(I_K^{\text{len } R \times \text{len } R}, i), R) = R_i$.

3. MORE ON THE DECOMPOSITION OF A VECTOR IN A BASIS

We now state a number of propositions:

- (17) $v_1 + w_1 \rightarrow b_1 = (v_1 \rightarrow b_1) + (w_1 \rightarrow b_1)$.

- (18) $a \cdot v_1 \rightarrow b_1 = a \cdot (v_1 \rightarrow b_1)$.
- (19) If $i \in \text{dom } b_1$, then $(b_1)_i \rightarrow b_1 = \text{Line}(I_K^{\text{len } b_1 \times \text{len } b_1}, i)$.
- (20) $0_{(V_1)} \rightarrow b_1 = \text{len } b_1 \mapsto 0_K$.
- (21) $\text{len } b_1 = \dim(V_1)$.
- (22)(i) $\text{rng}(b_1 \upharpoonright m)$ is a linearly independent subset of V_1 , and
 (ii) for every subset A of V_1 such that $A = \text{rng}(b_1 \upharpoonright m)$ holds $b_1 \upharpoonright m$ is an ordered basis of $\text{Lin}(A)$.
- (23)(i) $\text{rng}((b_1)_{\upharpoonright m})$ is a linearly independent subset of V_1 , and
 (ii) for every subset A of V_1 such that $A = \text{rng}((b_1)_{\upharpoonright m})$ holds $(b_1)_{\upharpoonright m}$ is an ordered basis of $\text{Lin}(A)$.
- (24) Let W_1, W_2 be subspaces of V_1 . Suppose $W_1 \cap W_2 = \mathbf{0}_{(V_1)}$. Let b_1 be an ordered basis of W_1 , b_2 be an ordered basis of W_2 , and b be an ordered basis of $W_1 + W_2$. Suppose $b = b_1 \hat{\ } b_2$. Let v, v_1, v_2 be vectors of $W_1 + W_2$, w_1 be a vector of W_1 , and w_2 be a vector of W_2 . If $v = v_1 + v_2$ and $v_1 = w_1$ and $v_2 = w_2$, then $v \rightarrow b = (w_1 \rightarrow b_1) \hat{\ } (w_2 \rightarrow b_2)$.
- (25) Let W_1 be a subspace of V_1 . Suppose $W_1 = \Omega_{(V_1)}$. Let w be a vector of W_1 , v be a vector of V_1 , and w_1 be an ordered basis of W_1 . If $v = w$ and $b_1 = w_1$, then $v \rightarrow b_1 = w \rightarrow w_1$.
- (26) Let W_1, W_2 be subspaces of V_1 . Suppose $W_1 \cap W_2 = \mathbf{0}_{(V_1)}$. Let w_1 be an ordered basis of W_1 and w_2 be an ordered basis of W_2 . Then $w_1 \hat{\ } w_2$ is an ordered basis of $W_1 + W_2$.

4. PROPERTIES OF MATRICES OF LINEAR TRANSFORMATIONS

Let us consider K, V_1, V_2, f, B_1, b_2 . Then $\text{AutMt}(f, B_1, b_2)$ is a matrix over K of dimension $\text{len } B_1 \times \text{len } b_2$.

Let S be a 1-sorted structure and let R be a binary relation. The functor $R \upharpoonright S$ is defined as follows:

(Def. 1) $R \upharpoonright S = R \upharpoonright \text{the carrier of } S$.

The following proposition is true

- (27) Let f be a linear transformation from V_1 to V_2 , W_1, W_2 be subspaces of V_1 , and U_1, U_2 be subspaces of V_2 . Suppose if $\dim(W_1) = 0$, then $\dim(U_1) = 0$ and if $\dim(W_2) = 0$, then $\dim(U_2) = 0$ and V_2 is the direct sum of U_1 and U_2 . Let f_1 be a linear transformation from W_1 to U_1 and f_2 be a linear transformation from W_2 to U_2 . Suppose $f_1 = f \upharpoonright W_1$ and $f_2 = f \upharpoonright W_2$. Let w_1 be an ordered basis of W_1 , w_2 be an ordered basis of W_2 , u_1 be an ordered basis of U_1 , and u_2 be an ordered basis of U_2 . Suppose $w_1 \hat{\ } w_2 = b_1$ and $u_1 \hat{\ } u_2 = b_2$. Then $\text{AutMt}(f, b_1, b_2) = \text{the } 0_K\text{-block diagonal of } \langle \text{AutMt}(f_1, w_1, u_1), \text{AutMt}(f_2, w_2, u_2) \rangle$.

Let us consider K , V_1 , V_2 , let f be a function from V_1 into V_2 , let B_1 be a finite sequence of elements of V_1 , and let b_2 be an ordered basis of V_2 . Let us assume that $\text{len } B_1 = \text{len } b_2$. The functor $\text{AutEqMt}(f, B_1, b_2)$ yielding a matrix over K of dimension $\text{len } B_1 \times \text{len } B_1$ is defined by:

(Def. 2) $\text{AutEqMt}(f, B_1, b_2) = \text{AutMt}(f, B_1, b_2)$.

The following propositions are true:

$$(28) \quad \text{AutMt}(\text{id}_{(V_1)}, b_1, b_1) = I_K^{\text{len } b_1 \times \text{len } b_1}.$$

$$(29) \quad \text{AutEqMt}(\text{id}_{(V_1)}, b_1, b'_1) \text{ is invertible and } \text{AutEqMt}(\text{id}_{(V_1)}, b'_1, b_1) = (\text{AutEqMt}(\text{id}_{(V_1)}, b_1, b'_1))^\sim.$$

$$(30) \quad \text{If } \text{len } p_1 = \text{len } p_2 \text{ and } \text{len } p_1 = \text{len } B_1 \text{ and } \text{len } p_1 > 0 \text{ and } j \in \text{dom } b_1 \text{ and for every } i \text{ such that } i \in \text{dom } p_2 \text{ holds } p_2(i) = ((B_1)_i \rightarrow b_1)(j), \text{ then } p_1 \cdot p_2 = (\sum \text{lmlt}(p_1, B_1) \rightarrow b_1)(j).$$

$$(31) \quad \text{If } \text{len } b_1 > 0 \text{ and } f \text{ is linear, then } \text{LineVec2Mx}(v_1 \rightarrow b_1) \cdot \text{AutMt}(f, b_1, b_2) = \text{LineVec2Mx}(f(v_1) \rightarrow b_2).$$

5. LINEAR TRANSFORMATIONS OF MATRICES

Let us consider K , V_1 , V_2 , b_1 , B_2 and let M be a matrix over K of dimension $\text{len } b_1 \times \text{len } B_2$. The functor $\text{Mx2Tran}(M, b_1, B_2)$ yielding a function from V_1 into V_2 is defined by:

(Def. 3) For every vector v of V_1 holds $(\text{Mx2Tran}(M, b_1, B_2))(v) = \sum \text{lmlt}(\text{Line}(\text{LineVec2Mx}(v \rightarrow b_1) \cdot M, 1), B_2)$.

Next we state two propositions:

$$(32) \quad \text{For every matrix } M \text{ over } K \text{ of dimension } \text{len } b_1 \times \text{len } b_2 \text{ such that } \text{len } b_1 > 0 \text{ holds } \text{LineVec2Mx}((\text{Mx2Tran}(M, b_1, b_2))(v_1) \rightarrow b_2) = \text{LineVec2Mx}(v_1 \rightarrow b_1) \cdot M.$$

$$(33) \quad \text{For every matrix } M \text{ over } K \text{ of dimension } \text{len } b_1 \times \text{len } B_2 \text{ such that } \text{len } b_1 = 0 \text{ holds } (\text{Mx2Tran}(M, b_1, B_2))(v_1) = 0_{(V_2)}.$$

Let us consider K , V_1 , V_2 , b_1 , B_2 and let M be a matrix over K of dimension $\text{len } b_1 \times \text{len } B_2$. Then $\text{Mx2Tran}(M, b_1, B_2)$ is a linear transformation from V_1 to V_2 .

Next we state three propositions:

$$(34) \quad \text{If } f \text{ is linear, then } \text{Mx2Tran}(\text{AutMt}(f, b_1, b_2), b_1, b_2) = f.$$

$$(35) \quad \text{For all matrices } A, B \text{ over } K \text{ such that } i \in \text{dom } A \text{ and } \text{width } A = \text{len } B \text{ holds } \text{LineVec2Mx } \text{Line}(A, i) \cdot B = \text{LineVec2Mx } \text{Line}(A \cdot B, i).$$

$$(36) \quad \text{For every matrix } M \text{ over } K \text{ of dimension } \text{len } b_1 \times \text{len } b_2 \text{ holds } \text{AutMt}(\text{Mx2Tran}(M, b_1, b_2), b_1, b_2) = M.$$

Let us consider n , m , K , let A be a matrix over K of dimension $n \times m$, and let B be a matrix over K . Then $A + B$ is a matrix over K of dimension $n \times m$.

The following propositions are true:

- (37) For all matrices A, B over K of dimension $\text{len } b_1 \times \text{len } B_2$ holds $\text{Mx2Tran}(A + B, b_1, B_2) = \text{Mx2Tran}(A, b_1, B_2) + \text{Mx2Tran}(B, b_1, B_2)$.
- (38) For every matrix A over K of dimension $\text{len } b_1 \times \text{len } B_2$ holds $a \cdot \text{Mx2Tran}(A, b_1, B_2) = \text{Mx2Tran}(a \cdot A, b_1, B_2)$.
- (39) For all matrices A, B over K of dimension $\text{len } b_1 \times \text{len } b_2$ such that $\text{Mx2Tran}(A, b_1, b_2) = \text{Mx2Tran}(B, b_1, b_2)$ holds $A = B$.
- (40) Let A be a matrix over K of dimension $\text{len } b_1 \times \text{len } b_2$ and B be a matrix over K of dimension $\text{len } b_2 \times \text{len } B_3$. Suppose $\text{width } A = \text{len } B$. Let A_1 be a matrix over K of dimension $\text{len } b_1 \times \text{len } B_3$. If $A_1 = A \cdot B$, then $\text{Mx2Tran}(A_1, b_1, B_3) = \text{Mx2Tran}(B, b_2, B_3) \cdot \text{Mx2Tran}(A, b_1, b_2)$.
- (41) Let A be a matrix over K of dimension $\text{len } b_1 \times \text{len } b_2$. Suppose $\text{len } b_1 > 0$ and $\text{len } b_2 > 0$. Then $v_1 \in \ker \text{Mx2Tran}(A, b_1, b_2)$ if and only if $v_1 \rightarrow b_1 \in$ the space of solutions of A^T .
- (42) V_1 is trivial iff $\dim(V_1) = 0$.
- (43) Let V_1, V_2 be vector spaces over K and f be a linear transformation from V_1 to V_2 . Then f is one-to-one if and only if $\ker f = \mathbf{0}_{(V_1)}$.

Let us consider K and let V_1 be a vector space over K . Then $\text{id}_{(V_1)}$ is a linear transformation from V_1 to V_1 .

Let us consider K , let V_1, V_2 be vector spaces over K , and let f, g be linear transformations from V_1 to V_2 . Then $f + g$ is a linear transformation from V_1 to V_2 .

Let us consider K , let V_1, V_2 be vector spaces over K , let f be a linear transformation from V_1 to V_2 , and let us consider a . Then $a \cdot f$ is a linear transformation from V_1 to V_2 .

Let us consider K , let V_1, V_2, V_3 be vector spaces over K , let f_3 be a linear transformation from V_1 to V_2 , and let f_4 be a linear transformation from V_2 to V_3 . Then $f_4 \cdot f_3$ is a linear transformation from V_1 to V_3 .

One can prove the following propositions:

- (44) For every matrix A over K of dimension $\text{len } b_1 \times \text{len } b_2$ such that $\text{rk}(A) = \text{len } b_1$ holds $\text{Mx2Tran}(A, b_1, b_2)$ is one-to-one.
- (45) $\text{MX2FinS}(I_K^{n \times n})$ is an ordered basis of the n -dimension vector space over K .
- (46) Let M be an ordered basis of the $\text{len } b_2$ -dimension vector space over K . Suppose $M = \text{MX2FinS}(I_K^{\text{len } b_2 \times \text{len } b_2})$. Let v_1 be a vector of the $\text{len } b_2$ -dimension vector space over K . Then $v_1 \rightarrow M = v_1$.
- (47) Let M be an ordered basis of the $\text{len } b_2$ -dimension vector space over K . Suppose $M = \text{MX2FinS}(I_K^{\text{len } b_2 \times \text{len } b_2})$. Let A be a matrix over K of dimension $\text{len } b_1 \times \text{len } M$. If $A = \text{AutMt}(f, b_1, b_2)$ and f is linear, then $(\text{Mx2Tran}(A, b_1, M))(v_1) = f(v_1) \rightarrow b_2$.

Let K be an add-associative right zeroed right complementable Abelian associative well unital distributive non empty double loop structure, let V_1, V_2 be Abelian add-associative right zeroed right complementable vector space-like non empty vector space structures over K , let W be a subspace of V_1 , and let f be a function from V_1 into V_2 . Then $f|W$ is a function from W into V_2 .

Let K be a field, let V_1, V_2 be vector spaces over K , let W be a subspace of V_1 , and let f be a linear transformation from V_1 to V_2 . Then $f|W$ is a linear transformation from W to V_2 .

6. THE MAIN THEOREMS

The following propositions are true:

- (48) For every linear transformation f from V_1 to V_2 holds $\text{rank } f = \text{rk}(\text{AutMt}(f, b_1, b_2))$.
- (49) For every matrix M over K of dimension $\text{len } b_1 \times \text{len } b_2$ holds $\text{rank Mx2Tran}(M, b_1, b_2) = \text{rk}(M)$.
- (50) For every linear transformation f from V_1 to V_2 such that $\dim(V_1) = \dim(V_2)$ holds $\ker f$ is non trivial iff $\text{Det AutEqMt}(f, b_1, b_2) = 0_K$.
- (51) Let f be a linear transformation from V_1 to V_2 and g be a linear transformation from V_2 to V_3 . If $g| \text{im } f$ is one-to-one, then $\text{rank}(g \cdot f) = \text{rank } f$ and $\text{nullity}(g \cdot f) = \text{nullity } f$.

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