

Ramsey's Theorem

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Summary. The goal of this article is to formalize two versions of Ramsey's theorem. The theorems are not phrased in the usually pictorial representation of a coloured graph but use a set-theoretic terminology. After some useful lemma, the second section presents a generalization of Ramsey's theorem on infinite set closely following the book [9]. The last section includes the formalization of the theorem in a more known version (see [1]).

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The notation and terminology used here are introduced in the following papers: [15], [16], [17], [4], [3], [6], [12], [7], [2], [5], [8], [14], [13], [10], and [11].

1. PRELIMINARIES

For simplicity, we adopt the following convention: n, m, k are natural numbers, X, Y, Z are sets, f is a function from X into Y , and H is a subset of X .

Let us consider X, Y, H and let P be a partition of $[X]^Y$. We say that H is homogeneous for P if and only if:

(Def. 1) There exists an element p of P such that $[H]^Y \subseteq p$.

Let us consider n and let X be an infinite set. One can check that $[X]^n$ is non empty.

Let us consider n, X, Y, f . Let us assume that f is one-to-one and $\overline{n} \subseteq \overline{X}$ and X is non empty and Y is non empty. The functor $f||^n$ yields a function from $[X]^n$ into $[Y]^n$ and is defined by:

(Def. 2) For every element x of $[X]^n$ holds $(f||^n)(x) = f^\circ x$.

Next we state four propositions:

- (1) If f is one-to-one and $\overline{n} \subseteq \overline{X}$ and X is non empty and Y is non empty, then $[f \circ H]^n = (f| |^n)^\circ([H]^n)$.
- (2) If X is infinite and $X \subseteq \omega$, then $\overline{X} = \omega$.
- (3) If X is infinite, then $X \cup Y$ is infinite.
- (4) If X is infinite and Y is finite, then $X \setminus Y$ is infinite.

Let X be an infinite set and let Y be a set. Note that $X \cup Y$ is infinite.

Let X be an infinite set and let Y be a finite set. One can verify that $X \setminus Y$ is infinite.

The following propositions are true:

- (5) $[X]^0 = \{0\}$.
- (6) For every finite set X such that $\text{card } X < n$ holds $[X]^n$ is empty.
- (7) If $X \subseteq Y$, then $[X]^Z \subseteq [Y]^Z$.
- (8) If X is finite and Y is finite and $\overline{Y} = X$, then $[Y]^X = \{Y\}$.
- (9) If X is non empty and Y is non empty, then f is constant iff there exists an element y of Y such that $\text{rng } f = \{y\}$.
- (10) For every finite set X such that $k \leq \text{card } X$ there exists a subset Y of X such that $\text{card } Y = k$.
- (11) If $m \geq 1$, then $n + 1 \leq \binom{n+m}{m}$.
- (12) If $m \geq 1$ and $n \geq 1$, then $m + 1 \leq \binom{n+m}{m}$.
- (13) Let X be a non empty set, p_1, p_2 be elements of X , P be a partition of X , and A be an element of P . Suppose $p_1 \in A$ and (the projection onto P)(p_1) = (the projection onto P)(p_2). Then $p_2 \in A$.

2. INFINITE RAMSEY THEOREM

We now state two propositions:

- (14) Let F be a function from $[X]^n$ into k . Suppose $k \neq 0$ and X is infinite. Then there exists H such that H is infinite and $F| [H]^n$ is constant.
- (15) Let X be an infinite set and P be a partition of $[X]^n$. If $\overline{P} = k$, then there exists a subset of X which is infinite and homogeneous for P .

3. RAMSEY'S THEOREM

The scheme *BinInd2* concerns a binary predicate $\mathcal{P}$, and states that:

$$\mathcal{P}[m, n]$$

provided the following conditions are satisfied:

- $\mathcal{P}[0, n]$ and $\mathcal{P}[n, 0]$, and
- If $\mathcal{P}[m + 1, n]$ and $\mathcal{P}[m, n + 1]$, then $\mathcal{P}[m + 1, n + 1]$.

We now state two propositions:

- (16) Suppose $m \geq 2$ and $n \geq 2$. Then there exists a natural number r such that
- (i) $r \leq \binom{m+n-1}{m-1}^2$,
 - (ii) $r \geq 2$, and
 - (iii) for every finite set X and for every function F from $[X]^2$ into $\text{Seg } 2$ such that $\text{card } X \geq r$ there exists a subset S of X such that $\text{card } S \geq m$ and $\text{rng}(F \upharpoonright [S]^2) = \{1\}$ or $\text{card } S \geq n$ and $\text{rng}(F \upharpoonright [S]^2) = \{2\}$.
- (17) Let m be a natural number. Then there exists a natural number r such that for every finite set X and for every partition P of $[X]^2$ if $\text{card } X \geq r$ and $\overline{P} = 2$, then there exists a subset S of X such that $\text{card } S \geq m$ and S is homogeneous for P .

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