

The Lebesgue Monotone Convergence Theorem

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Summary. In this article we prove the Monotone Convergence Theorem [16].

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The notation and terminology used in this paper have been introduced in the following articles: [10], [20], [2], [7], [21], [6], [8], [9], [1], [17], [18], [3], [4], [5], [13], [14], [15], [19], [11], [12], and [22].

1. PRELIMINARIES

For simplicity, we adopt the following rules: X is a non empty set, S is a σ -field of subsets of X , M is a σ -measure on S , E is an element of S , F, G are sequences of partial functions from X into $\overline{\mathbb{R}}$, I is a sequence of extended reals, f, g are partial functions from X to $\overline{\mathbb{R}}$, s_1, s_2, s_3 are sequences of extended reals, p is an extended real number, n, m are natural numbers, x is an element of X , and z, D are sets.

Next we state a number of propositions:

- (1) If f is without $+\infty$ and g is without $+\infty$, then $\text{dom}(f + g) = \text{dom } f \cap \text{dom } g$.
- (2) If f is without $+\infty$ and g is without $-\infty$, then $\text{dom}(f - g) = \text{dom } f \cap \text{dom } g$.

- (3) If f is without $-\infty$ and g is without $-\infty$, then $f + g$ is without $-\infty$.
- (4) If f is without $+\infty$ and g is without $+\infty$, then $f + g$ is without $+\infty$.
- (5) If f is without $-\infty$ and g is without $+\infty$, then $f - g$ is without $-\infty$.
- (6) If f is without $+\infty$ and g is without $-\infty$, then $f - g$ is without $+\infty$.
- (7)(i) If s_1 is convergent to finite number, then there exists a real number g such that $\lim s_1 = g$ and for every real number p such that $0 < p$ there exists a natural number n such that for every natural number m such that $n \leq m$ holds $|s_1(m) - \lim s_1| < p$,
- (ii) if s_1 is convergent to $+\infty$, then $\lim s_1 = +\infty$, and
- (iii) if s_1 is convergent to $-\infty$, then $\lim s_1 = -\infty$.
- (8) If s_1 is non-negative, then s_1 is not convergent to $-\infty$.
- (9) If s_1 is convergent and for every natural number k holds $s_1(k) \leq p$, then $\lim s_1 \leq p$.
- (10) If s_1 is convergent and for every natural number k holds $p \leq s_1(k)$, then $p \leq \lim s_1$.
- (11) Suppose that
 - (i) s_2 is convergent,
 - (ii) s_3 is convergent,
 - (iii) s_2 is non-negative,
 - (iv) s_3 is non-negative, and
 - (v) for every natural number k holds $s_1(k) = s_2(k) + s_3(k)$.
 Then s_1 is non-negative and convergent and $\lim s_1 = \lim s_2 + \lim s_3$.
- (12) Suppose for every natural number n holds $G(n) = F(n) \upharpoonright D$ and $x \in D$.
Then
 - (i) if $F \# x$ is convergent to $+\infty$, then $G \# x$ is convergent to $+\infty$,
 - (ii) if $F \# x$ is convergent to $-\infty$, then $G \# x$ is convergent to $-\infty$,
 - (iii) if $F \# x$ is convergent to finite number, then $G \# x$ is convergent to finite number, and
 - (iv) if $F \# x$ is convergent, then $G \# x$ is convergent.
- (13) If $E = \text{dom } f$ and f is measurable on E and f is non-negative and $M(E \cap \text{EQ-dom}(f, +\infty)) \neq 0$, then $\int f \, dM = +\infty$.
- (14) $\int \chi_{E,X} \, dM = M(E)$ and $\int \chi_{E,X} \upharpoonright E \, dM = M(E)$.
- (15) Suppose that
 - (i) $E \subseteq \text{dom } f$,
 - (ii) $E \subseteq \text{dom } g$,
 - (iii) f is measurable on E ,
 - (iv) g is measurable on E ,
 - (v) f is non-negative, and
 - (vi) for every element x of X such that $x \in E$ holds $f(x) \leq g(x)$.
 Then $\int f \upharpoonright E \, dM \leq \int g \upharpoonright E \, dM$.

2. SELECTED PROPERTIES OF EXTENDED REAL SEQUENCE

Let f be an extended real-valued function and let x be a set. Then $f(x)$ is an element of $\overline{\mathbb{R}}$.

Let s be an extended real-valued function. The functor $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}$ yields a sequence of extended reals and is defined by:

- (Def. 1) $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(0) = s(0)$ and for every natural number n holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(n+1) = (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(n) + s(n+1)$.

Let s be an extended real-valued function. We say that s is summable if and only if:

- (Def. 2) $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}$ is convergent.

Let s be an extended real-valued function. The functor $\sum s$ yielding an extended real number is defined as follows:

- (Def. 3) $\sum s = \lim((\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}})$.

Next we state several propositions:

- (16) If s_1 is non-negative, then $(\sum_{\alpha=0}^{\kappa} (s_1)(\alpha))_{\kappa \in \mathbb{N}}$ is non-negative and $(\sum_{\alpha=0}^{\kappa} (s_1)(\alpha))_{\kappa \in \mathbb{N}}$ is non-decreasing.
- (17) If for every natural number n holds $0 < s_1(n)$, then for every natural number m holds $0 < (\sum_{\alpha=0}^{\kappa} (s_1)(\alpha))_{\kappa \in \mathbb{N}}(m)$.
- (18) If F has the same dom and for every natural number n holds $G(n) = F(n) \upharpoonright D$, then G has the same dom.
- (19) Suppose that
 - (i) $D \subseteq \text{dom } F(0)$,
 - (ii) for every natural number n holds $G(n) = F(n) \upharpoonright D$, and
 - (iii) for every element x of X such that $x \in D$ holds $F \# x$ is convergent.

Then $\lim F \upharpoonright D = \lim G$.

- (20) Suppose F has the same dom and $E \subseteq \text{dom } F(0)$ and for every natural number m holds $F(m)$ is measurable on E and $G(m) = F(m) \upharpoonright E$. Then $G(n)$ is measurable on E .
- (21) Suppose that
 - (i) $E \subseteq \text{dom } F(0)$,
 - (ii) G has the same dom,
 - (iii) for every element x of X such that $x \in E$ holds $F \# x$ is summable, and
 - (iv) for every natural number n holds $G(n) = F(n) \upharpoonright E$.

Let x be an element of X . If $x \in E$, then $G \# x$ is summable.

3. PARTIAL SUMS OF FUNCTIONAL SEQUENCE AND THEIR PROPERTIES

Let X be a non empty set and let F be a sequence of partial functions from X into $\overline{\mathbb{R}}$. The functor $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}$ yields a sequence of partial functions from X into $\overline{\mathbb{R}}$ and is defined as follows:

- (Def. 4) $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(0) = F(0)$ and for every natural number n holds $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n+1) = (\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n) + F(n+1)$.

Let X be a set and let F be a sequence of partial functions from X into $\overline{\mathbb{R}}$. We say that F is additive if and only if:

- (Def. 5) For all natural numbers n, m such that $n \neq m$ and for every set x such that $x \in \text{dom } F(n) \cap \text{dom } F(m)$ holds $F(n)(x) \neq +\infty$ or $F(m)(x) \neq -\infty$.

Next we state a number of propositions:

- (22) If $z \in \text{dom}(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n)$ and $m \leq n$, then $z \in \text{dom}(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(m)$ and $z \in \text{dom } F(m)$.
- (23) If $z \in \text{dom}(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n)$ and $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n)(z) = +\infty$, then there exists a natural number m such that $m \leq n$ and $F(m)(z) = +\infty$.
- (24) If F is additive and $z \in \text{dom}(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n)$ and $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n)(z) = +\infty$ and $m \leq n$, then $F(m)(z) \neq -\infty$.
- (25) If $z \in \text{dom}(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n)$ and $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n)(z) = -\infty$, then there exists a natural number m such that $m \leq n$ and $F(m)(z) = -\infty$.
- (26) If F is additive and $z \in \text{dom}(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n)$ and $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n)(z) = -\infty$ and $m \leq n$, then $F(m)(z) \neq +\infty$.
- (27) If F is additive, then $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n)^{-1}(\{-\infty\}) \cap F(n+1)^{-1}(\{+\infty\}) = \emptyset$ and $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n)^{-1}(\{+\infty\}) \cap F(n+1)^{-1}(\{-\infty\}) = \emptyset$.
- (28) If F is additive, then $\text{dom}(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n) = \bigcap \{\text{dom } F(k); k \text{ ranges over elements of } \mathbb{N}: k \leq n\}$.
- (29) If F is additive and has the same dom, then $\text{dom}(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n) = \text{dom } F(0)$.
- (30) If for every natural number n holds $F(n)$ is non-negative, then F is additive.
- (31) If F is additive and for every n holds $G(n) = F(n) \upharpoonright D$, then G is additive.
- (32) If F is additive and has the same dom and $D \subseteq \text{dom } F(0)$ and $x \in D$, then $(\sum_{\alpha=0}^{\kappa} (F \# x)(\alpha))_{\kappa \in \mathbb{N}}(n) = ((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}} \# x)(n)$.
- (33) Suppose F is additive and has the same dom and $D \subseteq \text{dom } F(0)$ and $x \in D$. Then
 - (i) $(\sum_{\alpha=0}^{\kappa} (F \# x)(\alpha))_{\kappa \in \mathbb{N}}$ is convergent to finite number iff $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}} \# x$ is convergent to finite number,
 - (ii) $(\sum_{\alpha=0}^{\kappa} (F \# x)(\alpha))_{\kappa \in \mathbb{N}}$ is convergent to $+\infty$ iff $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}} \# x$ is convergent to $+\infty$,

- (iii) $(\sum_{\alpha=0}^{\kappa}(F\#x)(\alpha))_{\kappa \in \mathbb{N}}$ is convergent to $-\infty$ iff $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}\#x$ is convergent to $-\infty$, and
- (iv) $(\sum_{\alpha=0}^{\kappa}(F\#x)(\alpha))_{\kappa \in \mathbb{N}}$ is convergent iff $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}\#x$ is convergent.
- (34) If F is additive and has the same dom and $\text{dom } f \subseteq \text{dom } F(0)$ and $x \in \text{dom } f$ and $F\#x$ is summable and $f(x) = \sum F\#x$, then $f(x) = \lim((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}\#x)$.
- (35) Suppose that for every natural number m holds $F(m)$ is simple function in S . Then F is additive and $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n)$ is simple function in S .
- (36) If for every natural number m holds $F(m)$ is non-negative, then $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n)$ is non-negative.
- (37) If F has the same dom and $x \in \text{dom } F(0)$ and for every natural number k holds $F(k)$ is non-negative and $n \leq m$, then $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n)(x) \leq (\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(m)(x)$.
- (38) Suppose F has the same dom and $x \in \text{dom } F(0)$ and for every natural number m holds $F(m)$ is non-negative. Then $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}\#x$ is non-decreasing and $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}\#x$ is convergent.
- (39) If for every natural number m holds $F(m)$ is without $-\infty$, then $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n)$ is without $-\infty$.
- (40) If for every natural number m holds $F(m)$ is without $+\infty$, then $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n)$ is without $+\infty$.
- (41) Suppose that for every natural number n holds $F(n)$ is measurable on E and $F(n)$ is without $-\infty$. Then $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(m)$ is measurable on E .
- (42) Suppose that
 - (i) F is additive and has the same dom,
 - (ii) G is additive and has the same dom,
 - (iii) $x \in \text{dom } F(0) \cap \text{dom } G(0)$, and
 - (iv) for every natural number k and for every element y of X such that $y \in \text{dom } F(0) \cap \text{dom } G(0)$ holds $F(k)(y) \leq G(k)(y)$.
 Then $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n)(x) \leq (\sum_{\alpha=0}^{\kappa} G(\alpha))_{\kappa \in \mathbb{N}}(n)(x)$.
- (43) Let X be a non empty set and F be a sequence of partial functions from X into $\overline{\mathbb{R}}$. If F is additive and has the same dom, then $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}$ has the same dom.
- (44) Suppose that
 - (i) $\text{dom } F(0) = E$,
 - (ii) F is additive and has the same dom,
 - (iii) for every natural number n holds $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n)$ is measurable on E , and
 - (iv) for every element x of X such that $x \in E$ holds $F\#x$ is summable.

Then $\lim((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}})$ is measurable on E .

- (45) Suppose that for every natural number n holds $F(n)$ is integrable on M . Let m be a natural number. Then $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(m)$ is integrable on M .

- (46) Suppose that

- (i) $E = \text{dom } F(0)$,
- (ii) F is additive and has the same dom, and
- (iii) for every natural number n holds $F(n)$ is measurable on E and $F(n)$ is non-negative and $I(n) = \int F(n) dM$.

Then $\int (\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(m) dM = (\sum_{\alpha=0}^{\kappa} I(\alpha))_{\kappa \in \mathbb{N}}(m)$.

4. SEQUENCE OF MEASURABLE FUNCTIONS

Next we state two propositions:

- (47) Suppose that

- (i) $E \subseteq \text{dom } f$,
- (ii) f is non-negative,
- (iii) f is measurable on E ,
- (iv) F is additive,
- (v) for every n holds $F(n)$ is simple function in S and $F(n)$ is non-negative and $E \subseteq \text{dom } F(n)$, and
- (vi) for every x such that $x \in E$ holds $F\#x$ is summable and $f(x) = \sum F\#x$.

Then there exists a sequence I of extended reals such that for every n holds $I(n) = \int F(n) \upharpoonright E dM$ and I is summable and $\int f \upharpoonright E dM = \sum I$.

- (48) Suppose $E \subseteq \text{dom } f$ and f is non-negative and f is measurable on E . Then there exists a sequence g of partial functions from X into $\overline{\mathbb{R}}$ such that

- (i) g is additive,
- (ii) for every natural number n holds $g(n)$ is simple function in S and $g(n)$ is non-negative and $g(n)$ is measurable on E ,
- (iii) for every element x of X such that $x \in E$ holds $g\#x$ is summable and $f(x) = \sum g\#x$, and
- (iv) there exists a sequence I of extended reals such that for every natural number n holds $I(n) = \int g(n) \upharpoonright E dM$ and I is summable and $\int f \upharpoonright E dM = \sum I$.

Let X be a non empty set. Observe that there exists a sequence of partial functions from X into $\overline{\mathbb{R}}$ which is additive and has the same dom.

Let C, D, X be non empty sets, let F be a function from $C \times D$ into $X \rightarrow \overline{\mathbb{R}}$, let c be an element of C , and let d be an element of D . Then $F(c, d)$ is a partial function from X to $\overline{\mathbb{R}}$.

Let C, D, X be non empty sets, let F be a function from $C \times D$ into X , and let c be an element of C . The functor $\text{curry}(F, c)$ yields a function from D into X and is defined as follows:

(Def. 6) For every element d of D holds $(\text{curry}(F, c))(d) = F(c, d)$.

Let C, D, X be non empty sets, let F be a function from $C \times D$ into X , and let d be an element of D . The functor $\text{curry}'(F, d)$ yields a function from C into X and is defined as follows:

(Def. 7) For every element c of C holds $(\text{curry}'(F, d))(c) = F(c, d)$.

Let X, Y be sets, let F be a function from $\mathbb{N} \times \mathbb{N}$ into $X \rightarrowtail Y$, and let n be a natural number. The functor $\text{curry}(F, n)$ yielding a sequence of partial functions from X into Y is defined by:

(Def. 8) For every natural number m holds $(\text{curry}(F, n))(m) = F(n, m)$.

The functor $\text{curry}'(F, n)$ yields a sequence of partial functions from X into Y and is defined by:

(Def. 9) For every natural number m holds $(\text{curry}'(F, n))(m) = F(m, n)$.

Let X be a non empty set, let F be a function from $\mathbb{N}$ into $(X \rightarrowtail \overline{\mathbb{R}})^{\mathbb{N}}$, and let n be a natural number. Then $F(n)$ is a sequence of partial functions from X into $\overline{\mathbb{R}}$.

The following four propositions are true:

(49) Suppose $E = \text{dom } F(0)$ and F has the same dom and for every natural number n holds $F(n)$ is non-negative and $F(n)$ is measurable on E . Then there exists a function F_1 from $\mathbb{N}$ into $(X \rightarrowtail \overline{\mathbb{R}})^{\mathbb{N}}$ such that for every natural number n holds

- (i) for every natural number m holds $F_1(n)(m)$ is simple function in S and $\text{dom } F_1(n)(m) = \text{dom } F(n)$,
- (ii) for every natural number m holds $F_1(n)(m)$ is non-negative,
- (iii) for all natural numbers j, k such that $j \leq k$ and for every element x of X such that $x \in \text{dom } F(n)$ holds $F_1(n)(j)(x) \leq F_1(n)(k)(x)$, and
- (iv) for every element x of X such that $x \in \text{dom } F(n)$ holds $F_1(n)\#x$ is convergent and $\lim(F_1(n)\#x) = F(n)(x)$.

(50) Suppose that

- (i) $E = \text{dom } F(0)$,
- (ii) F is additive and has the same dom, and
- (iii) for every natural number n holds $F(n)$ is measurable on E and $F(n)$ is non-negative.

Then there exists a sequence I of extended reals such that for every natural number n holds

$$I(n) = \int F(n) \, dM \text{ and } \int (\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n) \, dM = (\sum_{\alpha=0}^{\kappa} I(\alpha))_{\kappa \in \mathbb{N}}(n).$$

(51) Suppose that

- (i) $E \subseteq \text{dom } F(0)$,

- (ii) F is additive and has the same dom,
- (iii) for every natural number n holds $F(n)$ is non-negative and $F(n)$ is measurable on E , and
- (iv) for every element x of X such that $x \in E$ holds $F \# x$ is summable.

Then there exists a sequence I of extended reals such that for every natural number n holds $I(n) = \int F(n) \upharpoonright E \, dM$ and I is summable and $\int \lim((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}) \upharpoonright E \, dM = \sum I$.

(52) Suppose that

- (i) $E = \text{dom } F(0)$,
- (ii) $F(0)$ is non-negative,
- (iii) F has the same dom,
- (iv) for every natural number n holds $F(n)$ is measurable on E ,
- (v) for all natural numbers n, m such that $n \leq m$ and for every element x of X such that $x \in E$ holds $F(n)(x) \leq F(m)(x)$, and
- (vi) for every element x of X such that $x \in E$ holds $F \# x$ is convergent.

Then there exists a sequence I of extended reals such that for every natural number n holds $I(n) = \int F(n) \, dM$ and I is convergent and $\int \lim F \, dM = \lim I$.

REFERENCES

- [1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(1):41–46, 1990.
- [2] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(1):91–96, 1990.
- [3] Józef Białas. Infimum and supremum of the set of real numbers. Measure theory. *Formalized Mathematics*, 2(1):163–171, 1991.
- [4] Józef Białas. Series of positive real numbers. Measure theory. *Formalized Mathematics*, 2(1):173–183, 1991.
- [5] Józef Białas. The σ -additive measure theory. *Formalized Mathematics*, 2(2):263–270, 1991.
- [6] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(1):175–180, 1990.
- [7] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(1):55–65, 1990.
- [8] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(1):153–164, 1990.
- [9] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(2):357–367, 1990.
- [10] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(1):47–53, 1990.
- [11] Noboru Endou and Yasunari Shidama. Integral of measurable function. *Formalized Mathematics*, 14(2):53–70, 2006.
- [12] Noboru Endou, Yasunari Shidama, and Keiko Narita. Egoroff's theorem. *Formalized Mathematics*, 16(1):57–63, 2008.
- [13] Noboru Endou, Katsumi Wasaki, and Yasunari Shidama. Basic properties of extended real numbers. *Formalized Mathematics*, 9(3):491–494, 2001.
- [14] Noboru Endou, Katsumi Wasaki, and Yasunari Shidama. Definitions and basic properties of measurable functions. *Formalized Mathematics*, 9(3):495–500, 2001.
- [15] Noboru Endou, Katsumi Wasaki, and Yasunari Shidama. The measurability of extended real valued functions. *Formalized Mathematics*, 9(3):525–529, 2001.
- [16] P. R. Halmos. *Measure Theory*. Springer-Verlag, 1987.
- [17] Andrzej Nędzusiak. σ -fields and probability. *Formalized Mathematics*, 1(2):401–407, 1990.
- [18] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(1):147–152, 1990.

- [19] Beata Perkowski. Functional sequence from a domain to a domain. *Formalized Mathematics*, 3(1):17–21, 1992.
- [20] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(1):67–71, 1990.
- [21] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(1):181–186, 1990.
- [22] Hiroshi Yamazaki, Noboru Endou, Yasunari Shidama, and Hiroyuki Okazaki. Inferior limit, superior limit and convergence of sequences of extended real numbers. *Formalized Mathematics*, 15(4):231–236, 2007.

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