

Inverse Trigonometric Functions

Arctan and Arccot

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Summary. This article describes definitions of inverse trigonometric functions arctan, arccot and their main properties, as well as several differentiation formulas of arctan and arccot.

MML identifier: SIN_C0S9, version: 7.8.10 4.100.1011

The articles [17], [1], [2], [18], [3], [13], [19], [7], [15], [5], [9], [12], [16], [4], [6], [8], [11], [14], and [10] provide the notation and terminology for this paper.

1. FUNCTION ARCTAN AND ARCCOT

For simplicity, we adopt the following convention: x , r , s , h denote real numbers, n denotes an element of $\mathbb{N}$, Z denotes an open subset of $\mathbb{R}$, and f , f_1 , f_2 denote partial functions from $\mathbb{R}$ to $\mathbb{R}$.

The following propositions are true:

- (1) $] -\frac{\pi}{2}, \frac{\pi}{2}[\subseteq \text{dom}(\text{the function } \tan)$.
- (2) $]0, \pi[\subseteq \text{dom}(\text{the function } \cot)$.
- (3)(i) The function $\tan$ is differentiable on $] -\frac{\pi}{2}, \frac{\pi}{2}[$, and
 (ii) for every x such that $x \in] -\frac{\pi}{2}, \frac{\pi}{2}[$ holds $(\text{the function } \tan)'(x) = \frac{1}{(\cos x)^2}$.
- (4) The function $\cot$ is differentiable on $]0, \pi[$ and for every x such that $x \in]0, \pi[$ holds $(\text{the function } \cot)'(x) = -\frac{1}{(\sin x)^2}$.
- (5) The function $\tan$ is continuous on $] -\frac{\pi}{2}, \frac{\pi}{2}[$.
- (6) The function $\cot$ is continuous on $]0, \pi[$.

- (7) The function $\tan$ is increasing on $] -\frac{\pi}{2}, \frac{\pi}{2}[$.
- (8) The function $\cot$ is decreasing on $]0, \pi[$.
- (9) (The function $\tan$) $\restriction] -\frac{\pi}{2}, \frac{\pi}{2}[$ is one-to-one.
- (10) (The function $\cot$) $\restriction]0, \pi[$ is one-to-one.

Let us mention that (the function $\tan$) $\restriction] -\frac{\pi}{2}, \frac{\pi}{2}[$ is one-to-one and (the function $\cot$) $\restriction]0, \pi[$ is one-to-one.

The partial function the function $\arctan$ from $\mathbb{R}$ to $\mathbb{R}$ is defined as follows:

- (Def. 1) The function $\arctan = ((\text{the function } \tan)\restriction] -\frac{\pi}{2}, \frac{\pi}{2}[)^{-1}$.

The partial function the function arccot from $\mathbb{R}$ to $\mathbb{R}$ is defined by:

- (Def. 2) The function $\operatorname{arccot} = ((\text{the function } \cot)\restriction]0, \pi[)^{-1}$.

Let r be a real number. The functor $\arctan r$ is defined by:

- (Def. 3) $\arctan r = (\text{the function } \arctan)(r)$.

The functor $\operatorname{arccot} r$ is defined by:

- (Def. 4) $\operatorname{arccot} r = (\text{the function } \operatorname{arccot})(r)$.

Let r be a real number. Then $\arctan r$ is a real number. Then $\operatorname{arccot} r$ is a real number.

We now state two propositions:

- (11) $\operatorname{rng}(\text{the function } \arctan) =] -\frac{\pi}{2}, \frac{\pi}{2}[$.
- (12) $\operatorname{rng}(\text{the function } \operatorname{arccot}) =]0, \pi[$.

Let us mention that the function $\arctan$ is one-to-one and the function arccot is one-to-one.

Let r be a real number. Then $\tan r$ is a real number. Then $\cot r$ is a real number.

Next we state a number of propositions:

- (13) For every real number x such that $x \in] -\frac{\pi}{2}, \frac{\pi}{2}[$ holds (the function $\tan$)(x) = $\tan x$.
- (14) For every real number x such that $x \in]0, \pi[$ holds (the function $\cot$)(x) = $\cot x$.
- (15) For every real number x such that $\cos x \neq 0$ holds (the function $\tan$)(x) = $\tan x$.
- (16) For every real number x such that (the function $\sin$)(x) $\neq 0$ holds (the function $\cot$)(x) = $\cot x$.
- (17) $\tan(-\frac{\pi}{4}) = -1$.
- (18) $\cot(\frac{\pi}{4}) = 1$ and $\cot(\frac{3}{4} \cdot \pi) = -1$.
- (19) For every real number x such that $x \in [-\frac{\pi}{4}, \frac{\pi}{4}]$ holds $\tan x \in [-1, 1]$.
- (20) For every real number x such that $x \in [\frac{\pi}{4}, \frac{3}{4} \cdot \pi]$ holds $\cot x \in [-1, 1]$.
- (21) $\operatorname{rng}((\text{the function } \tan)\restriction [-\frac{\pi}{4}, \frac{\pi}{4}]) = [-1, 1]$.
- (22) $\operatorname{rng}((\text{the function } \cot)\restriction [\frac{\pi}{4}, \frac{3}{4} \cdot \pi]) = [-1, 1]$.

$$(23) \quad [-1, 1] \subseteq \text{dom}(\text{the function } \arctan).$$

$$(24) \quad [-1, 1] \subseteq \text{dom}(\text{the function } \text{arccot}).$$

Let us observe that $(\text{the function } \tan) \upharpoonright [-\frac{\pi}{4}, \frac{\pi}{4}]$ is one-to-one and $(\text{the function } \cot) \upharpoonright [\frac{\pi}{4}, \frac{3}{4} \cdot \pi]$ is one-to-one.

The following propositions are true:

$$(25) \quad (\text{The function } \arctan) \upharpoonright [-1, 1] = ((\text{the function } \tan) \upharpoonright [-\frac{\pi}{4}, \frac{\pi}{4}])^{-1}.$$

$$(26) \quad (\text{The function } \text{arccot}) \upharpoonright [-1, 1] = ((\text{the function } \cot) \upharpoonright [\frac{\pi}{4}, \frac{3}{4} \cdot \pi])^{-1}.$$

$$(27) \quad ((\text{The function } \tan) \upharpoonright [-\frac{\pi}{4}, \frac{\pi}{4}] \text{ qua function}) \cdot ((\text{the function } \arctan) \upharpoonright [-1, 1]) = \text{id}_{[-1, 1]}.$$

$$(28) \quad ((\text{The function } \cot) \upharpoonright [\frac{\pi}{4}, \frac{3}{4} \cdot \pi] \text{ qua function}) \cdot ((\text{the function } \text{arccot}) \upharpoonright [-1, 1]) = \text{id}_{[-1, 1]}.$$

$$(29) \quad ((\text{The function } \tan) \upharpoonright [-\frac{\pi}{4}, \frac{\pi}{4}]) \cdot ((\text{the function } \arctan) \upharpoonright [-1, 1]) = \text{id}_{[-1, 1]}.$$

$$(30) \quad ((\text{The function } \cot) \upharpoonright [\frac{\pi}{4}, \frac{3}{4} \cdot \pi]) \cdot ((\text{the function } \text{arccot}) \upharpoonright [-1, 1]) = \text{id}_{[-1, 1]}.$$

$$(31) \quad (\text{The function } \arctan \text{ qua function}) \cdot ((\text{the function } \tan) \upharpoonright]-\frac{\pi}{2}, \frac{\pi}{2}[) = \text{id}_{]-\frac{\pi}{2}, \frac{\pi}{2}[}.$$

$$(32) \quad (\text{The function } \text{arccot}) \cdot ((\text{the function } \cot) \upharpoonright]0, \pi[) = \text{id}_{]0, \pi[}.$$

$$(33) \quad (\text{The function } \arctan \text{ qua function}) \cdot ((\text{the function } \tan) \upharpoonright]-\frac{\pi}{2}, \frac{\pi}{2}[) = \text{id}_{]-\frac{\pi}{2}, \frac{\pi}{2}[}.$$

$$(34) \quad (\text{The function } \text{arccot} \text{ qua function}) \cdot ((\text{the function } \cot) \upharpoonright]0, \pi[) = \text{id}_{]0, \pi[}.$$

$$(35) \quad \text{If } -\frac{\pi}{2} < r < \frac{\pi}{2}, \text{ then } \arctan \tan r = r.$$

$$(36) \quad \text{If } 0 < r < \pi, \text{ then } \text{arccot } \cot r = r.$$

$$(37) \quad \arctan(-1) = -\frac{\pi}{4}.$$

$$(38) \quad \text{arccot}(-1) = \frac{3}{4} \cdot \pi.$$

$$(39) \quad \arctan 1 = \frac{\pi}{4}.$$

$$(40) \quad \text{arccot } 1 = \frac{\pi}{4}.$$

$$(41) \quad \tan 0 = 0.$$

$$(42) \quad \cot(\frac{\pi}{2}) = 0.$$

$$(43) \quad \arctan 0 = 0.$$

$$(44) \quad \text{arccot } 0 = \frac{\pi}{2}.$$

$$(45) \quad \text{The function } \arctan \text{ is increasing on } (\text{the function } \tan) \circ]-\frac{\pi}{2}, \frac{\pi}{2}[.$$

$$(46) \quad \text{The function } \text{arccot} \text{ is decreasing on } (\text{the function } \cot) \circ]0, \pi[.$$

$$(47) \quad \text{The function } \arctan \text{ is increasing on } [-1, 1].$$

$$(48) \quad \text{The function } \text{arccot} \text{ is decreasing on } [-1, 1].$$

$$(49) \quad \text{For every real number } x \text{ such that } x \in [-1, 1] \text{ holds } \arctan x \in [-\frac{\pi}{4}, \frac{\pi}{4}].$$

$$(50) \quad \text{For every real number } x \text{ such that } x \in [-1, 1] \text{ holds } \text{arccot } x \in [\frac{\pi}{4}, \frac{3}{4} \cdot \pi].$$

$$(51) \quad \text{If } -1 \leq r \leq 1, \text{ then } \tan \arctan r = r.$$

$$(52) \quad \text{If } -1 \leq r \leq 1, \text{ then } \cot \text{arccot } r = r.$$

- (53) The function $\arctan$ is continuous on $[-1, 1]$.
- (54) The function arccot is continuous on $[-1, 1]$.
- (55) $\operatorname{rng}((\text{the function } \arctan) \upharpoonright [-1, 1]) = [-\frac{\pi}{4}, \frac{\pi}{4}]$.
- (56) $\operatorname{rng}((\text{the function } \operatorname{arccot}) \upharpoonright [-1, 1]) = [\frac{\pi}{4}, \frac{3}{4} \cdot \pi]$.
- (57) If $-1 \leq r \leq 1$ and $\arctan r = -\frac{\pi}{4}$, then $r = -1$.
- (58) If $-1 \leq r \leq 1$ and $\operatorname{arccot} r = \frac{3}{4} \cdot \pi$, then $r = -1$.
- (59) If $-1 \leq r \leq 1$ and $\arctan r = 0$, then $r = 0$.
- (60) If $-1 \leq r \leq 1$ and $\operatorname{arccot} r = \frac{\pi}{2}$, then $r = 0$.
- (61) If $-1 \leq r \leq 1$ and $\arctan r = \frac{\pi}{4}$, then $r = 1$.
- (62) If $-1 \leq r \leq 1$ and $\operatorname{arccot} r = \frac{\pi}{4}$, then $r = 1$.
- (63) If $-1 \leq r \leq 1$, then $-\frac{\pi}{4} \leq \arctan r \leq \frac{\pi}{4}$.
- (64) If $-1 \leq r \leq 1$, then $\frac{\pi}{4} \leq \operatorname{arccot} r \leq \frac{3}{4} \cdot \pi$.
- (65) If $-1 < r < 1$, then $-\frac{\pi}{4} < \arctan r < \frac{\pi}{4}$.
- (66) If $-1 < r < 1$, then $\frac{\pi}{4} < \operatorname{arccot} r < \frac{3}{4} \cdot \pi$.
- (67) If $-1 \leq r \leq 1$, then $\arctan r = -\arctan(-r)$.
- (68) If $-1 \leq r \leq 1$, then $\operatorname{arccot} r = \pi - \operatorname{arccot}(-r)$.
- (69) If $-1 \leq r \leq 1$, then $\cot \arctan r = \frac{1}{r}$.
- (70) If $-1 \leq r \leq 1$, then $\tan \operatorname{arccot} r = \frac{1}{r}$.
- (71) The function $\arctan$ is differentiable on $(\text{the function } \tan) \circ]-\frac{\pi}{2}, \frac{\pi}{2}[$.
- (72) The function arccot is differentiable on $(\text{the function } \cot) \circ]0, \pi[$.
- (73) The function $\arctan$ is differentiable on $] -1, 1[$.
- (74) The function arccot is differentiable on $] -1, 1[$.
- (75) If $-1 \leq r \leq 1$, then $(\text{the function } \arctan)'(r) = \frac{1}{1+r^2}$.
- (76) If $-1 \leq r \leq 1$, then $(\text{the function } \operatorname{arccot})'(r) = -\frac{1}{1+r^2}$.
- (77) The function $\arctan$ is continuous on $(\text{the function } \tan) \circ]-\frac{\pi}{2}, \frac{\pi}{2}[$.
- (78) The function arccot is continuous on $(\text{the function } \cot) \circ]0, \pi[$.
- (79) $\operatorname{dom}(\text{the function } \arctan)$ is open.
- (80) $\operatorname{dom}(\text{the function } \operatorname{arccot})$ is open.

2. SEVERAL DIFFERENTIATION FORMULAS OF ARCTAN AND ARCCOT

We now state a number of propositions:

- (81) Suppose $Z \subseteq]-1, 1[$. Then the function $\arctan$ is differentiable on Z and for every x such that $x \in Z$ holds $(\text{the function } \arctan)'_{|Z}(x) = \frac{1}{1+x^2}$.
- (82) Suppose $Z \subseteq]-1, 1[$. Then the function arccot is differentiable on Z and for every x such that $x \in Z$ holds $(\text{the function } \operatorname{arccot})'_{|Z}(x) = -\frac{1}{1+x^2}$.

- (83) Suppose $Z \subseteq]-1, 1[$. Then
- (i) r the function $\arctan$ is differentiable on Z , and
 - (ii) for every x such that $x \in Z$ holds $(r \text{ the function } \arctan)'_{|Z}(x) = \frac{r}{1+x^2}$.
- (84) Suppose $Z \subseteq]-1, 1[$. Then
- (i) r the function arccot is differentiable on Z , and
 - (ii) for every x such that $x \in Z$ holds $(r \text{ the function } \operatorname{arccot})'_{|Z}(x) = -\frac{r}{1+x^2}$.
- (85) Suppose f is differentiable in x and $-1 < f(x) < 1$. Then (the function $\arctan$) $\cdot f$ is differentiable in x and $((\text{the function } \arctan) \cdot f)'(x) = \frac{f'(x)}{1+f(x)^2}$.
- (86) Suppose f is differentiable in x and $-1 < f(x) < 1$. Then (the function arccot) $\cdot f$ is differentiable in x and $((\text{the function } \operatorname{arccot}) \cdot f)'(x) = -\frac{f'(x)}{1+f(x)^2}$.
- (87) Suppose $Z \subseteq \operatorname{dom}((\text{the function } \arctan) \cdot f)$ and for every x such that $x \in Z$ holds $f(x) = r \cdot x + s$ and $-1 < f(x) < 1$. Then
- (i) (the function $\arctan$) $\cdot f$ is differentiable on Z , and
 - (ii) for every x such that $x \in Z$ holds $((\text{the function } \arctan) \cdot f)'_{|Z}(x) = \frac{r}{1+(r \cdot x + s)^2}$.
- (88) Suppose $Z \subseteq \operatorname{dom}((\text{the function } \operatorname{arccot}) \cdot f)$ and for every x such that $x \in Z$ holds $f(x) = r \cdot x + s$ and $-1 < f(x) < 1$. Then
- (i) (the function arccot) $\cdot f$ is differentiable on Z , and
 - (ii) for every x such that $x \in Z$ holds $((\text{the function } \operatorname{arccot}) \cdot f)'_{|Z}(x) = -\frac{r}{1+(r \cdot x + s)^2}$.
- (89) Suppose $Z \subseteq \operatorname{dom}((\text{the function } \ln) \cdot (\text{the function } \arctan))$ and $Z \subseteq]-1, 1[$ and for every x such that $x \in Z$ holds $\arctan x > 0$. Then
- (i) (the function $\ln$) $\cdot$ (the function $\arctan$) is differentiable on Z , and
 - (ii) for every x such that $x \in Z$ holds $((\text{the function } \ln) \cdot (\text{the function } \arctan))'_{|Z}(x) = \frac{1}{(1+x^2) \cdot \arctan x}$.
- (90) Suppose $Z \subseteq \operatorname{dom}((\text{the function } \ln) \cdot (\text{the function } \operatorname{arccot}))$ and $Z \subseteq]-1, 1[$ and for every x such that $x \in Z$ holds $\operatorname{arccot} x > 0$. Then
- (i) (the function $\ln$) $\cdot$ (the function arccot) is differentiable on Z , and
 - (ii) for every x such that $x \in Z$ holds $((\text{the function } \ln) \cdot (\text{the function } \operatorname{arccot}))'_{|Z}(x) = -\frac{1}{(1+x^2) \cdot \operatorname{arccot} x}$.
- (91) Suppose $Z \subseteq \operatorname{dom}((\square^n) \cdot \text{the function } \arctan)$ and $Z \subseteq]-1, 1[$. Then
- (i) $(\square^n) \cdot \text{the function } \arctan$ is differentiable on Z , and
 - (ii) for every x such that $x \in Z$ holds $((\square^n) \cdot \text{the function } \arctan)'_{|Z}(x) = \frac{n \cdot (\arctan x)^{n-1}}{1+x^2}$.
- (92) Suppose $Z \subseteq \operatorname{dom}((\square^n) \cdot \text{the function } \operatorname{arccot})$ and $Z \subseteq]-1, 1[$. Then
- (i) $(\square^n) \cdot \text{the function } \operatorname{arccot}$ is differentiable on Z , and

- (ii) for every x such that $x \in Z$ holds $((\square^n) \cdot \text{the function arccot})'_{|Z}(x) = -\frac{n \cdot (\text{arccot } x)^{n-1}}{1+x^2}$.
- (93) Suppose $Z \subseteq \text{dom}(\frac{1}{2}((\square^2) \cdot \text{the function arctan}))$ and $Z \subseteq]-1, 1[$. Then
 - (i) $\frac{1}{2}((\square^2) \cdot \text{the function arctan})$ is differentiable on Z , and
 - (ii) for every x such that $x \in Z$ holds $(\frac{1}{2}((\square^2) \cdot \text{the function arctan}))'_{|Z}(x) = \frac{\text{arctan } x}{1+x^2}$.
- (94) Suppose $Z \subseteq \text{dom}(\frac{1}{2}((\square^2) \cdot \text{the function arccot}))$ and $Z \subseteq]-1, 1[$. Then
 - (i) $\frac{1}{2}((\square^2) \cdot \text{the function arccot})$ is differentiable on Z , and
 - (ii) for every x such that $x \in Z$ holds $(\frac{1}{2}((\square^2) \cdot \text{the function arccot}))'_{|Z}(x) = -\frac{\text{arccot } x}{1+x^2}$.
- (95) Suppose $Z \subseteq]-1, 1[$. Then
 - (i) id_Z the function arctan is differentiable on Z , and
 - (ii) for every x such that $x \in Z$ holds $(\text{id}_Z \text{ the function arctan})'_{|Z}(x) = \text{arctan } x + \frac{x}{1+x^2}$.
- (96) Suppose $Z \subseteq]-1, 1[$. Then
 - (i) id_Z the function arccot is differentiable on Z , and
 - (ii) for every x such that $x \in Z$ holds $(\text{id}_Z \text{ the function arccot})'_{|Z}(x) = \text{arccot } x - \frac{x}{1+x^2}$.
- (97) Suppose $Z \subseteq \text{dom}(f \text{ the function arctan})$ and $Z \subseteq]-1, 1[$ and for every x such that $x \in Z$ holds $f(x) = r \cdot x + s$. Then
 - (i) f the function arctan is differentiable on Z , and
 - (ii) for every x such that $x \in Z$ holds $(f \text{ the function arctan})'_{|Z}(x) = r \cdot \text{arctan } x + \frac{r \cdot x + s}{1+x^2}$.
- (98) Suppose $Z \subseteq \text{dom}(f \text{ the function arccot})$ and $Z \subseteq]-1, 1[$ and for every x such that $x \in Z$ holds $f(x) = r \cdot x + s$. Then
 - (i) f the function arccot is differentiable on Z , and
 - (ii) for every x such that $x \in Z$ holds $(f \text{ the function arccot})'_{|Z}(x) = r \cdot \text{arccot } x - \frac{r \cdot x + s}{1+x^2}$.
- (99) Suppose $Z \subseteq \text{dom}(\frac{1}{2}((\text{the function arctan}) \cdot f))$ and for every x such that $x \in Z$ holds $f(x) = 2 \cdot x$ and $-1 < f(x) < 1$. Then
 - (i) $\frac{1}{2}((\text{the function arctan}) \cdot f)$ is differentiable on Z , and
 - (ii) for every x such that $x \in Z$ holds $(\frac{1}{2}((\text{the function arctan}) \cdot f))'_{|Z}(x) = \frac{1}{1+(2 \cdot x)^2}$.
- (100) Suppose $Z \subseteq \text{dom}(\frac{1}{2}((\text{the function arccot}) \cdot f))$ and for every x such that $x \in Z$ holds $f(x) = 2 \cdot x$ and $-1 < f(x) < 1$. Then
 - (i) $\frac{1}{2}((\text{the function arccot}) \cdot f)$ is differentiable on Z , and
 - (ii) for every x such that $x \in Z$ holds $(\frac{1}{2}((\text{the function arccot}) \cdot f))'_{|Z}(x) = -\frac{1}{1+(2 \cdot x)^2}$.
- (101) Suppose $Z \subseteq \text{dom}(f_1 + f_2)$ and for every x such that $x \in Z$ holds $f_1(x) = 1$ and $f_2 = \square^2$. Then $f_1 + f_2$ is differentiable on Z and for every

- x such that $x \in Z$ holds $(f_1 + f_2)'_{|Z}(x) = 2 \cdot x$.
- (102) Suppose $Z \subseteq \text{dom}(\frac{1}{2}((\text{the function } \ln) \cdot (f_1 + f_2)))$ and $f_2 = \square^2$ and for every x such that $x \in Z$ holds $f_1(x) = 1$. Then
- (i) $\frac{1}{2}((\text{the function } \ln) \cdot (f_1 + f_2))$ is differentiable on Z , and
 - (ii) for every x such that $x \in Z$ holds $(\frac{1}{2}((\text{the function } \ln) \cdot (f_1 + f_2)))'_{|Z}(x) = \frac{x}{1+x^2}$.
- (103) Suppose that
- (i) $Z \subseteq \text{dom}(\text{id}_Z \text{ the function } \arctan - \frac{1}{2}((\text{the function } \ln) \cdot (f_1 + f_2)))$,
 - (ii) $Z \subseteq]-1, 1[$,
 - (iii) $f_2 = \square^2$, and
 - (iv) for every x such that $x \in Z$ holds $f_1(x) = 1$.
- Then
- (v) $\text{id}_Z \text{ the function } \arctan - \frac{1}{2}((\text{the function } \ln) \cdot (f_1 + f_2))$ is differentiable on Z , and
 - (vi) for every x such that $x \in Z$ holds $(\text{id}_Z \text{ the function } \arctan - \frac{1}{2}((\text{the function } \ln) \cdot (f_1 + f_2)))'_{|Z}(x) = \arctan x$.
- (104) Suppose that
- (i) $Z \subseteq \text{dom}(\text{id}_Z \text{ the function } \text{arccot} + \frac{1}{2}((\text{the function } \ln) \cdot (f_1 + f_2)))$,
 - (ii) $Z \subseteq]-1, 1[$,
 - (iii) $f_2 = \square^2$, and
 - (iv) for every x such that $x \in Z$ holds $f_1(x) = 1$.
- Then
- (v) $\text{id}_Z \text{ the function } \text{arccot} + \frac{1}{2}((\text{the function } \ln) \cdot (f_1 + f_2))$ is differentiable on Z , and
 - (vi) for every x such that $x \in Z$ holds $(\text{id}_Z \text{ the function } \text{arccot} + \frac{1}{2}((\text{the function } \ln) \cdot (f_1 + f_2)))'_{|Z}(x) = \text{arccot } x$.
- (105) Suppose $Z \subseteq \text{dom}(\text{id}_Z ((\text{the function } \arctan) \cdot f))$ and for every x such that $x \in Z$ holds $f(x) = \frac{x}{r}$ and $-1 < f(x) < 1$. Then
- (i) $\text{id}_Z ((\text{the function } \arctan) \cdot f)$ is differentiable on Z , and
 - (ii) for every x such that $x \in Z$ holds $(\text{id}_Z ((\text{the function } \arctan) \cdot f))'_{|Z}(x) = \arctan(\frac{x}{r}) + \frac{x}{r \cdot (1+(\frac{x}{r})^2)}$.
- (106) Suppose $Z \subseteq \text{dom}(\text{id}_Z ((\text{the function } \text{arccot}) \cdot f))$ and for every x such that $x \in Z$ holds $f(x) = \frac{x}{r}$ and $-1 < f(x) < 1$. Then
- (i) $\text{id}_Z ((\text{the function } \text{arccot}) \cdot f)$ is differentiable on Z , and
 - (ii) for every x such that $x \in Z$ holds $(\text{id}_Z ((\text{the function } \text{arccot}) \cdot f))'_{|Z}(x) = \text{arccot}(\frac{x}{r}) - \frac{x}{r \cdot (1+(\frac{x}{r})^2)}$.
- (107) Suppose $Z \subseteq \text{dom}(f_1 + f_2)$ and for every x such that $x \in Z$ holds $f_1(x) = 1$ and $f_2 = (\square^2) \cdot f$ and for every x such that $x \in Z$ holds $f(x) = \frac{x}{r}$. Then $f_1 + f_2$ is differentiable on Z and for every x such that $x \in Z$ holds $(f_1 + f_2)'_{|Z}(x) = \frac{2 \cdot x}{r^2}$.

(108) Suppose that

- (i) $Z \subseteq \text{dom}(\frac{r}{2}((\text{the function } \ln) \cdot (f_1 + f_2)))$,
- (ii) for every x such that $x \in Z$ holds $f_1(x) = 1$,
- (iii) $r \neq 0$,
- (iv) $f_2 = (\square^2) \cdot f$, and
- (v) for every x such that $x \in Z$ holds $f(x) = \frac{x}{r}$.

Then

- (vi) $\frac{r}{2}((\text{the function } \ln) \cdot (f_1 + f_2))$ is differentiable on Z , and
- (vii) for every x such that $x \in Z$ holds $(\frac{r}{2}((\text{the function } \ln) \cdot (f_1 + f_2)))'_{|Z}(x) = \frac{x}{r \cdot (1 + (\frac{x}{r})^2)}$.

(109) Suppose that

- (i) $Z \subseteq \text{dom}(\text{id}_Z((\text{the function } \arctan) \cdot f) - \frac{r}{2}((\text{the function } \ln) \cdot (f_1 + f_2)))$,
- (ii) $r \neq 0$,
- (iii) for every x such that $x \in Z$ holds $f(x) = \frac{x}{r}$ and $-1 < f(x) < 1$,
- (iv) for every x such that $x \in Z$ holds $f_1(x) = 1$,
- (v) $f_2 = (\square^2) \cdot f$, and
- (vi) for every x such that $x \in Z$ holds $f(x) = \frac{x}{r}$.

Then

- (vii) $\text{id}_Z((\text{the function } \arctan) \cdot f) - \frac{r}{2}((\text{the function } \ln) \cdot (f_1 + f_2))$ is differentiable on Z , and
- (viii) for every x such that $x \in Z$ holds $(\text{id}_Z((\text{the function } \arctan) \cdot f) - \frac{r}{2}((\text{the function } \ln) \cdot (f_1 + f_2)))'_{|Z}(x) = \arctan(\frac{x}{r})$.

(110) Suppose that

- (i) $Z \subseteq \text{dom}(\text{id}_Z((\text{the function } \text{arccot}) \cdot f) + \frac{r}{2}((\text{the function } \ln) \cdot (f_1 + f_2)))$,
- (ii) $r \neq 0$,
- (iii) for every x such that $x \in Z$ holds $f(x) = \frac{x}{r}$ and $-1 < f(x) < 1$,
- (iv) for every x such that $x \in Z$ holds $f_1(x) = 1$,
- (v) $f_2 = (\square^2) \cdot f$, and
- (vi) for every x such that $x \in Z$ holds $f(x) = \frac{x}{r}$.

Then

- (vii) $\text{id}_Z((\text{the function } \text{arccot}) \cdot f) + \frac{r}{2}((\text{the function } \ln) \cdot (f_1 + f_2))$ is differentiable on Z , and
- (viii) for every x such that $x \in Z$ holds $(\text{id}_Z((\text{the function } \text{arccot}) \cdot f) + \frac{r}{2}((\text{the function } \ln) \cdot (f_1 + f_2)))'_{|Z}(x) = \text{arccot}(\frac{x}{r})$.

(111) Suppose $Z \subseteq \text{dom}((\text{the function } \arctan) \cdot \frac{1}{f})$ and for every x such that $x \in Z$ holds $f(x) = x$ and $-1 < (\frac{1}{f})(x) < 1$. Then

- (i) $(\text{the function } \arctan) \cdot \frac{1}{f}$ is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds $((\text{the function } \arctan) \cdot \frac{1}{f})'_{|Z}(x) = -\frac{1}{1+x^2}$.

(112) Suppose $Z \subseteq \text{dom}((\text{the function } \text{arccot}) \cdot \frac{1}{f})$ and for every x such that $x \in Z$ holds $f(x) = x$ and $-1 < (\frac{1}{f})(x) < 1$. Then

- (i) $(\text{the function } \text{arccot}) \cdot \frac{1}{f}$ is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds $((\text{the function } \text{arccot}) \cdot \frac{1}{f})'_Z(x) = \frac{1}{1+x^2}$.

(113) Suppose that

- (i) $Z \subseteq \text{dom}((\text{the function } \text{arctan}) \cdot f)$,
- (ii) $f = f_1 + h f_2$,
- (iii) for every x such that $x \in Z$ holds $-1 < f(x) < 1$,
- (iv) for every x such that $x \in Z$ holds $f_1(x) = r + s \cdot x$, and
- (v) $f_2 = \square^2$.

Then

- (vi) $(\text{the function } \text{arctan}) \cdot (f_1 + h f_2)$ is differentiable on Z , and
- (vii) for every x such that $x \in Z$ holds $((\text{the function } \text{arctan}) \cdot (f_1 + h f_2))'_Z(x) = \frac{s+2 \cdot h \cdot x}{1+(r+s \cdot x+h \cdot x^2)^2}$.

(114) Suppose that

- (i) $Z \subseteq \text{dom}((\text{the function } \text{arccot}) \cdot f)$,
- (ii) $f = f_1 + h f_2$,
- (iii) for every x such that $x \in Z$ holds $-1 < f(x) < 1$,
- (iv) for every x such that $x \in Z$ holds $f_1(x) = r + s \cdot x$, and
- (v) $f_2 = \square^2$.

Then

- (vi) $(\text{the function } \text{arccot}) \cdot (f_1 + h f_2)$ is differentiable on Z , and
- (vii) for every x such that $x \in Z$ holds $((\text{the function } \text{arccot}) \cdot (f_1 + h f_2))'_Z(x) = -\frac{s+2 \cdot h \cdot x}{1+(r+s \cdot x+h \cdot x^2)^2}$.

(115) Suppose $Z \subseteq \text{dom}((\text{the function } \text{arctan}) \cdot (\text{the function } \exp))$ and for every x such that $x \in Z$ holds $\exp x < 1$. Then

- (i) $(\text{the function } \text{arctan}) \cdot (\text{the function } \exp)$ is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds $((\text{the function } \text{arctan}) \cdot (\text{the function } \exp))'_Z(x) = \frac{\exp x}{1+(\exp x)^2}$.

(116) Suppose $Z \subseteq \text{dom}((\text{the function } \text{arccot}) \cdot (\text{the function } \exp))$ and for every x such that $x \in Z$ holds $\exp x < 1$. Then

- (i) $(\text{the function } \text{arccot}) \cdot (\text{the function } \exp)$ is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds $((\text{the function } \text{arccot}) \cdot (\text{the function } \exp))'_Z(x) = -\frac{\exp x}{1+(\exp x)^2}$.

(117) Suppose that

- (i) $Z \subseteq \text{dom}((\text{the function } \text{arctan}) \cdot (\text{the function } \ln))$, and
- (ii) for every x such that $x \in Z$ holds $-1 < (\text{the function } \ln)(x)$ and $(\text{the function } \ln)(x) < 1$.

Then

- (iii) (the function $\arctan$) $\cdot$ (the function $\ln$) is differentiable on Z , and
 - (iv) for every x such that $x \in Z$ holds ((the function $\arctan$) $\cdot$ (the function $\ln$))' $\big|_Z(x) = \frac{1}{x \cdot (1 + (\text{the function } \ln)(x)^2)}$.
- (118) Suppose that
- (i) $Z \subseteq \text{dom}((\text{the function } \text{arccot}) \cdot (\text{the function } \ln))$, and
 - (ii) for every x such that $x \in Z$ holds $-1 < (\text{the function } \ln)(x)$ and $(\text{the function } \ln)(x) < 1$.
- Then
- (iii) (the function arccot) $\cdot$ (the function $\ln$) is differentiable on Z , and
 - (iv) for every x such that $x \in Z$ holds ((the function arccot) $\cdot$ (the function $\ln$))' $\big|_Z(x) = -\frac{1}{x \cdot (1 + (\text{the function } \ln)(x)^2)}$.
- (119) Suppose $Z \subseteq \text{dom}((\text{the function } \exp) \cdot (\text{the function } \arctan))$ and $Z \subseteq]-1, 1[$. Then
- (i) (the function $\exp$) $\cdot$ (the function $\arctan$) is differentiable on Z , and
 - (ii) for every x such that $x \in Z$ holds ((the function $\exp$) $\cdot$ (the function $\arctan$))' $\big|_Z(x) = \frac{\exp \arctan x}{1+x^2}$.
- (120) Suppose $Z \subseteq \text{dom}((\text{the function } \exp) \cdot (\text{the function } \text{arccot}))$ and $Z \subseteq]-1, 1[$. Then
- (i) (the function $\exp$) $\cdot$ (the function arccot) is differentiable on Z , and
 - (ii) for every x such that $x \in Z$ holds ((the function $\exp$) $\cdot$ (the function arccot))' $\big|_Z(x) = -\frac{\exp \text{arccot } x}{1+x^2}$.
- (121) Suppose $Z \subseteq \text{dom}((\text{the function } \arctan) - \text{id}_Z)$ and $Z \subseteq]-1, 1[$. Then
- (i) (the function $\arctan$) $- \text{id}_Z$ is differentiable on Z , and
 - (ii) for every x such that $x \in Z$ holds ((the function $\arctan$) $- \text{id}_Z$)' $\big|_Z(x) = -\frac{x^2}{1+x^2}$.
- (122) Suppose $Z \subseteq \text{dom}(-\text{the function } \text{arccot} - \text{id}_Z)$ and $Z \subseteq]-1, 1[$. Then
- (i) $-\text{the function } \text{arccot} - \text{id}_Z$ is differentiable on Z , and
 - (ii) for every x such that $x \in Z$ holds $(-\text{the function } \text{arccot} - \text{id}_Z)$ ' $\big|_Z(x) = -\frac{x^2}{1+x^2}$.
- (123) Suppose $Z \subseteq]-1, 1[$. Then
- (i) (the function $\exp$) (the function $\arctan$) is differentiable on Z , and
 - (ii) for every x such that $x \in Z$ holds ((the function $\exp$) (the function $\arctan$))' $\big|_Z(x) = \exp x \cdot \arctan x + \frac{\exp x}{1+x^2}$.
- (124) Suppose $Z \subseteq]-1, 1[$. Then
- (i) (the function $\exp$) (the function arccot) is differentiable on Z , and
 - (ii) for every x such that $x \in Z$ holds ((the function $\exp$) (the function arccot))' $\big|_Z(x) = \exp x \cdot \text{arccot } x - \frac{\exp x}{1+x^2}$.
- (125) Suppose $Z \subseteq \text{dom}(\frac{1}{r}((\text{the function } \arctan) \cdot f) - \text{id}_Z)$ and for every x such that $x \in Z$ holds $f(x) = r \cdot x$ and $r \neq 0$ and $-1 < f(x) < 1$. Then
- (i) $\frac{1}{r}((\text{the function } \arctan) \cdot f) - \text{id}_Z$ is differentiable on Z , and

- (ii) for every x such that $x \in Z$ holds $(\frac{1}{r}((\text{the function arctan}) \cdot f) - \text{id}_Z)'|_Z(x) = -\frac{(r \cdot x)^2}{1+(r \cdot x)^2}$.
- (126) Suppose $Z \subseteq \text{dom}((-\frac{1}{r})((\text{the function arccot}) \cdot f) - \text{id}_Z)$ and for every x such that $x \in Z$ holds $f(x) = r \cdot x$ and $r \neq 0$ and $-1 < f(x) < 1$. Then
- (i) $(-\frac{1}{r})((\text{the function arccot}) \cdot f) - \text{id}_Z$ is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds $((-\frac{1}{r})((\text{the function arccot}) \cdot f) - \text{id}_Z)'|_Z(x) = -\frac{(r \cdot x)^2}{1+(r \cdot x)^2}$.
- (127) Suppose $Z \subseteq \text{dom}((\text{the function ln}) (\text{the function arctan}))$ and $Z \subseteq]-1, 1[$. Then
- (i) $(\text{the function ln}) (\text{the function arctan})$ is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds $((\text{the function ln}) (\text{the function arctan}))'|_Z(x) = \frac{\arctan x}{x} + \frac{(\text{the function ln})(x)}{1+x^2}$.
- (128) Suppose $Z \subseteq \text{dom}((\text{the function ln}) (\text{the function arccot}))$ and $Z \subseteq]-1, 1[$. Then
- (i) $(\text{the function ln}) (\text{the function arccot})$ is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds $((\text{the function ln}) (\text{the function arccot}))'|_Z(x) = \frac{\text{arccot } x}{x} - \frac{(\text{the function ln})(x)}{1+x^2}$.
- (129) Suppose $Z \subseteq \text{dom}(\frac{1}{f} \text{ the function arctan})$ and $Z \subseteq]-1, 1[$ and for every x such that $x \in Z$ holds $f(x) = x$. Then
- (i) $\frac{1}{f}$ the function arctan is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds $(\frac{1}{f} \text{ the function arctan})'|_Z(x) = -\frac{\arctan x}{x^2} + \frac{1}{x \cdot (1+x^2)}$.
- (130) Suppose $Z \subseteq \text{dom}(\frac{1}{f} \text{ the function arccot})$ and $Z \subseteq]-1, 1[$ and for every x such that $x \in Z$ holds $f(x) = x$. Then
- (i) $\frac{1}{f}$ the function arccot is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds $(\frac{1}{f} \text{ the function arccot})'|_Z(x) = -\frac{\text{arccot } x}{x^2} - \frac{1}{x \cdot (1+x^2)}$.

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Received March 18, 2008
