

Convex Sets and Convex Combinations on Complex Linear Spaces

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Summary. In this article, convex sets, convex combinations and convex hulls on complex linear spaces are introduced.

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The articles [19], [18], [9], [23], [24], [6], [25], [7], [20], [3], [22], [17], [2], [11], [8], [1], [5], [10], [14], [15], [4], [16], [21], [12], and [13] provide the terminology and notation for this paper.

1. COMPLEX LINEAR COMBINATIONS

Let V be a non empty zero structure. An element of $\mathbb{C}^{\text{the carrier of } V}$ is said to be a $\mathbb{C}$ -linear combination of V if:

(Def. 1) There exists a finite subset T of V such that for every element v of V such that $v \notin T$ holds $\text{it}(v) = 0$.

Let V be a non empty additive loop structure and let L be an element of $\mathbb{C}^{\text{the carrier of } V}$. The support of L yielding a subset of V is defined by:

(Def. 2) The support of $L = \{v \in V: L(v) \neq 0_{\mathbb{C}}\}$.

Let V be a non empty additive loop structure and let L be a $\mathbb{C}$ -linear combination of V . One can check that the support of L is finite.

The following proposition is true

- (1) Let V be a non empty additive loop structure, L be a $\mathbb{C}$ -linear combination of V , and v be an element of V . Then $L(v) = 0_{\mathbb{C}}$ if and only if $v \notin$ the support of L .

Let V be a non empty additive loop structure. The functor $\text{ZeroCLC } V$ yields a $\mathbb{C}$ -linear combination of V and is defined by:

- (Def. 3) The support of $\text{ZeroCLC } V = \emptyset$.

Let V be a non empty additive loop structure. Note that the support of $\text{ZeroCLC } V$ is empty.

We now state the proposition

- (2) For every non empty additive loop structure V and for every element v of V holds $(\text{ZeroCLC } V)(v) = 0_{\mathbb{C}}$.

Let V be a non empty additive loop structure and let A be a subset of V . A $\mathbb{C}$ -linear combination of V is said to be a $\mathbb{C}$ -linear combination of A if:

- (Def. 4) The support of it $\subseteq A$.

Next we state three propositions:

- (3) Let V be a non empty additive loop structure, A, B be subsets of V , and l be a $\mathbb{C}$ -linear combination of A . If $A \subseteq B$, then l is a $\mathbb{C}$ -linear combination of B .
- (4) Let V be a non empty additive loop structure and A be a subset of V . Then $\text{ZeroCLC } V$ is a $\mathbb{C}$ -linear combination of A .
- (5) Let V be a non empty additive loop structure and l be a $\mathbb{C}$ -linear combination of $\emptyset_{\text{the carrier of } V}$. Then $l = \text{ZeroCLC } V$.

In the sequel i is a natural number.

Let V be a non empty CLS structure, let F be a finite sequence of elements of the carrier of V , and let f be a function from the carrier of V into $\mathbb{C}$. The functor $f F$ yields a finite sequence of elements of the carrier of V and is defined as follows:

- (Def. 5) $\text{len}(f F) = \text{len } F$ and for every i such that $i \in \text{dom}(f F)$ holds $(f F)(i) = f(F_i) \cdot F_i$.

For simplicity, we follow the rules: V denotes a non empty CLS structure, v, v_1, v_2, v_3 denote vectors of V , A denotes a subset of V , l denotes a $\mathbb{C}$ -linear combination of A , x denotes a set, a, b denote complex numbers, F denotes a finite sequence of elements of the carrier of V , and f denotes a function from the carrier of V into $\mathbb{C}$.

The following propositions are true:

- (6) If $x \in \text{dom } F$ and $v = F(x)$, then $(f F)(x) = f(v) \cdot v$.
- (7) $f \varepsilon_{(\text{the carrier of } V)} = \varepsilon_{(\text{the carrier of } V)}$.
- (8) $f \langle v \rangle = \langle f(v) \cdot v \rangle$.
- (9) $f \langle v_1, v_2 \rangle = \langle f(v_1) \cdot v_1, f(v_2) \cdot v_2 \rangle$.

$$(10) \quad f \langle v_1, v_2, v_3 \rangle = \langle f(v_1) \cdot v_1, f(v_2) \cdot v_2, f(v_3) \cdot v_3 \rangle.$$

In the sequel L, L_1, L_2, L_3 are $\mathbb{C}$ -linear combinations of V .

Let V be an Abelian add-associative right zeroed right complementable non empty CLS structure and let L be a $\mathbb{C}$ -linear combination of V . The functor $\sum L$ yields an element of V and is defined by the condition (Def. 6).

(Def. 6) There exists a finite sequence F of elements of the carrier of V such that F is one-to-one and $\text{rng } F = \text{the support of } L$ and $\sum L = \sum L F$.

One can prove the following propositions:

(11) For every Abelian add-associative right zeroed right complementable non empty CLS structure V holds $\sum \text{ZeroCLC } V = 0_V$.

(12) Let V be a complex linear space and A be a subset of V . Suppose $A \neq \emptyset$. Then A is linearly closed if and only if for every $\mathbb{C}$ -linear combination l of A holds $\sum l \in A$.

(13) Let V be an Abelian add-associative right zeroed right complementable non empty CLS structure and l be a $\mathbb{C}$ -linear combination of $\emptyset_{\text{the carrier of } V}$. Then $\sum l = 0_V$.

(14) Let V be a complex linear space, v be a vector of V , and l be a $\mathbb{C}$ -linear combination of $\{v\}$. Then $\sum l = l(v) \cdot v$.

(15) Let V be a complex linear space and v_1, v_2 be vectors of V . Suppose $v_1 \neq v_2$. Let l be a $\mathbb{C}$ -linear combination of $\{v_1, v_2\}$. Then $\sum l = l(v_1) \cdot v_1 + l(v_2) \cdot v_2$.

(16) Let V be an Abelian add-associative right zeroed right complementable non empty CLS structure and L be a $\mathbb{C}$ -linear combination of V . If the support of $L = \emptyset$, then $\sum L = 0_V$.

(17) Let V be a complex linear space, L be a $\mathbb{C}$ -linear combination of V , and v be a vector of V . If the support of $L = \{v\}$, then $\sum L = L(v) \cdot v$.

(18) Let V be a complex linear space, L be a $\mathbb{C}$ -linear combination of V , and v_1, v_2 be vectors of V . If the support of $L = \{v_1, v_2\}$ and $v_1 \neq v_2$, then $\sum L = L(v_1) \cdot v_1 + L(v_2) \cdot v_2$.

Let V be a non empty additive loop structure and let L_1, L_2 be $\mathbb{C}$ -linear combinations of V . Let us observe that $L_1 = L_2$ if and only if:

(Def. 7) For every element v of V holds $L_1(v) = L_2(v)$.

Let V be a non empty additive loop structure and let L_1, L_2 be $\mathbb{C}$ -linear combinations of V . Then $L_1 + L_2$ is a $\mathbb{C}$ -linear combination of V and it can be characterized by the condition:

(Def. 8) For every element v of V holds $(L_1 + L_2)(v) = L_1(v) + L_2(v)$.

One can prove the following propositions:

(19) The support of $L_1 + L_2 \subseteq (\text{the support of } L_1) \cup (\text{the support of } L_2)$.

- (20) Suppose L_1 is a $\mathbb{C}$ -linear combination of A and L_2 is a $\mathbb{C}$ -linear combination of A . Then $L_1 + L_2$ is a $\mathbb{C}$ -linear combination of A .

Let us consider V , A and let L_1, L_2 be $\mathbb{C}$ -linear combinations of A . Then $L_1 + L_2$ is a $\mathbb{C}$ -linear combination of A .

The following three propositions are true:

- (21) For every non empty additive loop structure V and for all $\mathbb{C}$ -linear combinations L_1, L_2 of V holds $L_1 + L_2 = L_2 + L_1$.
 (22) $L_1 + (L_2 + L_3) = (L_1 + L_2) + L_3$.
 (23) $L + \text{ZeroCLC } V = L$.

Let us consider V , a and let us consider L . The functor $a \cdot L$ yielding a $\mathbb{C}$ -linear combination of V is defined as follows:

- (Def. 9) For every v holds $(a \cdot L)(v) = a \cdot L(v)$.

One can prove the following propositions:

- (24) If $a \neq 0_{\mathbb{C}}$, then the support of $a \cdot L =$ the support of L .
 (25) $0_{\mathbb{C}} \cdot L = \text{ZeroCLC } V$.
 (26) If L is a $\mathbb{C}$ -linear combination of A , then $a \cdot L$ is a $\mathbb{C}$ -linear combination of A .
 (27) $(a + b) \cdot L = a \cdot L + b \cdot L$.
 (28) $a \cdot (L_1 + L_2) = a \cdot L_1 + a \cdot L_2$.
 (29) $a \cdot (b \cdot L) = (a \cdot b) \cdot L$.
 (30) $1_{\mathbb{C}} \cdot L = L$.

Let us consider V , L . The functor $-L$ yielding a $\mathbb{C}$ -linear combination of V is defined as follows:

- (Def. 10) $-L = (-1_{\mathbb{C}}) \cdot L$.

We now state three propositions:

- (31) $(-L)(v) = -L(v)$.
 (32) If $L_1 + L_2 = \text{ZeroCLC } V$, then $L_2 = -L_1$.
 (33) $--L = L$.

Let us consider V and let us consider L_1, L_2 . The functor $L_1 - L_2$ yields a $\mathbb{C}$ -linear combination of V and is defined by:

- (Def. 11) $L_1 - L_2 = L_1 + -L_2$.

One can prove the following propositions:

- (34) $(L_1 - L_2)(v) = L_1(v) - L_2(v)$.
 (35) The support of $L_1 - L_2 \subseteq (\text{the support of } L_1) \cup (\text{the support of } L_2)$.
 (36) Suppose L_1 is a $\mathbb{C}$ -linear combination of A and L_2 is a $\mathbb{C}$ -linear combination of A . Then $L_1 - L_2$ is a $\mathbb{C}$ -linear combination of A .
 (37) $L - L = \text{ZeroCLC } V$.

Let us consider V . The functor $\mathbb{C}\text{-LinComb } V$ yields a set and is defined as follows:

(Def. 12) $x \in \mathbb{C}\text{-LinComb } V$ iff x is a $\mathbb{C}$ -linear combination of V .

Let us consider V . One can verify that $\mathbb{C}\text{-LinComb } V$ is non empty.

In the sequel e, e_1, e_2 denote elements of $\mathbb{C}\text{-LinComb } V$.

Let us consider V and let us consider e . The functor ${}^@e$ yields a $\mathbb{C}$ -linear combination of V and is defined as follows:

(Def. 13) ${}^@e = e$.

Let us consider V and let us consider L . The functor ${}^@L$ yielding an element of $\mathbb{C}\text{-LinComb } V$ is defined by:

(Def. 14) ${}^@L = L$.

Let us consider V . The functor $\mathbb{C}\text{-LCAdd } V$ yields a binary operation on $\mathbb{C}\text{-LinComb } V$ and is defined by:

(Def. 15) For all e_1, e_2 holds $(\mathbb{C}\text{-LCAdd } V)(e_1, e_2) = ({}^@e_1) + {}^@e_2$.

Let us consider V . The functor $\mathbb{C}\text{-LCMult } V$ yields a function from $\mathbb{C} \times \mathbb{C}\text{-LinComb } V$ into $\mathbb{C}\text{-LinComb } V$ and is defined as follows:

(Def. 16) For all a, e holds $(\mathbb{C}\text{-LCMult } V)(\langle a, e \rangle) = a \cdot ({}^@e)$.

Let us consider V . The functor $\text{LC-CLSpace } V$ yielding a complex linear space is defined by:

(Def. 17) $\text{LC-CLSpace } V = \langle \mathbb{C}\text{-LinComb } V, {}^@ZeroCLC V, \mathbb{C}\text{-LCAdd } V, \mathbb{C}\text{-LCMult } V \rangle$.

Let us consider V . Note that $\text{LC-CLSpace } V$ is strict and non empty.

We now state four propositions:

$$(38) \quad L_1 {}^{\text{LC-CLSpace } V} + L_2 {}^{\text{LC-CLSpace } V} = L_1 + L_2.$$

$$(39) \quad a \cdot L {}^{\text{LC-CLSpace } V} = a \cdot L.$$

$$(40) \quad -L {}^{\text{LC-CLSpace } V} = -L.$$

$$(41) \quad L_1 {}^{\text{LC-CLSpace } V} - L_2 {}^{\text{LC-CLSpace } V} = L_1 - L_2.$$

Let us consider V and let us consider A . The functor $\text{LC-CLSpace } A$ yielding a strict subspace of $\text{LC-CLSpace } V$ is defined as follows:

(Def. 18) The carrier of $\text{LC-CLSpace } A = \{l\}$.

2. PRELIMINARIES FOR COMPLEX CONVEX SETS

Let V be a complex linear space and let W be a subspace of V . The functor $\text{Up}(W)$ yields a subset of V and is defined by:

(Def. 19) $\text{Up}(W) =$ the carrier of W .

Let V be a complex linear space and let W be a subspace of V . One can check that $\text{Up}(W)$ is non empty.

Let V be a non empty CLS structure and let S be a subset of V . We say that S is affine if and only if the condition (Def. 20) is satisfied.

(Def. 20) Let x, y be vectors of V and z be a complex number. If there exists a real number a such that $a = z$ and $x, y \in S$, then $(1_{\mathbb{C}} - z) \cdot x + z \cdot y \in S$.

Let V be a complex linear space. The functor Ω_V yields a strict subspace of V and is defined as follows:

(Def. 21) $\Omega_V =$ the CLS structure of V .

Let V be a non empty CLS structure. Observe that Ω_V is affine and $\emptyset_V$ is affine.

Let V be a non empty CLS structure. One can check that there exists a subset of V which is non empty and affine and there exists a subset of V which is empty and affine.

We now state three propositions:

(42) For every real number a and for every complex number z holds $\Re(a \cdot z) = a \cdot \Re(z)$.

(43) For every real number a and for every complex number z holds $\Im(a \cdot z) = a \cdot \Im(z)$.

(44) For every real number a and for every complex number z such that $0 \leq a \leq 1$ holds $|a \cdot z| = a \cdot |z|$ and $|(1_{\mathbb{C}} - a) \cdot z| = (1_{\mathbb{C}} - a) \cdot |z|$.

3. COMPLEX CONVEX SETS

Let V be a non empty CLS structure, let M be a subset of V , and let r be an element of $\mathbb{C}$. The functor $r \cdot M$ yielding a subset of V is defined by:

(Def. 22) $r \cdot M = \{r \cdot v; v \text{ ranges over elements of } V: v \in M\}$.

Let V be a non empty CLS structure and let M be a subset of V . We say that M is convex if and only if the condition (Def. 23) is satisfied.

(Def. 23) Let u, v be vectors of V and z be a complex number. Suppose there exists a real number r such that $z = r$ and $0 < r < 1$ and $u, v \in M$. Then $z \cdot u + (1_{\mathbb{C}} - z) \cdot v \in M$.

One can prove the following propositions:

(45) Let V be a complex linear space-like non empty CLS structure, M be a subset of V , and z be a complex number. If M is convex, then $z \cdot M$ is convex.

(46) Let V be an Abelian add-associative complex linear space-like non empty CLS structure and M, N be subsets of V . If M is convex and N is convex, then $M + N$ is convex.

(47) Let V be a complex linear space and M, N be subsets of V . If M is convex and N is convex, then $M - N$ is convex.

- (48) Let V be a non empty CLS structure and M be a subset of V . Then M is convex if and only if for every complex number z such that there exists a real number r such that $z = r$ and $0 < r < 1$ holds $z \cdot M + (1_{\mathbb{C}} - z) \cdot M \subseteq M$.
- (49) Let V be an Abelian non empty CLS structure and M be a subset of V . Suppose M is convex. Let z be a complex number. If there exists a real number r such that $z = r$ and $0 < r < 1$, then $(1_{\mathbb{C}} - z) \cdot M + z \cdot M \subseteq M$.
- (50) Let V be an Abelian add-associative complex linear space-like non empty CLS structure and M, N be subsets of V . Suppose M is convex and N is convex. Let z be a complex number. If there exists a real number r such that $z = r$, then $z \cdot M + (1_{\mathbb{C}} - z) \cdot N$ is convex.
- (51) For every complex linear space-like non empty CLS structure V and for every subset M of V holds $1_{\mathbb{C}} \cdot M = M$.
- (52) For every complex linear space V and for every non empty subset M of V holds $0_{\mathbb{C}} \cdot M = \{0_V\}$.
- (53) For every add-associative non empty additive loop structure V and for all subsets M_1, M_2, M_3 of V holds $(M_1 + M_2) + M_3 = M_1 + (M_2 + M_3)$.
- (54) Let V be a complex linear space-like non empty CLS structure, M be a subset of V , and z_1, z_2 be complex numbers. Then $z_1 \cdot (z_2 \cdot M) = (z_1 \cdot z_2) \cdot M$.
- (55) Let V be a complex linear space-like non empty CLS structure, M_1, M_2 be subsets of V , and z be a complex number. Then $z \cdot (M_1 + M_2) = z \cdot M_1 + z \cdot M_2$.
- (56) Let V be a complex linear space, M be a subset of V , and v be a vector of V . Then M is convex if and only if $v + M$ is convex.
- (57) For every complex linear space V holds $\text{Up}(\mathbf{0}_V)$ is convex.
- (58) For every complex linear space V holds $\text{Up}(\Omega_V)$ is convex.
- (59) For every non empty CLS structure V and for every subset M of V such that $M = \emptyset$ holds M is convex.
- (60) Let V be an Abelian add-associative complex linear space-like non empty CLS structure, M_1, M_2 be subsets of V , and z_1, z_2 be complex numbers. If M_1 is convex and M_2 is convex, then $z_1 \cdot M_1 + z_2 \cdot M_2$ is convex.
- (61) Let V be a complex linear space-like non empty CLS structure, M be a subset of V , and z_1, z_2 be complex numbers. Then $(z_1 + z_2) \cdot M \subseteq z_1 \cdot M + z_2 \cdot M$.
- (62) Let V be a non empty CLS structure, M, N be subsets of V , and z be a complex number. If $M \subseteq N$, then $z \cdot M \subseteq z \cdot N$.
- (63) For every non empty CLS structure V and for every empty subset M of V and for every complex number z holds $z \cdot M = \emptyset$.
- (64) Let V be a non empty additive loop structure, M be an empty subset of V , and N be a subset of V . Then $M + N = \emptyset$.

- (65) For every right zeroed non empty additive loop structure V and for every subset M of V holds $M + \{0_V\} = M$.
- (66) Let V be a complex linear space, M be a subset of V , and z_1, z_2 be complex numbers. Suppose there exist real numbers r_1, r_2 such that $z_1 = r_1$ and $z_2 = r_2$ and $r_1 \geq 0$ and $r_2 \geq 0$ and M is convex. Then $z_1 \cdot M + z_2 \cdot M = (z_1 + z_2) \cdot M$.
- (67) Let V be an Abelian add-associative complex linear space-like non empty CLS structure, M_1, M_2, M_3 be subsets of V , and z_1, z_2, z_3 be complex numbers. If M_1 is convex and M_2 is convex and M_3 is convex, then $z_1 \cdot M_1 + z_2 \cdot M_2 + z_3 \cdot M_3$ is convex.
- (68) Let V be a non empty CLS structure and F be a family of subsets of V . Suppose that for every subset M of V such that $M \in F$ holds M is convex. Then $\bigcap F$ is convex.
- (69) For every non empty CLS structure V and for every subset M of V such that M is affine holds M is convex.

Let V be a non empty CLS structure. One can check that there exists a subset of V which is non empty and convex.

Let V be a non empty CLS structure. Observe that there exists a subset of V which is empty and convex.

One can prove the following propositions:

- (70) Let V be a complex unitary space-like non empty complex unitary space structure, M be a subset of V , v be a vector of V , and r be a real number. If $M = \{u; u \text{ ranges over vectors of } V: \Re((u|v)) \geq r\}$, then M is convex.
- (71) Let V be a complex unitary space-like non empty complex unitary space structure, M be a subset of V , v be a vector of V , and r be a real number. If $M = \{u; u \text{ ranges over vectors of } V: \Re((u|v)) > r\}$, then M is convex.
- (72) Let V be a complex unitary space-like non empty complex unitary space structure, M be a subset of V , v be a vector of V , and r be a real number. If $M = \{u; u \text{ ranges over vectors of } V: \Re((u|v)) \leq r\}$, then M is convex.
- (73) Let V be a complex unitary space-like non empty complex unitary space structure, M be a subset of V , v be a vector of V , and r be a real number. If $M = \{u; u \text{ ranges over vectors of } V: \Re((u|v)) < r\}$, then M is convex.
- (74) Let V be a complex unitary space-like non empty complex unitary space structure, M be a subset of V , v be a vector of V , and r be a real number. If $M = \{u; u \text{ ranges over vectors of } V: \Im((u|v)) \geq r\}$, then M is convex.
- (75) Let V be a complex unitary space-like non empty complex unitary space structure, M be a subset of V , v be a vector of V , and r be a real number. If $M = \{u; u \text{ ranges over vectors of } V: \Im((u|v)) > r\}$, then M is convex.
- (76) Let V be a complex unitary space-like non empty complex unitary space structure, M be a subset of V , v be a vector of V , and r be a real number.

- If $M = \{u; u \text{ ranges over vectors of } V: \Im((u|v)) \leq r\}$, then M is convex.
- (77) Let V be a complex unitary space-like non empty complex unitary space structure, M be a subset of V , v be a vector of V , and r be a real number. If $M = \{u; u \text{ ranges over vectors of } V: \Im((u|v)) < r\}$, then M is convex.
- (78) Let V be a complex unitary space-like non empty complex unitary space structure, M be a subset of V , v be a vector of V , and r be a real number. If $M = \{u; u \text{ ranges over vectors of } V: |(u|v)| \leq r\}$, then M is convex.
- (79) Let V be a complex unitary space-like non empty complex unitary space structure, M be a subset of V , v be a vector of V , and r be a real number. If $M = \{u; u \text{ ranges over vectors of } V: |(u|v)| < r\}$, then M is convex.

4. COMPLEX CONVEX COMBINATIONS

Let V be a complex linear space and let L be a $\mathbb{C}$ -linear combination of V . We say that L is convex if and only if the condition (Def. 24) is satisfied.

- (Def. 24) There exists a finite sequence F of elements of the carrier of V such that
- (i) F is one-to-one,
 - (ii) $\text{rng } F = \text{the support of } L$, and
 - (iii) there exists a finite sequence f of elements of $\mathbb{R}$ such that $\text{len } f = \text{len } F$ and $\sum f = 1$ and for every natural number n such that $n \in \text{dom } f$ holds $f(n) = L(F(n))$ and $f(n) \geq 0$.

We now state several propositions:

- (80) Let V be a complex linear space and L be a $\mathbb{C}$ -linear combination of V . If L is convex, then the support of $L \neq \emptyset$.
- (81) Let V be a complex linear space, L be a $\mathbb{C}$ -linear combination of V , and v be a vector of V . Suppose L is convex and there exists a real number r such that $r = L(v)$ and $r \leq 0$. Then $v \notin \text{the support of } L$.
- (82) For every complex linear space V and for every $\mathbb{C}$ -linear combination L of V such that L is convex holds $L \neq \text{ZeroCLC } V$.
- (83) Let V be a complex linear space, v be a vector of V , and L be a $\mathbb{C}$ -linear combination of V . Suppose L is convex and the support of $L = \{v\}$. Then there exists a real number r such that $r = L(v)$ and $r = 1$ and $\sum L = L(v) \cdot v$.
- (84) Let V be a complex linear space, v_1, v_2 be vectors of V , and L be a $\mathbb{C}$ -linear combination of V . Suppose L is convex and the support of $L = \{v_1, v_2\}$ and $v_1 \neq v_2$. Then there exist real numbers r_1, r_2 such that $r_1 = L(v_1)$ and $r_2 = L(v_2)$ and $r_1 + r_2 = 1$ and $r_1 \geq 0$ and $r_2 \geq 0$ and $\sum L = L(v_1) \cdot v_1 + L(v_2) \cdot v_2$.

- (85) Let V be a complex linear space, v_1, v_2, v_3 be vectors of V , and L be a $\mathbb{C}$ -linear combination of V . Suppose L is convex and the support of $L = \{v_1, v_2, v_3\}$ and $v_1 \neq v_2 \neq v_3 \neq v_1$. Then
- (i) there exist real numbers r_1, r_2, r_3 such that $r_1 = L(v_1)$ and $r_2 = L(v_2)$ and $r_3 = L(v_3)$ and $r_1 + r_2 + r_3 = 1$ and $r_1 \geq 0$ and $r_2 \geq 0$ and $r_3 \geq 0$, and
 - (ii) $\sum L = L(v_1) \cdot v_1 + L(v_2) \cdot v_2 + L(v_3) \cdot v_3$.
- (86) Let V be a complex linear space, v be a vector of V , and L be a $\mathbb{C}$ -linear combination of $\{v\}$. Suppose L is convex. Then there exists a real number r such that $r = L(v)$ and $r = 1$ and $\sum L = L(v) \cdot v$.
- (87) Let V be a complex linear space, v_1, v_2 be vectors of V , and L be a $\mathbb{C}$ -linear combination of $\{v_1, v_2\}$. Suppose $v_1 \neq v_2$ and L is convex. Then there exist real numbers r_1, r_2 such that $r_1 = L(v_1)$ and $r_2 = L(v_2)$ and $r_1 \geq 0$ and $r_2 \geq 0$ and $\sum L = L(v_1) \cdot v_1 + L(v_2) \cdot v_2$.
- (88) Let V be a complex linear space, v_1, v_2, v_3 be vectors of V , and L be a $\mathbb{C}$ -linear combination of $\{v_1, v_2, v_3\}$. Suppose $v_1 \neq v_2 \neq v_3 \neq v_1$ and L is convex. Then
- (i) there exist real numbers r_1, r_2, r_3 such that $r_1 = L(v_1)$ and $r_2 = L(v_2)$ and $r_3 = L(v_3)$ and $r_1 + r_2 + r_3 = 1$ and $r_1 \geq 0$ and $r_2 \geq 0$ and $r_3 \geq 0$, and
 - (ii) $\sum L = L(v_1) \cdot v_1 + L(v_2) \cdot v_2 + L(v_3) \cdot v_3$.

5. COMPLEX CONVEX HULL

Let V be a non empty CLS structure and let M be a subset of V . The functor Convex-Family M yielding a family of subsets of V is defined by:

- (Def. 25) For every subset N of V holds $N \in \text{Convex-Family } M$ iff N is convex and $M \subseteq N$.

Let V be a non empty CLS structure and let M be a subset of V . The functor $\text{conv } M$ yielding a convex subset of V is defined as follows:

- (Def. 26) $\text{conv } M = \bigcap \text{Convex-Family } M$.

The following proposition is true

- (89) Let V be a non empty CLS structure, M be a subset of V , and N be a convex subset of V . If $M \subseteq N$, then $\text{conv } M \subseteq N$.

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