

Banach Algebra of Bounded Functionals

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Summary. In this article, we describe some basic properties of the Banach algebra which is constructed from all bounded functionals.

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The notation and terminology used here are introduced in the following papers: [7], [24], [4], [2], [5], [3], [21], [16], [23], [22], [13], [15], [6], [1], [20], [25], [8], [12], [11], [10], [9], [14], [17], [19], and [18].

1. SOME PROPERTIES OF RINGS

Let V be a non empty additive loop structure and let V_1 be a subset of V . We say that V_1 has inverse if and only if:

(Def. 1) For every element v of V such that $v \in V_1$ holds $-v \in V_1$.

Let V be a non empty additive loop structure and let V_1 be a subset of V . We say that V_1 is additively-closed if and only if:

(Def. 2) V_1 is add closed and has inverse.

Let V be a non empty additive loop structure. One can verify that Ω_V is add closed and has inverse.

Let V be a non empty double loop structure. One can verify that every subset of V which is additively-closed is also add closed and has inverse and every subset of V which is add closed and has inverse is also additively-closed.

Let V be a non empty additive loop structure. Observe that there exists a subset of V which is add closed and non empty and has inverse.

Let V be a ring. A ring is called a subring of V if it satisfies the conditions (Def. 3).

- (Def. 3)(i) The carrier of it $\subseteq$ the carrier of V ,
(ii) the addition of it = (the addition of V) $\upharpoonright$ (the carrier of it),
(iii) the multiplication of it = (the multiplication of V) $\upharpoonright$ (the carrier of it),
(iv) $1_{it} = 1_V$, and
(v) $0_{it} = 0_V$.

For simplicity, we follow the rules: X is a non empty set, x is an element of X , d_1, d_2 are elements of X , A is a binary operation on X , M is a function from $X \times X$ into X , V is a ring, and V_1 is a subset of V .

We now state the proposition

- (1) Suppose $V_1 = X$ and $A = (\text{the addition of } V) \upharpoonright (V_1)$ and $M = (\text{the multiplication of } V) \upharpoonright (V_1)$ and $d_1 = 1_V$ and $d_2 = 0_V$ and V_1 has inverse. Then $\langle X, A, M, d_1, d_2 \rangle$ is a subring of V .

Let V be a ring. One can check that there exists a subring of V which is strict.

Let V be a non empty multiplicative loop with zero structure and let V_1 be a subset of V . We say that V_1 is multiplicatively-closed if and only if:

- (Def. 4) $1_V \in V_1$ and for all elements v, u of V such that $v, u \in V_1$ holds $v \cdot u \in V_1$.

Let V be a non empty additive loop structure and let V_1 be a subset of V . Let us assume that V_1 is add closed and non empty. The functor $\text{Add}(V_1, V)$ yielding a binary operation on V_1 is defined as follows:

- (Def. 5) $\text{Add}(V_1, V) = (\text{the addition of } V) \upharpoonright (V_1)$.

Let V be a non empty multiplicative loop with zero structure and let V_1 be a subset of V . Let us assume that V_1 is multiplicatively-closed and non empty. The functor $\text{mult}(V_1, V)$ yields a binary operation on V_1 and is defined as follows:

- (Def. 6) $\text{mult}(V_1, V) = (\text{the multiplication of } V) \upharpoonright (V_1)$.

Let V be an add-associative right zeroed right complementable non empty double loop structure and let V_1 be a subset of V . Let us assume that V_1 is add closed and non empty and has inverse. The functor $\text{Zero}(V_1, V)$ yields an element of V_1 and is defined by:

- (Def. 7) $\text{Zero}(V_1, V) = 0_V$.

Let V be a non empty multiplicative loop with zero structure and let V_1 be a subset of V . Let us assume that V_1 is multiplicatively-closed and non empty. The functor $\text{One}(V_1, V)$ yields an element of V_1 and is defined as follows:

- (Def. 8) $\text{One}(V_1, V) = 1_V$.

We now state the proposition

- (2) If V_1 is additively-closed, multiplicatively-closed, and non empty, then $\langle V_1, \text{Add}(V_1, V), \text{mult}(V_1, V), \text{One}(V_1, V), \text{Zero}(V_1, V) \rangle$ is a ring.

2. SOME PROPERTIES OF ALGEBRAS

In the sequel V is an algebra, V_1 is a subset of V , M_1 is a function from $\mathbb{R} \times X$ into X , and a is a real number.

Let V be an algebra. An algebra is called a subalgebra of V if it satisfies the conditions (Def. 9).

- (Def. 9)(i) The carrier of it $\subseteq$ the carrier of V ,
(ii) the addition of it = (the addition of V) $\upharpoonright$ (the carrier of it),
(iii) the multiplication of it = (the multiplication of V) $\upharpoonright$ (the carrier of it),
(iv) the external multiplication of it = (the external multiplication of V) $\upharpoonright$ ($\mathbb{R} \times$ the carrier of it),
(v) $1_{\text{it}} = 1_V$, and
(vi) $0_{\text{it}} = 0_V$.

The following proposition is true

- (3) Suppose that $V_1 = X$ and $d_1 = 0_V$ and $d_2 = 1_V$ and $A =$ (the addition of V) $\upharpoonright$ (V_1) and $M =$ (the multiplication of V) $\upharpoonright$ (V_1) and $M_1 =$ (the external multiplication of V) $\upharpoonright$ ($\mathbb{R} \times V_1$) and V_1 has inverse. Then $\langle X, M, A, M_1, d_2, d_1 \rangle$ is a subalgebra of V .

Let V be an algebra. Observe that there exists a subalgebra of V which is strict.

Let V be an algebra and let V_1 be a subset of V . We say that V_1 is additively-linearly-closed if and only if:

- (Def. 10) V_1 is add closed and has inverse and for every real number a and for every element v of V such that $v \in V_1$ holds $a \cdot v \in V_1$.

Let V be an algebra. One can check that every subset of V which is additively-linearly-closed is also additively-closed.

Let V be an algebra and let V_1 be a subset of V . Let us assume that V_1 is additively-linearly-closed and non empty. The functor $\text{Mult}(V_1, V)$ yielding a function from $\mathbb{R} \times V_1$ into V_1 is defined by:

- (Def. 11) $\text{Mult}(V_1, V) =$ (the external multiplication of V) $\upharpoonright$ ($\mathbb{R} \times V_1$).

Let V be a non empty RLS structure. We say that V is scalar-multiplication-cancelable if and only if:

- (Def. 12) For every real number a and for every element v of V such that $a \cdot v = 0_V$ holds $a = 0$ or $v = 0_V$.

One can prove the following propositions:

- (4) Let V be an add-associative right zeroed right complementable algebra-like non empty algebra structure and a be a real number. Then $a \cdot 0_V = 0_V$.

- (5) Let V be an Abelian add-associative right zeroed right complementable algebra-like non empty algebra structure. Suppose V is scalar-multiplication-cancelable. Then V is a real linear space.
- (6) Suppose V_1 is additively-linearly-closed, multiplicatively-closed, and non empty.
Then $\langle V_1, \text{mult}(V_1, V), \text{Add}(V_1, V), \text{Mult}(V_1, V), \text{One}(V_1, V), \text{Zero}(V_1, V) \rangle$ is a subalgebra of V .

Let X be a non empty set. Observe that $\text{RAlgebra } X$ is Abelian, add-associative, right zeroed, right complementable, commutative, associative, right unital, right distributive, and algebra-like.

One can prove the following two propositions:

- (7) $\text{RAlgebra } X$ is a real linear space.
- (8) Let V be an algebra and V_1 be a subalgebra of V . Then
 - (i) for all elements v_1, w_1 of V_1 and for all elements v, w of V such that $v_1 = v$ and $w_1 = w$ holds $v_1 + w_1 = v + w$,
 - (ii) for all elements v_1, w_1 of V_1 and for all elements v, w of V such that $v_1 = v$ and $w_1 = w$ holds $v_1 \cdot w_1 = v \cdot w$,
 - (iii) for every element v_1 of V_1 and for every element v of V and for every real number a such that $v_1 = v$ holds $a \cdot v_1 = a \cdot v$,
 - (iv) $\mathbf{1}_{(V_1)} = \mathbf{1}_V$, and
 - (v) $0_{(V_1)} = 0_V$.

3. BANACH ALGEBRA OF BOUNDED FUNCTIONALS

Let X be a non empty set. The functor $\text{BoundedFunctions } X$ yielding a non empty subset of $\text{RAlgebra } X$ is defined as follows:

(Def. 13) $\text{BoundedFunctions } X = \{f : X \rightarrow \mathbb{R} : f \text{ is bounded on } X\}$.

We now state the proposition

- (9) $\text{BoundedFunctions } X$ is additively-linearly-closed and multiplicatively-closed.

Let us consider X . Note that $\text{BoundedFunctions } X$ is additively-linearly-closed and multiplicatively-closed.

The following proposition is true

- (10) $\langle \text{BoundedFunctions } X, \text{mult}(\text{BoundedFunctions } X, \text{RAlgebra } X), \text{Add}(\text{BoundedFunctions } X, \text{RAlgebra } X), \text{Mult}(\text{BoundedFunctions } X, \text{RAlgebra } X), \text{One}(\text{BoundedFunctions } X, \text{RAlgebra } X), \text{Zero}(\text{BoundedFunctions } X, \text{RAlgebra } X) \rangle$ is a subalgebra of $\text{RAlgebra } X$.

Let X be a non empty set. The $\mathbb{R}$ -algebra of bounded functions on X yields an algebra and is defined by:

- (Def. 14) The $\mathbb{R}$ -algebra of bounded functions on $X = \langle \text{BoundedFunctions } X, \text{mult}(\text{BoundedFunctions } X, \text{RAlgebra } X), \text{Add}(\text{BoundedFunctions } X, \text{RAlgebra } X), \text{Mult}(\text{BoundedFunctions } X, \text{RAlgebra } X), \text{One}(\text{BoundedFunctions } X, \text{RAlgebra } X), \text{Zero}(\text{BoundedFunctions } X, \text{RAlgebra } X) \rangle$.

The following proposition is true

- (11) The $\mathbb{R}$ -algebra of bounded functions on X is a real linear space.

We adopt the following rules: F, G, H are vectors of the $\mathbb{R}$ -algebra of bounded functions on X and f, g, h are functions from X into $\mathbb{R}$.

Next we state several propositions:

- (12) If $f = F$ and $g = G$ and $h = H$, then $H = F + G$ iff for every element x of X holds $h(x) = f(x) + g(x)$.
- (13) If $f = F$ and $g = G$, then $G = a \cdot F$ iff for every element x of X holds $g(x) = a \cdot f(x)$.
- (14) If $f = F$ and $g = G$ and $h = H$, then $H = F \cdot G$ iff for every element x of X holds $h(x) = f(x) \cdot g(x)$.
- (15) $0_{\text{the } \mathbb{R}\text{-algebra of bounded functions on } X} = X \mapsto 0$.
- (16) $1_{\text{the } \mathbb{R}\text{-algebra of bounded functions on } X} = X \mapsto 1$.

Let X be a non empty set and let F be a set. Let us assume that $F \in \text{BoundedFunctions } X$. The functor $\text{modetrans}(F, X)$ yielding a function from X into $\mathbb{R}$ is defined by:

- (Def. 15) $\text{modetrans}(F, X) = F$ and $\text{modetrans}(F, X)$ is bounded on X .

Let X be a non empty set and let f be a function from X into $\mathbb{R}$. The functor $\text{PreNorms}(f)$ yielding a non empty subset of $\mathbb{R}$ is defined as follows:

- (Def. 16) $\text{PreNorms}(f) = \{|f(x)| : x \text{ ranges over elements of } X\}$.

Next we state three propositions:

- (17) If f is bounded on X , then $\text{PreNorms}(f)$ is non empty and upper bounded.
- (18) f is bounded on X iff $\text{PreNorms}(f)$ is upper bounded.
- (19) There exists a function N_1 from $\text{BoundedFunctions } X$ into $\mathbb{R}$ such that for every set F such that $F \in \text{BoundedFunctions } X$ holds $N_1(F) = \sup \text{PreNorms}(\text{modetrans}(F, X))$.

Let X be a non empty set. The functor $\text{BoundedFunctionsNorm } X$ yields a function from $\text{BoundedFunctions } X$ into $\mathbb{R}$ and is defined by:

- (Def. 17) For every set x such that $x \in \text{BoundedFunctions } X$ holds $(\text{BoundedFunctionsNorm } X)(x) = \sup \text{PreNorms}(\text{modetrans}(x, X))$.

We now state two propositions:

- (20) If f is bounded on X , then $\text{modetrans}(f, X) = f$.
- (21) If f is bounded on X , then $(\text{BoundedFunctionsNorm } X)(f) = \sup \text{PreNorms}(f)$.

Let X be a non empty set. The $\mathbb{R}$ -normed algebra of bounded functions on X yielding a normed algebra structure is defined as follows:

- (Def. 18) The $\mathbb{R}$ -normed algebra of bounded functions on $X =$
 $\langle \text{BoundedFunctions } X, \text{mult}(\text{BoundedFunctions } X, \text{RAlgebra } X),$
 $\text{Add}(\text{BoundedFunctions } X, \text{RAlgebra } X), \text{Mult}(\text{BoundedFunctions } X,$
 $\text{RAlgebra } X), \text{One}(\text{BoundedFunctions } X, \text{RAlgebra } X),$
 $\text{Zero}(\text{BoundedFunctions } X, \text{RAlgebra } X), \text{BoundedFunctionsNorm } X \rangle.$

Let X be a non empty set. Note that the $\mathbb{R}$ -normed algebra of bounded functions on X is non empty.

Let X be a non empty set. Observe that the $\mathbb{R}$ -normed algebra of bounded functions on X is unital.

We now state the proposition

- (22) Let W be a normed algebra structure and V be an algebra. If the algebra structure of $W = V$ and $1_V = 1_W$, then W is an algebra.

In the sequel F, G, H denote points of the $\mathbb{R}$ -normed algebra of bounded functions on X .

We now state a number of propositions:

- (23) The $\mathbb{R}$ -normed algebra of bounded functions on X is an algebra.
(24) $(\text{Mult}(\text{BoundedFunctions } X, \text{RAlgebra } X))(1, F) = F.$
(25) The $\mathbb{R}$ -normed algebra of bounded functions on X is a real linear space.
(26) $X \mapsto 0 = 0_{\text{the } \mathbb{R}\text{-normed algebra of bounded functions on } X}.$
(27) If $f = F$ and f is bounded on X , then $|f(x)| \leq \|F\|.$
(28) $0 \leq \|F\|.$
(29) $0 = \|(0_{\text{the } \mathbb{R}\text{-normed algebra of bounded functions on } X})\|.$
(30) If $f = F$ and $g = G$ and $h = H$, then $H = F + G$ iff for every element x of X holds $h(x) = f(x) + g(x).$
(31) If $f = F$ and $g = G$, then $G = a \cdot F$ iff for every element x of X holds $g(x) = a \cdot f(x).$
(32) If $f = F$ and $g = G$ and $h = H$, then $H = F \cdot G$ iff for every element x of X holds $h(x) = f(x) \cdot g(x).$
(33)(i) $\|F\| = 0$ iff $F = 0_{\text{the } \mathbb{R}\text{-normed algebra of bounded functions on } X},$
(ii) $\|a \cdot F\| = |a| \cdot \|F\|,$ and
(iii) $\|F + G\| \leq \|F\| + \|G\|.$
(34) The $\mathbb{R}$ -normed algebra of bounded functions on X is real normed space-like.

Let X be a non empty set.

Note that the $\mathbb{R}$ -normed algebra of bounded functions on X is real normed space-like, real linear space-like, Abelian, add-associative, right zeroed, and right complementable.

We now state three propositions:

- (35) If $f = F$ and $g = G$ and $h = H$, then $H = F - G$ iff for every element x of X holds $h(x) = f(x) - g(x)$.
- (36) Let X be a non empty set and s_1 be a sequence of the $\mathbb{R}$ -normed algebra of bounded functions on X . If s_1 is Cauchy sequence by norm, then s_1 is convergent.
- (37) The $\mathbb{R}$ -normed algebra of bounded functions on X is a real Banach space.
Let X be a non empty set.
Observe that the $\mathbb{R}$ -normed algebra of bounded functions on X is complete.
The following proposition is true
- (38) The $\mathbb{R}$ -normed algebra of bounded functions on X is a Banach algebra.

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