

Helly Property for Subtrees¹

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Summary. We prove, following [5, p. 92], that any family of subtrees of a finite tree satisfies the Helly property.

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The articles [12], [4], [10], [3], [2], [1], [11], [9], [8], [7], and [6] provide the notation and terminology for this paper.

1. GENERAL PRELIMINARIES

One can prove the following proposition

- (1) For every non empty finite sequence p holds $\langle p(1) \rangle \curvearrowright p = p$.

Let p, q be finite sequences. The functor $\text{maxPrefix}(p, q)$ yields a finite sequence and is defined by:

- (Def. 1) $\text{maxPrefix}(p, q) \preceq p$ and $\text{maxPrefix}(p, q) \preceq q$ and for every finite sequence r such that $r \preceq p$ and $r \preceq q$ holds $r \preceq \text{maxPrefix}(p, q)$.

Let us observe that the functor $\text{maxPrefix}(p, q)$ is commutative.

Next we state several propositions:

- (2) For all finite sequences p, q holds $p \preceq q$ iff $\text{maxPrefix}(p, q) = p$.
- (3) For all finite sequences p, q holds $\text{len maxPrefix}(p, q) \leq \text{len } p$.
- (4) For every non empty finite sequence p holds $\langle p(1) \rangle \preceq p$.
- (5) For all non empty finite sequences p, q such that $p(1) = q(1)$ holds $1 \leq \text{len maxPrefix}(p, q)$.

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- (6) For all finite sequences p, q and for every natural number j such that $j \leq \text{len maxPrefix}(p, q)$ holds $(\text{maxPrefix}(p, q))(j) = p(j)$.
- (7) For all finite sequences p, q and for every natural number j such that $j \leq \text{len maxPrefix}(p, q)$ holds $p(j) = q(j)$.
- (8) For all finite sequences p, q holds $p \not\preceq q$ iff $\text{len maxPrefix}(p, q) < \text{len } p$.
- (9) For all finite sequences p, q such that $p \not\preceq q$ and $q \not\preceq p$ holds $p(\text{len maxPrefix}(p, q) + 1) \neq q(\text{len maxPrefix}(p, q) + 1)$.

2. GRAPH PRELIMINARIES

Next we state three propositions:

- (10) For every graph G and for every walk W of G and for all natural numbers m, n holds $\text{len}(W.\text{cut}(m, n)) \leq \text{len } W$.
- (11) Let G be a graph, W be a walk of G , and m, n be natural numbers. If $W.\text{cut}(m, n)$ is non trivial, then W is non trivial.
- (12) Let G be a graph, W be a walk of G , and m, n, i be odd natural numbers. Suppose $m \leq n \leq \text{len } W$ and $i \leq \text{len}(W.\text{cut}(m, n))$. Then there exists an odd natural number j such that $(W.\text{cut}(m, n))(i) = W(j)$ and $j = (m + i) - 1$ and $j \leq \text{len } W$.

Let G be a graph. One can verify that every walk of G is non empty.

The following propositions are true:

- (13) For every graph G and for all walks W_1, W_2 of G such that $W_1 \preceq W_2$ holds $W_1.\text{vertices}() \subseteq W_2.\text{vertices}()$.
- (14) For every graph G and for all walks W_1, W_2 of G such that $W_1 \preceq W_2$ holds $W_1.\text{edges}() \subseteq W_2.\text{edges}()$.
- (15) For every graph G and for all walks W_1, W_2 of G holds $W_1 \preceq W_1.\text{append}(W_2)$.
- (16) For every graph G and for all trails W_1, W_2 of G such that $W_1.\text{last}() = W_2.\text{first}()$ and $W_1.\text{edges}()$ misses $W_2.\text{edges}()$ holds $W_1.\text{append}(W_2)$ is trail-like.
- (17) Let G be a graph and P_1, P_2 be paths of G . Suppose $P_1.\text{last}() = P_2.\text{first}()$ and P_1 is open and P_2 is open and $P_1.\text{edges}()$ misses $P_2.\text{edges}()$ and if $P_1.\text{first}() \in P_2.\text{vertices}()$, then $P_1.\text{first}() = P_2.\text{last}()$ and $P_1.\text{vertices}() \cap P_2.\text{vertices}() \subseteq \{P_1.\text{first}(), P_1.\text{last}()\}$. Then $P_1.\text{append}(P_2)$ is path-like.
- (18) Let G be a graph and P_1, P_2 be paths of G . Suppose $P_1.\text{last}() = P_2.\text{first}()$ and P_1 is open and P_2 is open and $P_1.\text{vertices}() \cap P_2.\text{vertices}() = \{P_1.\text{last}()\}$. Then $P_1.\text{append}(P_2)$ is open and path-like.
- (19) Let G be a graph and P_1, P_2 be paths of G . Suppose $P_1.\text{last}() = P_2.\text{first}()$ and $P_2.\text{last}() = P_1.\text{first}()$ and P_1 is open and P_2 is open and $P_1.\text{edges}()$

- misses $P_2.\text{edges}()$ and $P_1.\text{vertices}() \cap P_2.\text{vertices}() = \{P_1.\text{last}(), P_1.\text{first}()\}$. Then $P_1.\text{append}(P_2)$ is cycle-like.
- (20) Let G be a simple graph, W_1, W_2 be walks of G , and k be an odd natural number. Suppose $k \leq \text{len } W_1$ and $k \leq \text{len } W_2$ and for every odd natural number j such that $j \leq k$ holds $W_1(j) = W_2(j)$. Let j be a natural number. If $1 \leq j \leq k$, then $W_1(j) = W_2(j)$.
 - (21) For every graph G and for all walks W_1, W_2 of G such that $W_1.\text{first}() = W_2.\text{first}()$ holds $\text{len maxPrefix}(W_1, W_2)$ is odd.
 - (22) For every graph G and for all walks W_1, W_2 of G such that $W_1.\text{first}() = W_2.\text{first}()$ and $W_1 \not\subseteq W_2$ holds $\text{len maxPrefix}(W_1, W_2) + 2 \leq \text{len } W_1$.
 - (23) For every non-multi graph G and for all walks W_1, W_2 of G such that $W_1.\text{first}() = W_2.\text{first}()$ and $W_1 \not\subseteq W_2$ and $W_2 \not\subseteq W_1$ holds $W_1(\text{len maxPrefix}(W_1, W_2) + 2) \neq W_2(\text{len maxPrefix}(W_1, W_2) + 2)$.

3. TREES

A tree is a tree-like graph. Let G be a graph. A subtree of G is a tree-like subgraph of G .

Let T be a tree. Observe that every walk of T which is trail-like is also path-like.

One can prove the following proposition

- (24) For every tree T and for every path P of T such that P is non trivial holds P is open.

Let T be a tree. Note that every path of T which is non trivial is also open.

The following propositions are true:

- (25) Let T be a tree, P be a path of T , and i, j be odd natural numbers. If $i < j \leq \text{len } P$, then $P(i) \neq P(j)$.
- (26) Let T be a tree, a, b be vertices of T , and P_1, P_2 be paths of T . If P_1 is walk from a to b and P_2 is walk from a to b , then $P_1 = P_2$.

Let T be a tree and let a, b be vertices of T . The functor $T.\text{pathBetween}(a, b)$ yields a path of T and is defined as follows:

(Def. 2) $T.\text{pathBetween}(a, b)$ is walk from a to b .

One can prove the following propositions:

- (27) For every tree T and for all vertices a, b of T holds $(T.\text{pathBetween}(a, b)).\text{first}() = a$ and $(T.\text{pathBetween}(a, b)).\text{last}() = b$.
- (28) For every tree T and for all vertices a, b of T holds $a, b \in (T.\text{pathBetween}(a, b)).\text{vertices}()$.

Let T be a tree and let a be a vertex of T . Observe that $T.\text{pathBetween}(a, a)$ is closed.

Let T be a tree and let a be a vertex of T .

One can check that $T.\text{pathBetween}(a, a)$ is trivial.

We now state a number of propositions:

- (29) For every tree T and for every vertex a of T holds
 $(T.\text{pathBetween}(a, a)).\text{vertices}() = \{a\}$.
- (30) For every tree T and for all vertices a, b of T holds
 $(T.\text{pathBetween}(a, b)).\text{reverse}() = T.\text{pathBetween}(b, a)$.
- (31) For every tree T and for all vertices a, b of T holds
 $(T.\text{pathBetween}(a, b)).\text{vertices}() = (T.\text{pathBetween}(b, a)).\text{vertices}()$.
- (32) Let T be a tree, a, b be vertices of T , t be a subtree of T , and a', b' be vertices of t . If $a = a'$ and $b = b'$, then $T.\text{pathBetween}(a, b) = t.\text{pathBetween}(a', b')$.
- (33) Let T be a tree, a, b be vertices of T , and t be a subtree of T . Suppose $a \in \text{the vertices of } t$ and $b \in \text{the vertices of } t$. Then $(T.\text{pathBetween}(a, b)).\text{vertices}() \subseteq \text{the vertices of } t$.
- (34) Let T be a tree, P be a path of T , a, b be vertices of T , and i, j be odd natural numbers. If $i \leq j \leq \text{len } P$ and $P(i) = a$ and $P(j) = b$, then $T.\text{pathBetween}(a, b) = P.\text{cut}(i, j)$.
- (35) For every tree T and for all vertices a, b, c of T holds
 $c \in (T.\text{pathBetween}(a, b)).\text{vertices}()$ iff $T.\text{pathBetween}(a, b) = (T.\text{pathBetween}(a, c)).\text{append}((T.\text{pathBetween}(c, b)))$.
- (36) For every tree T and for all vertices a, b, c of T holds
 $c \in (T.\text{pathBetween}(a, b)).\text{vertices}()$ iff $T.\text{pathBetween}(a, c) \preceq T.\text{pathBetween}(a, b)$.
- (37) For every tree T and for all paths P_1, P_2 of T such that $P_1.\text{last}() = P_2.\text{first}()$ and $P_1.\text{vertices}() \cap P_2.\text{vertices}() = \{P_1.\text{last}()\}$ holds $P_1.\text{append}(P_2)$ is path-like.
- (38) For every tree T and for all vertices a, b, c of T holds
 $c \in (T.\text{pathBetween}(a, b)).\text{vertices}()$ iff $(T.\text{pathBetween}(a, c)).\text{vertices}() \cap (T.\text{pathBetween}(c, b)).\text{vertices}() = \{c\}$.
- (39) Let T be a tree, a, b, c, d be vertices of T , and P_1, P_2 be paths of T . Suppose $P_1 = T.\text{pathBetween}(a, b)$ and $P_2 = T.\text{pathBetween}(a, c)$ and $P_1 \not\subseteq P_2$ and $P_2 \not\subseteq P_1$ and $d = P_1(\text{len maxPrefix}(P_1, P_2))$. Then $(T.\text{pathBetween}(d, b)).\text{vertices}() \cap (T.\text{pathBetween}(d, c)).\text{vertices}() = \{d\}$.

Let T be a tree and let a, b, c be vertices of T . The functor $\text{middleVertex}(a, b, c)$ yielding a vertex of T is defined as follows:

- (Def. 3) $(T.\text{pathBetween}(a, b)).\text{vertices}() \cap (T.\text{pathBetween}(b, c)).\text{vertices}() \cap (T.\text{pathBetween}(c, a)).\text{vertices}() = \{\text{middleVertex}(a, b, c)\}$.

We now state a number of propositions:

- (40) For every tree T and for all vertices a, b, c of T holds $\text{middleVertex}(a, b, c) = \text{middleVertex}(a, c, b)$.
- (41) For every tree T and for all vertices a, b, c of T holds $\text{middleVertex}(a, b, c) = \text{middleVertex}(b, a, c)$.
- (42) For every tree T and for all vertices a, b, c of T holds $\text{middleVertex}(a, b, c) = \text{middleVertex}(b, c, a)$.
- (43) For every tree T and for all vertices a, b, c of T holds $\text{middleVertex}(a, b, c) = \text{middleVertex}(c, a, b)$.
- (44) For every tree T and for all vertices a, b, c of T holds $\text{middleVertex}(a, b, c) = \text{middleVertex}(c, b, a)$.
- (45) For every tree T and for all vertices a, b, c of T such that $c \in (T.\text{pathBetween}(a, b)).\text{vertices}()$ holds $\text{middleVertex}(a, b, c) = c$.
- (46) For every tree T and for every vertex a of T holds $\text{middleVertex}(a, a, a) = a$.
- (47) For every tree T and for all vertices a, b of T holds $\text{middleVertex}(a, a, b) = a$.
- (48) For every tree T and for all vertices a, b of T holds $\text{middleVertex}(a, b, a) = a$.
- (49) For every tree T and for all vertices a, b of T holds $\text{middleVertex}(a, b, b) = b$.
- (50) Let T be a tree, P_1, P_2 be paths of T , and a, b, c be vertices of T . If $P_1 = T.\text{pathBetween}(a, b)$ and $P_2 = T.\text{pathBetween}(a, c)$ and $b \notin P_2.\text{vertices}()$ and $c \notin P_1.\text{vertices}()$, then $\text{middleVertex}(a, b, c) = P_1(\text{len maxPrefix}(P_1, P_2))$.
- (51) Let T be a tree, P_1, P_2, P_3, P_4 be paths of T , and a, b, c be vertices of T . Suppose $P_1 = T.\text{pathBetween}(a, b)$ and $P_2 = T.\text{pathBetween}(a, c)$ and $P_3 = T.\text{pathBetween}(b, a)$ and $P_4 = T.\text{pathBetween}(b, c)$ and $b \notin P_2.\text{vertices}()$ and $c \notin P_1.\text{vertices}()$ and $a \notin P_4.\text{vertices}()$. Then $P_1(\text{len maxPrefix}(P_1, P_2)) = P_3(\text{len maxPrefix}(P_3, P_4))$.
- (52) Let T be a tree, a, b, c be vertices of T , and S be a non empty set. Suppose that for every set s such that $s \in S$ holds there exists a subtree t of T such that $s = \text{the vertices of } t$ but $a, b \in s$ or $a, c \in s$ or $b, c \in s$. Then $\bigcap S \neq \emptyset$.

4. THE HELLY PROPERTY

Let F be a set. We say that F has Helly property if and only if:

- (Def. 4) For every non empty set H such that $H \subseteq F$ and for all sets x, y such that $x, y \in H$ holds x meets y holds $\bigcap H \neq \emptyset$.

One can prove the following proposition

- (53) Let T be a tree and X be a finite set such that for every set x such that $x \in X$ there exists a subtree t of T such that $x =$ the vertices of t . Then X has Helly property.

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