

Uniform Boundedness Principle

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Summary. In this article at first, we proved the lemma of the inferior limit and the superior limit. Next, we proved the Baire category theorem (Banach space version) [20], [9], [3], quoted it and proved the uniform boundedness principle. Moreover, the proof of the Banach-Steinhaus theorem is added.

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The articles [17], [18], [15], [12], [19], [1], [21], [5], [8], [7], [16], [10], [6], [13], [4], [2], [14], and [11] provide the terminology and notation for this paper.

1. UNIFORM BOUNDEDNESS PRINCIPLE

The following two propositions are true:

- (1) For every sequence s_1 of real numbers and for every real number r such that s_1 is bounded and $0 \leq r$ holds $\liminf(r s_1) = r \cdot \liminf s_1$.
- (2) For every sequence s_1 of real numbers and for every real number r such that s_1 is bounded and $0 \leq r$ holds $\limsup(r s_1) = r \cdot \limsup s_1$.

Let X be a real Banach space. One can verify that $\text{MetricSpaceNorm } X$ is complete.

Let X be a real Banach space, let x_0 be a point of X , and let r be a real number. The functor $\text{Ball}(x_0, r)$ yielding a subset of X is defined as follows:

(Def. 1) $\text{Ball}(x_0, r) = \{x; x \text{ ranges over points of } X: \|x_0 - x\| < r\}$.

The following propositions are true:

- (3) Let X be a real Banach space and Y be a sequence of subsets of X . Suppose $\bigcup \text{rng } Y = \text{the carrier of } X$ and for every element n of $\mathbb{N}$ holds $Y(n)$ is closed. Then there exists an element n_0 of $\mathbb{N}$ and there exists

a real number r and there exists a point x_0 of X such that $0 < r$ and $\text{Ball}(x_0, r) \subseteq Y(n_0)$.

- (4) Let X, Y be real normed spaces and f be a bounded linear operator from X into Y . Then
 - (i) f is Lipschitzian on the carrier of X and continuous on the carrier of X , and
 - (ii) for every point x of X holds f is continuous in x .
- (5) Let X be a real Banach space, Y be a real normed space, and T be a subset of the real norm space of bounded linear operators from X into Y . Suppose that for every point x of X there exists a real number K such that $0 \leq K$ and for every point f of the real norm space of bounded linear operators from X into Y such that $f \in T$ holds $\|f(x)\| \leq K$. Then there exists a real number L such that
 - (i) $0 \leq L$, and
 - (ii) for every point f of the real norm space of bounded linear operators from X into Y such that $f \in T$ holds $\|f\| \leq L$.

Let X, Y be real normed spaces, let H be a function from $\mathbb{N}$ into the carrier of the real norm space of bounded linear operators from X into Y , and let x be a point of X . The functor $H\#x$ yields a sequence of Y and is defined by:

- (Def. 2) For every element n of $\mathbb{N}$ holds $(H\#x)(n) = H(n)(x)$.

The following proposition is true

- (6) Let X be a real Banach space, Y be a real normed space, v_1 be a sequence of the real norm space of bounded linear operators from X into Y , and t_1 be a function from X into Y . Suppose that for every point x of X holds $v_1\#x$ is convergent and $t_1(x) = \lim(v_1\#x)$. Then
 - (i) t_1 is a bounded linear operator from X into Y ,
 - (ii) for every point x of X holds $\|t_1(x)\| \leq \liminf\|v_1\| \cdot \|x\|$, and
 - (iii) for every point t_2 of the real norm space of bounded linear operators from X into Y such that $t_2 = t_1$ holds $\|t_2\| \leq \liminf\|v_1\|$.

2. BANACH-STEINHAUS THEOREM

We now state two propositions:

- (7) Let X be a real Banach space, X_0 be a subset of $\text{LinearTopSpaceNorm } X$, Y be a real Banach space, and v_1 be a sequence of the real norm space of bounded linear operators from X into Y . Suppose that
 - (i) X_0 is dense,
 - (ii) for every point x of X such that $x \in X_0$ holds $v_1\#x$ is convergent, and
 - (iii) for every point x of X there exists a real number K such that $0 \leq K$ and for every element n of $\mathbb{N}$ holds $\|(v_1\#x)(n)\| \leq K$.
 Let x be a point of X . Then $v_1\#x$ is convergent.

- (8) Let X, Y be real Banach spaces, X_0 be a subset of $\text{LinearTopSpaceNorm } X$, and v_1 be a sequence of the real norm space of bounded linear operators from X into Y . Suppose that
- (i) X_0 is dense,
 - (ii) for every point x of X such that $x \in X_0$ holds $v_1 \# x$ is convergent, and
 - (iii) for every point x of X there exists a real number K such that $0 \leq K$ and for every element n of $\mathbb{N}$ holds $\|(v_1 \# x)(n)\| \leq K$.

Then there exists a point t_1 of the real norm space of bounded linear operators from X into Y such that for every point x of X holds $v_1 \# x$ is convergent and $t_1(x) = \lim(v_1 \# x)$ and $\|t_1(x)\| \leq \liminf \|v_1\| \cdot \|x\|$ and $\|t_1\| \leq \liminf \|v_1\|$.

REFERENCES

- [1] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(1):91–96, 1990.
- [2] Czesław Byliński. Introduction to real linear topological spaces. *Formalized Mathematics*, 13(1):99–107, 2005.
- [3] N. J. Dunford and T. Schwartz. *Linear operators I*. Interscience Publ., 1958.
- [4] Noboru Endou, Yasunari Shidama, and Katsumasa Okamura. Baire's category theorem and some spaces generated from real normed space. *Formalized Mathematics*, 14(4):213–219, 2006.
- [5] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(1):35–40, 1990.
- [6] Stanisława Kanas, Adam Lecko, and Mariusz Startek. Metric spaces. *Formalized Mathematics*, 1(3):607–610, 1990.
- [7] Jarosław Kotowicz. Convergent sequences and the limit of sequences. *Formalized Mathematics*, 1(2):273–275, 1990.
- [8] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(2):269–272, 1990.
- [9] Isao Miyadera. *Functional Analysis*. Riko-Gaku-Sya, 1972.
- [10] Andrzej Nędzusiak. σ -fields and probability. *Formalized Mathematics*, 1(2):401–407, 1990.
- [11] Takaya Nishiyama, Keiji Ohkubo, and Yasunari Shidama. The continuous functions on normed linear spaces. *Formalized Mathematics*, 12(3):269–275, 2004.
- [12] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(1):223–230, 1990.
- [13] Jan Popiolek. Real normed space. *Formalized Mathematics*, 2(1):111–115, 1991.
- [14] Yasunari Shidama. Banach space of bounded linear operators. *Formalized Mathematics*, 12(1):39–48, 2004.
- [15] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(2):329–334, 1990.
- [16] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(2):291–296, 1990.
- [17] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(1):67–71, 1990.
- [18] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(1):73–83, 1990.
- [19] Mirosław Wysocki and Agata Darmochwał. Subsets of topological spaces. *Formalized Mathematics*, 1(1):231–237, 1990.
- [20] Kosaku Yoshida. *Functional Analysis*. Springer, 1980.
- [21] Bo Zhang, Hiroshi Yamazaki, and Yatsuka Nakamura. Inferior limit and superior limit of sequences of real numbers. *Formalized Mathematics*, 13(3):375–381, 2005.

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