

The First Mean Value Theorem for Integrals

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Summary. In this article, we prove the first mean value theorem for integrals [16]. The formalization of various theorems about the properties of the Lebesgue integral is also presented.

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The notation and terminology used in this paper are introduced in the following articles: [20], [2], [17], [6], [1], [4], [21], [22], [11], [3], [9], [8], [10], [18], [19], [5], [13], [12], [14], [15], and [7].

1. LEMMAS FOR EXTENDED REAL VALUED FUNCTIONS

For simplicity, we use the following convention: X is a non empty set, S is a σ -field of subsets of X , M is a σ -measure on S , f, g are partial functions from X to $\overline{\mathbb{R}}$, and E is an element of S .

One can prove the following three propositions:

- (1) If for every element x of X such that $x \in \text{dom } f$ holds $f(x) \leq g(x)$, then $g - f$ is non-negative.
- (2) For every set Y and for every partial function f from X to $\overline{\mathbb{R}}$ and for every real number r holds $(r f)|Y = r (f|Y)$.
- (3) Suppose f is integrable on M and g is integrable on M and $g - f$ is non-negative. Then there exists an element E of S such that $E = \text{dom } f \cap \text{dom } g$ and $\int f|E \, dM \leq \int g|E \, dM$.

2. σ -FINITE SETS

Let us consider X . One can verify that there exists a partial function from X to $\overline{\mathbb{R}}$ which is non-negative.

Let us consider X, f . Then $|f|$ is a non-negative partial function from X to $\overline{\mathbb{R}}$.

Next we state the proposition

- (4) Suppose f is integrable on M . Then there exists a function F from $\mathbb{N}$ into S such that
- (i) for every element n of $\mathbb{N}$ holds $F(n) = \text{dom } f \cap \text{GTE-dom}(|f|, \overline{\mathbb{R}}(\frac{1}{n+1}))$,
 - (ii) $\text{dom } f \setminus \text{EQ-dom}(f, 0_{\overline{\mathbb{R}}}) = \bigcup \text{rng } F$, and
 - (iii) for every element n of $\mathbb{N}$ holds $F(n) \in S$ and $M(F(n)) < +\infty$.

3. THE FIRST MEAN VALUE THEOREM FOR INTEGRALS

Let F be a binary relation. We introduce F is extreal-yielding as a synonym of F is extended real-valued.

Let k be a natural number and let x be an element of $\overline{\mathbb{R}}$. Then $k \mapsto x$ is a finite sequence of elements of $\overline{\mathbb{R}}$.

Let us note that there exists a finite sequence which is extreal-yielding.

The binary operation $\cdot_{\overline{\mathbb{R}}}$ on $\overline{\mathbb{R}}$ is defined by:

- (Def. 2)¹ For all elements x, y of $\overline{\mathbb{R}}$ holds $\cdot_{\overline{\mathbb{R}}}(x, y) = x \cdot y$.

One can check that $\cdot_{\overline{\mathbb{R}}}$ is commutative and associative.

One can prove the following proposition

- (5) $1_{\overline{\mathbb{R}}} = 1$.

One can check that $\cdot_{\overline{\mathbb{R}}}$ is unital.

Let F be an extreal-yielding finite sequence. The functor $\prod F$ yields an element of $\overline{\mathbb{R}}$ and is defined by:

- (Def. 3) There exists a finite sequence f of elements of $\overline{\mathbb{R}}$ such that $f = F$ and $\prod F = \cdot_{\overline{\mathbb{R}}} \otimes f$.

Let x be an element of $\overline{\mathbb{R}}$ and let n be a natural number. Note that $n \mapsto x$ is extreal-yielding.

Let x be an element of $\overline{\mathbb{R}}$ and let k be a natural number. The functor x^k is defined by:

- (Def. 4) $x^k = \prod(k \mapsto x)$.

Let x be an element of $\overline{\mathbb{R}}$ and let k be a natural number. Then x^k is an extended real number.

Let us note that $\varepsilon_{\overline{\mathbb{R}}}$ is extreal-yielding.

¹The definition (Def. 1) has been removed.

Let r be an element of $\overline{\mathbb{R}}$. Note that $\langle r \rangle$ is extreal-yielding.

We now state two propositions:

- (6) $\prod(\varepsilon_{\overline{\mathbb{R}}}) = 1$.
- (7) For every element r of $\overline{\mathbb{R}}$ holds $\prod\langle r \rangle = r$.

Let f, g be extreal-yielding finite sequences. Observe that $f \cap g$ is extreal-yielding.

We now state three propositions:

- (8) For every extreal-yielding finite sequence F and for every element r of $\overline{\mathbb{R}}$ holds $\prod(F \cap \langle r \rangle) = \prod F \cdot r$.
- (9) For every element x of $\overline{\mathbb{R}}$ holds $x^1 = x$.
- (10) For every element x of $\overline{\mathbb{R}}$ and for every natural number k holds $x^{k+1} = x^k \cdot x$.

Let k be a natural number and let us consider X, f . The functor f^k yields a partial function from X to $\overline{\mathbb{R}}$ and is defined by:

- (Def. 5) $\text{dom}(f^k) = \text{dom } f$ and for every element x of X such that $x \in \text{dom}(f^k)$ holds $f^k(x) = f(x)^k$.

Next we state several propositions:

- (11) For every element x of $\overline{\mathbb{R}}$ and for every real number y and for every natural number k such that $x = y$ holds $x^k = y^k$.
- (12) For every element x of $\overline{\mathbb{R}}$ and for every natural number k such that $0 \leq x$ holds $0 \leq x^k$.
- (13) For every natural number k such that $1 \leq k$ holds $+\infty^k = +\infty$.
- (14) Let k be a natural number and given X, S, f, E . If $E \subseteq \text{dom } f$ and f is measurable on E , then $|f|^k$ is measurable on E .
- (15) Suppose $\text{dom } f \cap \text{dom } g = E$ and f is finite and g is finite and f is measurable on E and g is measurable on E . Then $f g$ is measurable on E .
- (16) If $\text{rng } f$ is bounded, then f is finite.
- (17) Let M be a σ -measure on S , f, g be partial functions from X to $\overline{\mathbb{R}}$, E be an element of S , and F be a non empty subset of $\overline{\mathbb{R}}$. Suppose $\text{dom } f \cap \text{dom } g = E$ and $\text{rng } f = F$ and g is finite and f is measurable on E and $\text{rng } f$ is bounded and g is integrable on M . Then $(f g) \upharpoonright E$ is integrable on M and there exists an element c of $\mathbb{R}$ such that $c \geq \inf F$ and $c \leq \sup F$ and $\int (f |g|) \upharpoonright E dM = \overline{\mathbb{R}}(c) \cdot \int |g| \upharpoonright E dM$.

4. SELECTED PROPERTIES OF INTEGRALS

We use the following convention: E_1, E_2 denote elements of S , x, A denote sets, and a, b denote real numbers.

The following propositions are true:

- (18) $|f| \upharpoonright A = |f \upharpoonright A|$.
- (19) $\text{dom}(|f| + |g|) = \text{dom } f \cap \text{dom } g$ and $\text{dom } |f + g| \subseteq \text{dom } |f|$.
- (20) $|f| \upharpoonright \text{dom } |f + g| + |g| \upharpoonright \text{dom } |f + g| = (|f| + |g|) \upharpoonright \text{dom } |f + g|$.
- (21) If $x \in \text{dom } |f + g|$, then $|f + g|(x) \leq (|f| + |g|)(x)$.
- (22) Suppose f is integrable on M and g is integrable on M . Then there exists an element E of S such that $E = \text{dom}(f + g)$ and $\int |f + g| \upharpoonright E \, dM \leq \int |f| \upharpoonright E \, dM + \int |g| \upharpoonright E \, dM$.
- (23) $\max_+(\chi_{A,X}) = \chi_{A,X}$.
- (24) If $M(E) < +\infty$, then $\chi_{E,X}$ is integrable on M and $\int \chi_{E,X} \, dM = M(E)$ and $\int \chi_{E,X} \upharpoonright E \, dM = M(E)$.
- (25) If $M(E_1 \cap E_2) < +\infty$, then $\int \chi_{(E_1),X} \upharpoonright E_2 \, dM = M(E_1 \cap E_2)$.
- (26) Suppose f is integrable on M and $E \subseteq \text{dom } f$ and $M(E) < +\infty$ and for every element x of X such that $x \in E$ holds $a \leq f(x) \leq b$. Then $\mathbb{R}(a) \cdot M(E) \leq \int f \upharpoonright E \, dM \leq \mathbb{R}(b) \cdot M(E)$.

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