


Elementary Number Theory Problems. Part XVII

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Summary. This paper furthers the formalization of problems from Wa-
cław Sierpiński book “250 Problems in Elementary Number Theory” in the Mizar
system. The selected twelve items are 37, 101, 115, 117, 145, 157, 159, 161–163,
165, and 169.

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INTRODUCTION

In this paper we continue formalizing the proofs of selected problems from Sierpiński’s book “250 Problems in Elementary Number Theory” [22] using the Mizar formalism [8, 18]. The paper is a part of the project *Formalization of Elementary Number Theory in Mizar* [16].

In the preliminary section, we provided some valuable facts about the monotonicity of functions and some properties of finite sequences. Next, as a sequel to Problem 36, credited to [11], with its formal solution presented in [15], we show that for every positive integer s there exists a positive integer n with the sum of digits (in decimal system) equal to s which is divisible by s , which constitutes the solution to Problem 37. Later, with the example of the number 200, we falsify the claim from Problem 101 that one can obtain a prime out of every positive integer by changing only one of its digits [5], [9]. The formalized proof of Problem 115 follows a method attributed to A. Schinzel to show that for $k \neq 1$ there exist infinitely many positive integers n such that the number $2^{2^n} + k$ is

composite. The assertion $k \neq 1$ is important, since it is still not known if there exist infinitely many composite Fermat numbers [10]. However, there are known examples, e.g. $n = 5$, such that $2^{2^n} + 1$ is composite [21], [19].

Next we provide a proof of Problem 117 showing that specific forms of numbers, namely $2^{2^{2n+1}} + 3$, $2^{2^{4n+1}} + 7$, $2^{2^{6n+2}} + 13$, $2^{2^{10n+1}} + 19$, and $2^{2^{6n+2}} + 21$ are all composite for $n = 1, 2, \dots$ [2]. The only (ordered) solution of the equation in Problem 145 is evidently given by the prime divisors of the number $F_5 = 641 \cdot 6700417$ (famous discovery by Euler which falsified Fermat's conjecture about the primality of Fermat numbers; see Proth primes [4], [13]). Then we show the proof of Problem 157 as the equivalence of two theorems: theorem T_1 asserting that there are no positive integers x, y, z for which $x/y + y/z = z/x$ and theorem T_2 asserting that there are no solutions in positive integers u, v, w of the equation $u^3 + v^3 = w^3$ (also attributed to A. Schinzel [20]).

In order to prove Problem 159, instead of using the lemma found in Sierpiński's proof, we applied the general property of geometric and arithmetic means already proved in the article [7]. The next three problems formalized in our paper (161, 162 and 163) deal with finding solutions of the same equation, $1/x_1 + 1/x_2 + \dots + 1/x_s = 1$, for every positive integer s , when $s > 2$ and when $s \neq 2$, respectively [14]. It should be noted that the original proof of Problem 162 contained a typo ($t_{s+2} = 2x_s$ instead of $t_{s+1} = 2x_s$). Then we show that the numbers 1, 4, and all integers $s \geq 6$ form the solution of Problem 165. Finally, we use the inductive argument to justify that since Problem 169 is solvable for $s = 3$ and $s = 4$, it is also solvable for every $s \geq 3$ [3].

1. PRELIMINARIES

From now on A denotes a set, a, k, m, n denote natural numbers, s denotes a positive natural number, i, j denote integers, r denotes a real number, and $c, c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8$ denote complex numbers.

Let A be a finite set. One can check that \mathbb{Z}_A is finite. Now we state the propositions:

- (1) Let us consider objects a, b, c, d . Then $\text{dom}\langle a, b, c, d \rangle = \{1, 2, 3, 4\}$.
- (2) Let us consider objects a, b, c, d, e . Then $\text{dom}\langle a, b, c, d, e \rangle = \{1, 2, 3, 4, 5\}$.
- (3) Let us consider an object a , and a set A . If $\langle a \rangle$ is A -valued, then $a \in A$.

Let s be a positive natural number and f be an s -element finite sequence. Observe that $f \upharpoonright (s-1)$ is $(s-1)$ -element.

Let f be a non empty, positive yielding, real-valued finite sequence. Let us note that $f(1)$ is positive. Now we state the propositions:

- (4) Let us consider an s -element, complex-valued finite sequence f . Then $\sum(f \upharpoonright (s-1)) = \sum f - f(s)$.

- (5) Let us consider a non empty set D . Then D^5 = the set of all $\langle d_1, d_2, d_3, d_4, d_5 \rangle$ where d_1, d_2, d_3, d_4, d_5 are elements of D .
- (6) Let us consider a non empty set D , and a 5-tuple z of D . Then there exist elements d_1, d_2, d_3, d_4, d_5 of D such that $z = \langle d_1, d_2, d_3, d_4, d_5 \rangle$. The theorem is a consequence of (5).
- (7) $\langle c \rangle^2 = \langle c^2 \rangle$.
- (8) $\langle c_1, c_2 \rangle^2 = \langle c_1^2, c_2^2 \rangle$. The theorem is a consequence of (7).
- (9) $\langle c_1, c_2, c_3 \rangle^2 = \langle c_1^2, c_2^2, c_3^2 \rangle$. The theorem is a consequence of (8) and (7).
- (10) $\langle c_1, c_2, c_3, c_4 \rangle^2 = \langle c_1^2, c_2^2, c_3^2, c_4^2 \rangle$. The theorem is a consequence of (9) and (7).
- (11) $\langle c_1, c_2, c_3, c_4, c_5 \rangle^2 = \langle c_1^2, c_2^2, c_3^2, c_4^2, c_5^2 \rangle$. The theorem is a consequence of (9) and (8).
- (12) $\langle c_1, c_2, c_3, c_4, c_5, c_6 \rangle^2 = \langle c_1^2, c_2^2, c_3^2, c_4^2, c_5^2, c_6^2 \rangle$. The theorem is a consequence of (11) and (7).
- (13) $\langle c_1, c_2, c_3, c_4, c_5, c_6, c_7 \rangle^2 = \langle c_1^2, c_2^2, c_3^2, c_4^2, c_5^2, c_6^2, c_7^2 \rangle$. The theorem is a consequence of (11) and (8).
- (14) $\langle c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8 \rangle^2 = \langle c_1^2, c_2^2, c_3^2, c_4^2, c_5^2, c_6^2, c_7^2, c_8^2 \rangle$. The theorem is a consequence of (11) and (9).
- (15) $\langle c_1, c_2, c_3, c_4 \rangle^{-1} = \langle c_1^{-1}, c_2^{-1}, c_3^{-1}, c_4^{-1} \rangle$.
- (16) $\langle c_1, c_2, c_3, c_4, c_5 \rangle^{-1} = \langle c_1^{-1}, c_2^{-1}, c_3^{-1}, c_4^{-1}, c_5^{-1} \rangle$.
- (17) $\langle c_1, c_2, c_3, c_4, c_5, c_6 \rangle^{-1} = \langle c_1^{-1}, c_2^{-1}, c_3^{-1}, c_4^{-1}, c_5^{-1}, c_6^{-1} \rangle$. The theorem is a consequence of (16).
- (18) $\langle c_1, c_2, c_3, c_4, c_5, c_6, c_7 \rangle^{-1} = \langle c_1^{-1}, c_2^{-1}, c_3^{-1}, c_4^{-1}, c_5^{-1}, c_6^{-1}, c_7^{-1} \rangle$. The theorem is a consequence of (16).
- (19) $\langle c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8 \rangle^{-1} = \langle c_1^{-1}, c_2^{-1}, c_3^{-1}, c_4^{-1}, c_5^{-1}, c_6^{-1}, c_7^{-1}, c_8^{-1} \rangle$. The theorem is a consequence of (16).
- (20) $\sum \langle c_1, c_2, c_3, c_4 \rangle = c_1 + c_2 + c_3 + c_4$.
- (21) $\sum \langle c_1, c_2, c_3, c_4, c_5 \rangle = c_1 + c_2 + c_3 + c_4 + c_5$.
- (22) $\sum \langle c_1, c_2, c_3, c_4, c_5, c_6 \rangle = c_1 + c_2 + c_3 + c_4 + c_5 + c_6$. The theorem is a consequence of (21).
- (23) $\sum \langle c_1, c_2, c_3, c_4, c_5, c_6, c_7 \rangle = c_1 + c_2 + c_3 + c_4 + c_5 + c_6 + c_7$. The theorem is a consequence of (21).
- (24) $\sum \langle c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8 \rangle = c_1 + c_2 + c_3 + c_4 + c_5 + c_6 + c_7 + c_8$. The theorem is a consequence of (21).

- (25) If $m > 1$, then there exists a positive natural number n such that $m^{m^n} > r$.
- (26) Let us consider a non zero integer n . Then there exists a natural number k and there exists an odd integer j such that $n = j \cdot 2^k$.
- (27) If $i \equiv j \pmod{j}$, then $i \equiv 0 \pmod{j}$.
- (28) Let us consider natural numbers x, y, n . Suppose $n > 0$ and $x \geq y$. Then $x \equiv y \pmod{n}$ if and only if there exists a natural number k such that $x = k \cdot n + y$.
- (29) If $m \leq n$ and $i^m \equiv i^{m+k} \pmod{a}$, then $i^n \equiv i^{n+k} \pmod{a}$.
- (30) $3 \mid 2^{2^n} - 1$.
- (31) $5 \mid 2^{4^n} - 1$.
- (32) $7 \mid 2^{6^n} - 1$.
- (33) $11 \mid 2^{10^n} - 1$.
- (34) $9 \mid 2^{6^n} - 1$.

Let F be a rational-valued finite sequence. Note that $\sum F$ is rational and $\prod F$ is rational. Let n be a natural number and f be an $(n + 1)$ -element finite sequence. One can check that $f_{\uparrow 1}$ is n -element. Let D be a set and f be a D -valued finite sequence. Observe that $f_{\uparrow n}$ is D -valued. Let f be an increasing, real-valued finite sequence. Observe that $f_{\uparrow n}$ is increasing. Let f be a decreasing, real-valued finite sequence. One can verify that $f_{\uparrow n}$ is decreasing.

Let f be a non-increasing, real-valued finite sequence. Note that $f_{\uparrow n}$ is non-increasing. Let f be a non-decreasing, real-valued finite sequence. Observe that $f_{\uparrow n}$ is non-decreasing. Let m, n be natural numbers and f be a positive yielding, n -element finite sequence. Observe that $f_{\uparrow m}$ is positive yielding. Let us consider extended real-valued finite sequences f, g . Now we state the propositions:

- (35) If $f \wedge g$ is decreasing, then f is decreasing and g is decreasing.
 PROOF: f is decreasing. g is decreasing. \square
- (36) If $f \wedge g$ is non-increasing, then f is non-increasing and g is non-increasing.
 PROOF: f is non-increasing. g is non-increasing. \square
- (37) If $f \wedge g$ is non-decreasing, then f is non-decreasing and g is non-decreasing.
 PROOF: f is non-decreasing. g is non-decreasing. \square
- (38) Let us consider increasing, real-valued finite sequences p, q . If $p(\text{len } p) < q(1)$, then $p \wedge q$ is increasing.
- (39) Let us consider real numbers r_1, r_2 . If $r_1 < r_2$, then $\langle r_1, r_2 \rangle$ is increasing. The theorem is a consequence of (38).
- (40) Let us consider real numbers r_1, r_2, r_3 . Suppose $r_1 < r_2 < r_3$. Then $\langle r_1, r_2, r_3 \rangle$ is increasing. The theorem is a consequence of (39) and (38).

- (41) Let us consider real numbers r_1, r_2, r_3, r_4 . Suppose $r_1 < r_2 < r_3 < r_4$. Then $\langle r_1, r_2, r_3, r_4 \rangle$ is increasing. The theorem is a consequence of (40) and (38).

Let n be a natural number. Let us note that there exists a natural-valued finite sequence which is increasing, positive yielding, and n -element. Now we state the propositions:

- (42) Let us consider a positive yielding, real-valued finite sequence f . Suppose $\text{len } f \geq 2$. Let us consider a natural number k . If $k \in \text{dom } f$, then $f(k) < \sum f$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ if $\$1 \geq 2$, then for every positive yielding, real-valued finite sequence f such that $\text{len } f = \$1$ for every natural number k such that $k \in \text{dom } f$ holds $f(k) < \sum f$. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$. For every natural number i , $\mathcal{P}[i]$. \square

- (43) $n \mapsto c^{-1} = (n \mapsto c)^{-1}$.

Let a be a non positive real number and b be a real number. One can check that $\langle a, b \rangle$ is non positive yielding. Let a be a real number and b be a non positive real number. Let us observe that $\langle a, b \rangle$ is non positive yielding. Let a be a non positive real number and b, c be real numbers. Note that $\langle a, b, c \rangle$ is non positive yielding. Let b be a non positive real number and a, c be real numbers. Let us note that $\langle a, b, c \rangle$ is non positive yielding.

Let c be a non positive real number and a, b be real numbers. Observe that $\langle a, b, c \rangle$ is non positive yielding. Let a be a non positive real number and b, c, d be real numbers. Observe that $\langle a, b, c, d \rangle$ is non positive yielding. Let b be a non positive real number and a, c, d be real numbers. Note that $\langle a, b, c, d \rangle$ is non positive yielding. Let c be a non positive real number and a, b, d be real numbers. Observe that $\langle a, b, c, d \rangle$ is non positive yielding. Let d be a non positive real number and a, b, c be real numbers. One can verify that $\langle a, b, c, d \rangle$ is non positive yielding.

Let a be a non positive real number and b, c, d, e be real numbers. One can check that $\langle a, b, c, d, e \rangle$ is non positive yielding. Let b be a non positive real number and a, c, d, e be real numbers. Note that $\langle a, b, c, d, e \rangle$ is non positive yielding. Let c be a non positive real number and a, b, d, e be real numbers. Let us note that $\langle a, b, c, d, e \rangle$ is non positive yielding. Let d be a non positive real number and a, b, c, e be real numbers. Observe that $\langle a, b, c, d, e \rangle$ is non positive yielding. Let e be a non positive real number and a, b, c, d be real numbers. Let us observe that $\langle a, b, c, d, e \rangle$ is non positive yielding. Now we state the propositions:

- (44) Let us consider a positive, natural-valued, 1-element finite sequence f .

Then there exists a positive natural number a such that $f = \langle a \rangle$.

(45) Let us consider a positive, natural-valued, 2-element finite sequence f . Then there exist positive natural numbers a, b such that $f = \langle a, b \rangle$.

(46) Let us consider an increasing, positive, natural-valued, 2-element finite sequence f . Then there exist positive natural numbers a, b such that

- (i) $f = \langle a, b \rangle$, and
- (ii) $a < b$.

The theorem is a consequence of (45).

(47) Let us consider a positive, natural-valued, 3-element finite sequence f . Then there exist positive natural numbers a, b, c such that $f = \langle a, b, c \rangle$.

(48) Let us consider an increasing, positive, natural-valued, 3-element finite sequence f . Then there exist positive natural numbers a, b, c such that

- (i) $f = \langle a, b, c \rangle$, and
- (ii) $a < b < c$.

The theorem is a consequence of (47).

Let us consider complex-valued functions f, g . Now we state the propositions:

(49) If $c \cdot f = c \cdot g$ and $c \neq 0$, then $f = g$.

(50) $(f \cdot g)^{-1} = f^{-1} \cdot g^{-1}$.

(51) $(c_1 \cdot f) \cdot (c_2 \cdot g) = c_1 \cdot c_2 \cdot (f \cdot g)$.

(52) Let us consider complex-valued finite sequences $f_1, f_2, g_1, g_2, h_1, h_2$. Suppose $\text{len } f_1 = \text{len } g_1 = \text{len } h_1$. Then $((f_1 \wedge f_2) \cdot (g_1 \wedge g_2)) \cdot (h_1 \wedge h_2) = ((f_1 \cdot g_1) \cdot h_1) \wedge ((f_2 \cdot g_2) \cdot h_2)$.

(53) Let us consider a natural number n , and an object e . Then $\mathbb{Z}_{n+1} \mapsto e = \langle e \rangle \wedge (\mathbb{Z}_n \mapsto e)$.

PROOF: Set $p = \mathbb{Z}_n \mapsto e$. Set $q = \mathbb{Z}_{n+1} \mapsto e$. For every natural number k such that $k \in \text{dom } \langle e \rangle$ holds $q(k) = \langle e \rangle(k)$. For every natural number k such that $k \in \text{dom } p$ holds $q(\text{len } \langle e \rangle + k) = p(k)$. \square

2. PROBLEM 37

Now we state the propositions:

(54) Let us consider natural numbers n, b . Suppose $n > 0$ and $b > 1$. Then $\text{digits}(n \cdot b, b) = \langle 0 \rangle \wedge \text{digits}(n, b)$.

PROOF: Set $d = \text{digits}(n, b)$. Consider d' being a finite 0-sequence of \mathbb{N} such that $\text{dom } d' = \text{dom } d$ and for every natural number i such that $i \in \text{dom } d'$ holds $d'(i) = d(i) \cdot b^i$ and $n = \sum d'$. Define $\mathcal{A} = \text{len } d'$. Define

\mathcal{F} (natural number) = $b \cdot d'(\$_1)$. Consider d_1 being a finite 0-sequence such that $\text{len } d_1 = \mathcal{A}$ and for every natural number n such that $n \in \mathcal{A}$ holds $d_1(n) = \mathcal{F}(n)$. $\text{rng } d_1 \subseteq \mathbb{N}$. \square

(55) Let us consider natural numbers n, b . Suppose $n > 0$ and $b > 1$. Let us consider a natural number i . Then $\text{digits}(n \cdot b^i, b) = (i \mapsto 0) \wedge \text{digits}(n, b)$.
 PROOF: Define \mathcal{P} [natural number] $\equiv \text{digits}(n \cdot b^{\$1}, b) = (\$1 \mapsto 0) \wedge \text{digits}(n, b)$. For every natural number i , $\mathcal{P}[i]$. \square

(56) Let us consider a natural number s . Then there exists a natural number n such that

(i) $\sum \text{digits}(n, 10) = s$, and

(ii) $s \mid n$.

The theorem is a consequence of (55).

3. PROBLEM 101

Now we state the proposition:

(57) Let us consider sets a, b . Then $\text{Replace}(\langle a \rangle, 0, b) = \langle b \rangle$.

Let us consider sets a, b, c . Now we state the propositions:

(58) $\text{Replace}(\langle a, b \rangle, 0, c) = \langle c, b \rangle$.

(59) $\text{Replace}(\langle a, b \rangle, 1, c) = \langle a, c \rangle$.

Let us consider sets a, b, c, d . Now we state the propositions:

(60) $\text{Replace}(\langle a, b, c \rangle, 0, d) = \langle d, b, c \rangle$.

(61) $\text{Replace}(\langle a, b, c \rangle, 1, d) = \langle a, d, c \rangle$.

(62) $\text{Replace}(\langle a, b, c \rangle, 2, d) = \langle a, b, d \rangle$.

(63) $\text{digits}(201, 10) = \langle 1, 0, 2 \rangle$.

(64) $\text{digits}(202, 10) = \langle 2, 0, 2 \rangle$.

(65) $\text{digits}(203, 10) = \langle 3, 0, 2 \rangle$.

(66) $\text{digits}(205, 10) = \langle 5, 0, 2 \rangle$.

(67) $\text{digits}(206, 10) = \langle 6, 0, 2 \rangle$.

(68) Let us consider elements i, j of \mathbb{N} . Suppose $0 \leq j \leq 9$.

Then $\text{value}(\text{Replace}(\text{digits}(200, 10), i, j), 10)$ is not prime. The theorem is a consequence of (60), (63), (64), (65), (66), (67), and (62).

4. PROBLEM 115

Now we state the propositions:

(69) A CONSEQUENCE OF EULER'S THEOREM:

Let us consider natural numbers a, n, x, y . Suppose a and n are relatively prime and $a \neq 0$ and $n \neq 0$ and $x \equiv y \pmod{\text{Euler } n}$. Then $a^x \equiv a^y \pmod{n}$. The theorem is a consequence of (28).

(70) Let us consider a natural number a , and an integer k . Suppose $k \neq 1$. Then there exists a positive natural number n such that

- (i) $a < 2^{2^n} + k$, and
- (ii) $2^{2^n} + k$ is composite.

The theorem is a consequence of (26), (25), (29), and (69).

Let us consider an integer k . Now we state the propositions:

(71) Suppose $k \neq 1$. Then $\{n, \text{ where } n \text{ is a positive natural number : } 2^{2^n} + k \text{ is composite}\} \approx \{2^{2^n} + k, \text{ where } n \text{ is a positive natural number : } 2^{2^n} + k \text{ is composite}\}$.

PROOF: Set $X = \{n, \text{ where } n \text{ is a positive natural number : } 2^{2^n} + k \text{ is composite}\}$. Set $Y = \{2^{2^n} + k, \text{ where } n \text{ is a positive natural number : } 2^{2^n} + k \text{ is composite}\}$. Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ for every m such that $m = \$_1$ holds $\$_2 = 2^{2^m} + k$. For every object x such that $x \in X$ there exists an object y such that $y \in Y$ and $\mathcal{P}[x, y]$.

Consider f being a function from X into Y such that for every object x such that $x \in X$ holds $\mathcal{P}[x, f(x)]$. f is one-to-one. Consider n being a positive natural number such that $y = 2^{2^n} + k$ and $2^{2^n} + k$ is composite. \square

(72) If $k \neq 1$, then $\{2^{2^n} + k, \text{ where } n \text{ is a positive natural number : } 2^{2^n} + k \text{ is composite}\}$ is infinite. The theorem is a consequence of (70).

(73) If $k \neq 1$, then $\{n, \text{ where } n \text{ is a positive natural number : } 2^{2^n} + k \text{ is composite}\}$ is infinite. The theorem is a consequence of (71) and (72).

5. PROBLEM 117

Now we state the propositions:

(74) $7 \mid 2^{2^{2^{s+1}}} + 3$. The theorem is a consequence of (30) and (27).

(75) $2^{2^{2^{s+1}}} + 3$ is composite. The theorem is a consequence of (74).

(76) $11 \mid 2^{2^{4^{s+1}}} + 7$. The theorem is a consequence of (31) and (27).

(77) $2^{2^{4^{s+1}}} + 7$ is composite. The theorem is a consequence of (76).

- (78) $29 \mid 2^{2^{6 \cdot s+2}} + 13$. The theorem is a consequence of (32) and (27).
 (79) $2^{2^{6 \cdot s+2}} + 13$ is composite. The theorem is a consequence of (78).
 (80) $23 \mid 2^{2^{10 \cdot s+1}} + 19$. The theorem is a consequence of (33) and (27).
 (81) $2^{2^{10 \cdot s+1}} + 19$ is composite. The theorem is a consequence of (80).
 (82) $37 \mid 2^{2^{6 \cdot s+2}} + 21$. The theorem is a consequence of (34) and (27).
 (83) $2^{2^{6 \cdot s+2}} + 21$ is composite. The theorem is a consequence of (82).

6. PROBLEM 145

Now we state the propositions:

- (84) Fermat 5 = 641 · 6700417.
 (85) Let us consider positive natural numbers x, y . Then $x \cdot y + x + y = 2^{32}$ if and only if $x = 640$ and $y = 6700416$ or $x = 6700416$ and $y = 640$.
 PROOF: If $x \cdot y + x + y = 2^{32}$, then $x = 640$ and $y = 6700416$ or $x = 6700416$ and $y = 640$ by (84), [12, (59)]. \square

7. PROBLEM 157

Let a be a non positive integer and n be an odd natural number. One can check that a^n is non positive.

- (86) Let us consider natural numbers a, b, c, d, n . Suppose a, b, c are mutually coprime and $a \cdot b \cdot c = d^n$. Then there exists a natural number k such that $a = k^n$.
 (87) for every positive natural numbers $x, y, z, \frac{x}{y} + \frac{y}{z} \neq \frac{z}{x}$ if and only if for every positive natural numbers $u, v, w, u^3 + v^3 \neq w^3$.

PROOF: If for every positive natural numbers $x, y, z, \frac{x}{y} + \frac{y}{z} \neq \frac{z}{x}$, then for every positive natural numbers $u, v, w, u^3 + v^3 \neq w^3$. Set $a = x^2 \cdot z$. Set $b = y^2 \cdot x$. Set $d = \text{gcd}(a, b)$.

Consider a_1, b_1 being natural numbers such that $a = d \cdot a_1$ and $b = d \cdot b_1$ and a_1 and b_1 are relatively prime. Consider t being a natural number such that $x \cdot y \cdot z = d \cdot t$. Consider u being a natural number such that $u^3 = a_1$. Consider v being a natural number such that $v^3 = a_1 + b_1$. Consider w being a natural number such that $w^3 = b_1$. \square

8. PROBLEM 159

Let a, b, c be positive real numbers. One can check that $\langle a, b, c \rangle$ is positive yielding. Let a, b, c, d be positive real numbers. One can check that $\langle a, b, c, d \rangle$ is positive yielding. Now we state the proposition:

(88) Let us consider complex numbers a, b, c, d . Then $\prod \langle a, b, c, d \rangle = a \cdot b \cdot c \cdot d$.

Let us consider a positive real number a . Now we state the propositions:

(89) $\sqrt[3]{a} = \sqrt{a}$.

(90) $\text{GMean}\langle a \rangle = a$.

(91) Let us consider positive real numbers a, b . Then $(\text{GMean}\langle a, b \rangle)^2 = a \cdot b$.

(92) Let us consider positive real numbers a, b, c . Then $(\text{GMean}\langle a, b, c \rangle)^3 = a \cdot b \cdot c$.

(93) Let us consider positive real numbers a, b, c, d .

Then $(\text{GMean}\langle a, b, c, d \rangle)^4 = a \cdot b \cdot c \cdot d$. The theorem is a consequence of (88).

(94) Let us consider a real number a . Then $\text{Mean}\langle a \rangle = a$.

(95) Let us consider real numbers a, b . Then $\text{Mean}\langle a, b \rangle = \frac{a+b}{2}$.

(96) Let us consider real numbers a, b, c . Then $\text{Mean}\langle a, b, c \rangle = \frac{a+b+c}{3}$.

(97) Let us consider real numbers a, b, c, d . Then $\text{Mean}\langle a, b, c, d \rangle = \frac{a+b+c+d}{4}$.

The theorem is a consequence of (20).

(98) Let us consider objects w, x, y, z . Suppose $\langle w, x, y, z \rangle$ is constant. Then $w = x = y = z$.

(99) Let us consider positive real numbers a, b, c, d . If $a \cdot b \cdot c \cdot d \geq (\frac{a+b+c+d}{4})^4$, then $a = b = c = d$. The theorem is a consequence of (88), (93), (97), and (98).

(100) Let us consider a real number m . Suppose $0 \leq m < 4$. Then there exist no positive integers x, y, z, t such that $\frac{x}{y} + \frac{y}{z} + \frac{z}{t} + \frac{t}{x} = m$.

(101) $\{\langle x, y, z, t \rangle, \text{ where } x, y, z, t \text{ are positive integers : } \frac{x}{y} + \frac{y}{z} + \frac{z}{t} + \frac{t}{x} = 4\} = \{\langle x, y, z, t \rangle, \text{ where } x, y, z, t \text{ are positive integers : } x = y = z = t\}$. The theorem is a consequence of (99).

Let a, b be natural numbers. Let us note that $\langle a, b \rangle (\in \mathbb{N} \times \mathbb{N})$ reduces to $\langle a, b \rangle$. Let a, b, c be natural numbers. One can verify that $\langle a, b, c \rangle (\in \mathbb{N} \times \mathbb{N} \times \mathbb{N})$ reduces to $\langle a, b, c \rangle$. Let a, b, c, d be natural numbers. Let us observe that $\langle a, b, c, d \rangle (\in \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N})$ reduces to $\langle a, b, c, d \rangle$.

Let a be a positive natural number. Observe that $a (\in \mathbb{N}_+)$ reduces to a . Let a, b be positive natural numbers. Let us observe that $\langle a, b \rangle (\in \mathbb{N}_+ \times \mathbb{N}_+)$ reduces to $\langle a, b \rangle$. Let a, b, c be positive natural numbers. Note that $\langle a, b, c \rangle (\in \mathbb{N}_+ \times$

$\mathbb{N}_+ \times \mathbb{N}_+$) reduces to $\langle a, b, c \rangle$. Let a, b, c, d be positive natural numbers. One can check that $\langle a, b, c, d \rangle (\in \mathbb{N}_+ \times \mathbb{N}_+ \times \mathbb{N}_+ \times \mathbb{N}_+)$ reduces to $\langle a, b, c, d \rangle$.

The functor Cart4Id yielding a function from \mathbb{N}_+ into $\mathbb{N}_+ \times \mathbb{N}_+ \times \mathbb{N}_+ \times \mathbb{N}_+$ is defined by

(Def. 1) for every positive natural number n , $it(n) = \langle n, n, n, n \rangle$.

Observe that Cart4Id is one-to-one. Now we state the propositions:

(102) $\text{rng Cart4Id} \subseteq \{ \langle x, y, z, t \rangle, \text{ where } x, y, z, t \text{ are positive integers : } \frac{x}{y} + \frac{y}{z} + \frac{z}{t} + \frac{t}{x} = 4 \}$.

(103) $\{ \langle x, y, z, t \rangle, \text{ where } x, y, z, t \text{ are positive integers : } \frac{x}{y} + \frac{y}{z} + \frac{z}{t} + \frac{t}{x} = 4 \}$ is infinite. The theorem is a consequence of (102).

9. PROBLEM 161

Let R be a binary relation. The functor $\text{Permutations}(R)$ yielding a set is defined by the term

(Def. 2) the set of all $p \cdot R$ where p is a permutation of $\text{dom } R$.

Let f be a function. One can check that the functor $\text{Permutations}(f)$ yields a set and is defined by the term

(Def. 3) the set of all $f \cdot p$ where p is a permutation of $\text{dom } f$.

Let A be a set. We say that A is relational if and only if

(Def. 4) for every object a such that $a \in A$ holds a is a binary relation.

Let a, b be objects. One can check that $\{ \{ \langle a, b \rangle \} \}$ is relational and there exists a set which is non empty, finite, and relational.

Let A be a relational set. Let us note that every element of A is relation-like and every set which is functional is also relational.

Let R be a finite binary relation. One can check that $\text{Permutations}(R)$ is finite.

Let A be a relational set. The functor $\text{Permutations}(A)$ yielding a set is defined by the term

(Def. 5) $\{ \text{Permutations}(f), \text{ where } f \text{ is an element of } A : f \in A \}$.

Let A be a finite, relational set. One can check that $\text{Permutations}(A)$ is finite.

Let f be a complex-valued finite sequence. We say that f has a sum of reciprocals equal 1 if and only if

(Def. 6) $\sum f^{-1} = 1$.

Observe that (1) has a sum of reciprocals equal 1 and there exists a complex-valued finite sequence which has a sum of reciprocals equal 1.

Let s be a natural number. The functor $\text{SolutionsOfReciprocalsSum}(s)$ yielding a set is defined by the term

(Def. 7) $\{f, \text{ where } f \text{ is a positive yielding, } s\text{-element, natural-valued finite sequence : } f \text{ has a sum of reciprocals equal } 1\}$.

Let s be a zero natural number. Note that $\text{SolutionsOfReciprocalsSum}(s)$ is empty. Now we state the proposition:

$$(104) \quad \sum(s \mapsto s^{-1}) = 1.$$

Let s be a positive natural number. Let us note that $s \mapsto s$ has a sum of reciprocals equal 1. Now we state the proposition:

$$(105) \quad s \mapsto s \in \text{SolutionsOfReciprocalsSum}(s).$$

Let s be a positive natural number. Let us observe that $\text{SolutionsOfReciprocalsSum}(s)$ is non empty.

Let A, B be finite sequence-membered sets. The functor $A \frown B$ yielding a set is defined by the term

(Def. 8) $\{p \frown q, \text{ where } p \text{ is an element of } A, q \text{ is an element of } B : p \in A \text{ and } q \in B\}$.

One can verify that $A \frown B$ is finite sequence-membered.

Let us consider a finite sequence-membered set A . Now we state the propositions:

$$(106) \quad A \frown \emptyset \subseteq A.$$

$$(107) \quad \emptyset \frown A \subseteq A.$$

Let A, B be finite, finite sequence-membered sets. Let us note that $A \frown B$ is finite.

Let D, A be sets. We say that A is D -FinSequence-membered if and only if

(Def. 9) for every object f such that $f \in A$ holds f is a D -valued finite sequence.

Let D be a non empty set and d be an element of D . Let us note that $\{\langle d \rangle\}$ is D -FinSequence-membered and there exists a set which is D -FinSequence-membered.

Let A be a D -FinSequence-membered set. The functor $\text{decomp } A$ yielding a binary relation is defined by

(Def. 10) for every objects $x, y, \langle x, y \rangle \in it$ iff $x \in D$ and there exists a finite sequence f of elements of D such that $y = f$ and $\langle x \rangle \frown f \in A$.

Let D be a set. Let us note that every set which is D -FinSequence-membered is also finite sequence-membered.

Let A be a set. The functor $\text{OneFS}(A)$ yielding a many sorted set indexed by A is defined by

(Def. 11) for every object a such that $a \in A$ holds $it(a) = \langle a \rangle$.

Let us observe that $\text{OneFS}(A)$ is finite sequence-yielding.

Let f be a finite sequence-yielding function. One can check that $\text{rng } f$ is finite sequence-membered.

Let us consider n . Let f be an n -element, real-valued finite sequence. One can verify that $\text{sort}_a f$ is n -element and $\text{sort}_d f$ is n -element. Now we state the propositions:

(108) Let us consider complex-valued finite sequences f, g , and a permutation P of $\text{dom } g$. Suppose $f = g \cdot P$ and $\text{len } g \geq 1$. Then $f^{-1} = g^{-1} \cdot P$.

PROOF: Reconsider $k = g^{-1} \cdot P$ as a finite sequence of elements of \mathbb{C} . For every natural number i such that $i \in \text{dom}(f^{-1})$ holds $(f^{-1})(i) = k(i)$. \square

(109) Let us consider complex-valued finite sequences f, g . Suppose f and g are fiberwise equipotent. Then f^{-1} and g^{-1} are fiberwise equipotent. The theorem is a consequence of (108).

(110) Let us consider a natural number s , and a non-decreasing, positive, $(s + 1)$ -element, natural-valued finite sequence f . Then

- (i) $f(1) \leq \sum_{f^{-1}}^{s+1}$, and
- (ii) $\sum(f_{\uparrow 1})^{-1} = \sum f^{-1} - \frac{1}{f(1)}$.

PROOF: Define $\mathcal{Q}[\text{finite sequence of elements of } \mathbb{N}] \equiv$ for every non-decreasing, positive, natural-valued finite sequence F such that $F = \$_1$ holds $\sum \$_1^{-1} \leq \frac{\text{len } \$_1}{\$_1(1)}$. For every finite sequence p of elements of \mathbb{N} and for every element x of \mathbb{N} such that $\mathcal{Q}[p]$ holds $\mathcal{Q}[p \hat{\ } \langle x \rangle]$. For every finite sequence p of elements of \mathbb{N} , $\mathcal{Q}[p]$. \square

Let us consider a natural number s and a rational number w . Now we state the propositions:

(111) $\{f, \text{ where } f \text{ is a non-decreasing, positive, } s\text{-element, natural-valued finite sequence : } \sum f^{-1} = w\}$ is finite.

PROOF: Define $\mathcal{A}(\text{natural number, rational number}) = \{f, \text{ where } f \text{ is a non-decreasing, positive, } \$_1\text{-element, natural-valued finite sequence : } \sum f^{-1} = \$_2\}$. Define $\mathcal{P}[\text{natural number}] \equiv$ for every rational number q , $\mathcal{A}(\$_1, q)$ is finite. $\mathcal{P}[0]$. For every natural number s such that $\mathcal{P}[s]$ holds $\mathcal{P}[s + 1]$ by (110), [1, (7)]. For every natural number i , $\mathcal{P}[i]$. \square

(112) $\{f, \text{ where } f \text{ is a positive yielding, } s\text{-element, natural-valued finite sequence : } \sum f^{-1} = w\}$ is finite.

PROOF: Set $A = \{f, \text{ where } f \text{ is a positive, } s\text{-element, natural-valued finite sequence : } \sum f^{-1} = w\}$. Set $I = \{f, \text{ where } f \text{ is a non-decreasing, positive, } s\text{-element, natural-valued finite sequence : } \sum f^{-1} = w\}$. I is functional. Reconsider $I_1 = I$ as a functional set. I is finite. For every set X such that

$X \in \text{Permutations}(I_1)$ holds X is finite. Define $\mathcal{S}(\text{natural-valued finite sequence}) = \text{sort}_a \1 . $A \subseteq \bigcup \text{Permutations}(I_1)$. \square

Let s be a natural number. Note that $\text{SolutionsOfReciprocalsSum}(s)$ is finite.

10. PROBLEM 162

Observe that $\langle 2, 3 \rangle$ is increasing and $\langle 2, 3, 6 \rangle$ is increasing and $\langle 2, 3, 7, 42 \rangle$ is increasing and $\langle 2, 3, 8, 24 \rangle$ is increasing.

Let f be a complex-valued finite sequence. The functors: $\text{Sierp162FS}(f)$ and $\text{Sierp162FS2}(f)$ yielding finite sequences are defined by terms

(Def. 12) $\langle 2 \rangle \frown (2 \cdot f)$,

(Def. 13) $\langle 2, 3 \rangle \frown (6 \cdot f)$,

respectively.

One can check that $\text{Sierp162FS}(f)$ is complex-valued and $\text{Sierp162FS2}(f)$ is complex-valued. Let f be a real-valued finite sequence. One can verify that $\text{Sierp162FS}(f)$ is real-valued and $\text{Sierp162FS2}(f)$ is real-valued. Let f be a rational-valued finite sequence. Let us observe that $\text{Sierp162FS}(f)$ is rational-valued and $\text{Sierp162FS2}(f)$ is rational-valued.

Let f be an integer-valued finite sequence. Observe that $\text{Sierp162FS}(f)$ is integer-valued and $\text{Sierp162FS2}(f)$ is integer-valued. Let f be a natural-valued finite sequence. Let us note that $\text{Sierp162FS}(f)$ is natural-valued and $\text{Sierp162FS2}(f)$ is natural-valued. Let s be a natural number and f be an s -element, complex-valued finite sequence. Let us note that $\text{Sierp162FS}(f)$ is $(s+1)$ -element and $\text{Sierp162FS2}(f)$ is $(s+2)$ -element and there exists a natural-valued finite sequence which is increasing and s -element. Now we state the propositions:

(113) Let us consider a natural number s . Suppose $s \geq 2$. Let us consider a positive yielding, s -element, real-valued finite sequence f . Suppose f has a sum of reciprocals equal 1. Let us consider a natural number k . If $k \in \text{dom } f$, then $f(k) > 1$. The theorem is a consequence of (42).

(114) Let us consider a natural number s . Suppose $s \geq 2$. Let us consider an increasing, positive yielding, s -element, real-valued finite sequence f . Suppose f has a sum of reciprocals equal 1. Then $\text{Sierp162FS}(f)$ is increasing. The theorem is a consequence of (113).

(115) Let us consider an increasing, positive yielding, s -element, natural-valued finite sequence f . Then $\text{Sierp162FS2}(f)$ is increasing. The theorem is a consequence of (38).

Let us consider a complex-valued finite sequence f . Now we state the propositions:

- (116) If f has a sum of reciprocals equal 1, then $\text{Sierp162FS}(f)$ has a sum of reciprocals equal 1.
- (117) If f has a sum of reciprocals equal 1, then $\text{Sierp162FS2}(f)$ has a sum of reciprocals equal 1.
- (118) $\langle 2, 3, 6 \rangle$ has a sum of reciprocals equal 1.
- (119) $\langle 2, 3, 7, 42 \rangle$ has a sum of reciprocals equal 1. The theorem is a consequence of (20).
- (120) $\langle 2, 3, 8, 24 \rangle$ has a sum of reciprocals equal 1. The theorem is a consequence of (20).
- (121) Let us consider a natural number s . Suppose $s > 2$. Then there exists an increasing, positive yielding, s -element, natural-valued finite sequence f such that f has a sum of reciprocals equal 1.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ there exists an increasing, positive, s_1 -element, natural-valued finite sequence f such that f has a sum of reciprocals equal 1. $\mathcal{P}[3]$. For every natural number j such that $3 \leq j$ holds if $\mathcal{P}[j]$, then $\mathcal{P}[j + 1]$. For every natural number i such that $3 \leq i$ holds $\mathcal{P}[i]$. \square

Let s be a natural number. The functor $\text{IncreasingSolutionsOfReciprocalsSum}(s)$ yielding a set is defined by the term

(Def. 14) $\{f, \text{ where } f \text{ is an increasing, positive yielding, } s\text{-element, natural-valued finite sequence : } f \text{ has a sum of reciprocals equal } 1\}$.

Now we state the proposition:

- (122) Let us consider a natural number s . Then $\text{IncreasingSolutionsOfReciprocalsSum}(s) \subseteq \text{SolutionsOfReciprocalsSum}(s)$.

Let s be a positive natural number.

Let us observe that $\text{IncreasingSolutionsOfReciprocalsSum}(s)$ is finite. Now we state the propositions:

- (123) $\text{IncreasingSolutionsOfReciprocalsSum}(3) = \{\langle 2, 3, 6 \rangle\}$.

PROOF: Set $S = \text{IncreasingSolutionsOfReciprocalsSum}(3)$. $S \subseteq \{\langle 2, 3, 6 \rangle\}$. \square

- (124) $\{\langle 2, 3, 7, 42 \rangle, \langle 2, 3, 8, 24 \rangle\} \subseteq \text{IncreasingSolutionsOfReciprocalsSum}(4)$.

The theorem is a consequence of (119) and (120).

- (125) Let us consider a positive natural number s . Suppose $s > 2$. Then $\overline{\overline{\text{IncreasingSolutionsOfReciprocalsSum}(s)}} < \overline{\overline{\text{IncreasingSolutionsOfReciprocalsSum}(s + 1)}}$. The theorem is a consequence of (123), (124), (121), (114), (116), (115), (117), and (49).

11. PROBLEM 163

Let n be a natural number. One can verify that Triangle n is triangular. Let a be a triangular natural number. One can check that $\langle a \rangle$ is triangular-valued and there exists a finite sequence which is triangular-valued. Let f, g be triangular-valued finite sequences. One can check that $f \wedge g$ is triangular-valued. Let a, b be triangular natural numbers. One can verify that $\langle a, b \rangle$ is triangular-valued. Let a, b, c be triangular natural numbers.

Let us note that $\langle a, b, c \rangle$ is triangular-valued. Let a, b, c, d be triangular natural numbers. Let us note that $\langle a, b, c, d \rangle$ is triangular-valued. Let a, b, c, d, e be triangular natural numbers. One can verify that $\langle a, b, c, d, e \rangle$ is triangular-valued. Let a, b, c, d, e, f be triangular natural numbers. One can verify that $\langle a, b, c, d, e, f \rangle$ is triangular-valued. Now we state the propositions:

(126) $\langle \text{Triangle } 1 \rangle$ has a sum of reciprocals equal 1.

(127) $\langle \text{Triangle } 2, \text{Triangle } 2, \text{Triangle } 2 \rangle$ has a sum of reciprocals equal 1.

(128) $\langle \text{Triangle } 2, \text{Triangle } 2, \text{Triangle } 3, \text{Triangle } 3 \rangle$ has a sum of reciprocals equal 1. The theorem is a consequence of (15) and (20).

(129) $\langle \text{Triangle } 3, \text{Triangle } 3, \text{Triangle } 3, \text{Triangle } 3, \text{Triangle } 3, \text{Triangle } 3 \rangle$ has a sum of reciprocals equal 1. The theorem is a consequence of (17) and (22).

Let n, a be natural numbers. Assume $n \geq a + 1$. The functor $\text{TriangleShift2}(n, a)$ yielding a finite sequence of elements of \mathbb{N} is defined by

(Def. 15) $\text{len } it = n - a - 1$ and for every natural number i such that $1 \leq i \leq \text{len } it$ holds $it(i) = \text{Triangle}(a + i)$.

One can verify that $\text{TriangleShift2}(3, 2)$ is empty. Now we state the propositions:

(130) $\text{TriangleShift2}(4, 2) = \langle \text{Triangle } 3 \rangle$.

(131) If $n \geq a + 1$, then $\text{TriangleShift2}(n + 1, a) = \text{TriangleShift2}(n, a) \wedge \langle \text{Triangle } n \rangle$.

(132) If $a + 1 \leq n$, then $\text{TriangleShift2}(n, a)$ is triangular-valued.

Let us consider m and n . Note that $m \mapsto \text{Triangle } n$ is triangular-valued.

(133) If $3 \leq n$, then $\sum(\text{TriangleShift2}(n, 2))^{-1} = \frac{2}{3} - \frac{n+1}{\text{Triangle } n}$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv \sum(\text{TriangleShift2}(\$1, 2))^{-1} = \frac{2}{3} - \frac{\$1+1}{\mathcal{T}(\$1)}$. For every natural number j such that $3 \leq j$ holds if $\mathcal{P}[j]$, then $\mathcal{P}[j + 1]$ by [6, (10), (19)], [17, (63)], (131). For every natural number i such that $3 \leq i$ holds $\mathcal{P}[i]$. \square

(134) If $s \neq 2$, then there exists a natural-valued, s -element finite sequence f such that f is triangular-valued and has a sum of reciprocals equal 1. The theorem is a consequence of (127), (128), (132), (133), (43), and (129).

12. PROBLEM 165

Let f be a complex-valued finite sequence. We say that f has a sum of squares of reciprocals equal 1 if and only if

(Def. 16) $\sum (f^{-1})^2 = 1.$

Now we state the propositions:

(135) $\langle 1 \rangle$ has a sum of squares of reciprocals equal 1.

(136) $\langle 2, 2, 2, 2 \rangle$ has a sum of squares of reciprocals equal 1. The theorem is a consequence of (15) and (10).

(137) $\langle 2, 2, 2, 3, 3, 6 \rangle$ has a sum of squares of reciprocals equal 1. The theorem is a consequence of (17), (12), and (22).

(138) $\langle 2, 2, 2, 4, 4, 4, 4 \rangle$ has a sum of squares of reciprocals equal 1. The theorem is a consequence of (18), (13), and (23).

(139) $\langle 2, 2, 2, 3, 3, 7, 14, 21 \rangle$ has a sum of squares of reciprocals equal 1. The theorem is a consequence of (19), (14), and (24).

(140) Let us consider an s -element, natural-valued finite sequence f . Suppose f has a sum of squares of reciprocals equal 1. Then $(f \upharpoonright (s - 1)) \wedge \langle 2 \cdot f(s), 2 \cdot f(s), 2 \cdot f(s), 2 \cdot f(s) \rangle$ has a sum of squares of reciprocals equal 1. The theorem is a consequence of (15), (10), (20), and (4).

Let s be a positive natural number, a, b, c, d be positive real numbers, and f be an s -element, positive, real-valued finite sequence. One can check that $\langle a \cdot f(s), b \cdot f(s), c \cdot f(s), d \cdot f(s) \rangle$ is positive yielding. Now we state the proposition:

(141) Let us consider an s -element, positive, natural-valued finite sequence f . Suppose f has a sum of squares of reciprocals equal 1. Then there exists an $(s + 3 \cdot n)$ -element, positive, natural-valued finite sequence g such that g has a sum of squares of reciprocals equal 1.

PROOF: Set $g = (f \upharpoonright (s - 1)) \wedge \langle 2 \cdot f(s), 2 \cdot f(s), 2 \cdot f(s), 2 \cdot f(s) \rangle$. Define $\mathcal{P}[\text{natural number}] \equiv$ there exists an $(s + 3 \cdot \$_1)$ -element, positive, natural-valued finite sequence g such that g has a sum of squares of reciprocals equal 1. For every k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k + 1]$. For every k , $\mathcal{P}[k]$. \square

The scheme *STEP2a* deals with a natural number \mathcal{N} and a unary predicate \mathcal{P} and states that

(Sch. 1) For every natural number k such that $k \geq \mathcal{N}$ holds $\mathcal{P}[k]$ provided

- $\mathcal{P}[\mathcal{N}]$ and
- $\mathcal{P}[\mathcal{N} + 1]$ and

- for every natural number k such that $k \geq \mathcal{N}$ and $\mathcal{P}[k]$ and $\mathcal{P}[k + 1]$ holds $\mathcal{P}[k + 2]$.

The scheme *STEP2* deals with a unary predicate \mathcal{P} and states that

(Sch. 2) For every natural number k , $\mathcal{P}[k]$ provided

- for every natural number k , $\mathcal{P}[2 \cdot k]$ and $\mathcal{P}[2 \cdot k + 1]$.

The scheme *STEP3* deals with a unary predicate \mathcal{P} and states that

(Sch. 3) For every natural number k , $\mathcal{P}[k]$ provided

- for every natural number k , $\mathcal{P}[3 \cdot k]$ and $\mathcal{P}[3 \cdot k + 1]$ and $\mathcal{P}[3 \cdot k + 2]$.

The scheme *STEP4* deals with a unary predicate \mathcal{P} and states that

(Sch. 4) For every natural number k , $\mathcal{P}[k]$ provided

- for every natural number k , $\mathcal{P}[4 \cdot k]$ and $\mathcal{P}[4 \cdot k + 1]$ and $\mathcal{P}[4 \cdot k + 2]$ and $\mathcal{P}[4 \cdot k + 3]$.

Now we state the proposition:

- (142) there exists an s -element, positive, natural-valued finite sequence f such that f has a sum of squares of reciprocals equal 1 if and only if $s = 1$ or $s = 4$ or $s \geq 6$.

PROOF: Define $\mathcal{S}[\text{natural number}] \equiv$ there exists a $\$1$ -element, positive, natural-valued finite sequence f such that f has a sum of squares of reciprocals equal 1. If $\mathcal{S}[s]$, then $s = 1$ or $s = 4$ or $s \geq 6$. \square

13. PROBLEM 169

Let s be a natural number and f be an $(s+1)$ -element, complex-valued finite sequence. We say that f satisfies 169th Sierpiński problem if and only if

(Def. 17) $\sum(((f \upharpoonright s) \cdot (f \upharpoonright s)) \cdot (f \upharpoonright s))^{-1} = \frac{1}{f(s+1)^3}$.

Let a, b, c, d be objects. One can verify that $\langle a, b, c, d \rangle$ is $(3+1)$ -element.

Now we state the proposition:

- (143) $\langle 12, 15, 20, 10 \rangle$ satisfies 169th Sierpiński problem. The theorem is a consequence of (50).

Let a, b, c, d, e be objects. Let us note that $\langle a, b, c, d, e \rangle$ is $(4+1)$ -element.

Now we state the proposition:

(144) $\langle 455, 780, 1092, 5460, 420 \rangle$ satisfies 169th Sierpiński problem. The theorem is a consequence of (50), (15), and (20).

Let s be a positive natural number, n be a natural number, c be a complex number, f be an $(s+n)$ -element, complex-valued finite sequence, and g be a 4-element finite sequence. Let us note that $(c \cdot (f \upharpoonright (s-1))) \hat{\ } g$ is $(s+2+1)$ -element.

Now we state the proposition:

(145) Let us consider an $(s+1)$ -element, complex-valued finite sequence f . Suppose f satisfies 169th Sierpiński problem. Then $(10 \cdot (f \upharpoonright (s-1))) \hat{\ } \langle 12 \cdot f(s), 15 \cdot f(s), 20 \cdot f(s), 10 \cdot f(s+1) \rangle$ satisfies 169th Sierpiński problem. The theorem is a consequence of (50), (52), and (51).

Let s be a positive natural number, a, b, c, d be positive real numbers, and f be an $(s+1)$ -element, positive, real-valued finite sequence. One can verify that $\langle a \cdot f(s), b \cdot f(s), c \cdot f(s), d \cdot f(s+1) \rangle$ is positive yielding.

Now we state the proposition:

(146) Let us consider an $(s+1)$ -element, complex-valued finite sequence f . Suppose f satisfies 169th Sierpiński problem. Then $n \cdot f$ satisfies 169th Sierpiński problem. The theorem is a consequence of (51).

Let s be a positive natural number. Assume $s \geq 3$.

A solution of 169th Sierpiński problem of s is an $(s+1)$ -element, positive yielding, natural-valued finite sequence defined by

(Def. 18) *it* satisfies 169th Sierpiński problem.

Now we state the propositions:

(147) Suppose $s \geq 3$. Let us consider a solution of 169th Sierpiński problem f of s . Then $\text{rng SolutionsofSierp168}(f) \subseteq$ the set of all g where g is a solution of 169th Sierpiński problem of s . The theorem is a consequence of (146).

(148) Let us consider a positive natural number s . Suppose $s \geq 3$. Then the set of all f where f is a solution of 169th Sierpiński problem of s is infinite. The theorem is a consequence of (147).

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