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Free Product of Groups

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Summary. In this article the free product of groups is formalized in the Mizar system.

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Introduction

The concept of free groups and the free product of groups is widely known, cf. [1], [7], [12] for example. However, a formalization in the Mizar system (cf. [2], [4]) has not taken place until now, even if the overall hierarchy of algebraic structures in the MML seems to be quite rich [5], and formalization of group theory is also promising [11]. This article was primarily written as a necessary precursor to the formalization of the Seifert-Van Kampen theorem, hence the formalization loosely follows that of [6] and does not go into much detail about the properties of the free product or free groups. Anyway, we are motivated by another similar developments in another proof-assistants [3], [8], having in mind that this could result in reusing fundamental groups as described formally in [9] or even more categorical viewpoint as in [10].

After the preliminaries the *free atoms* of a family of groups $\{G_i\}_{i\in I}$ are introduced: they are the set of all pairs of the form (i, g) with $i \in I$ and $g \in G_i$. This choice allows for the G_i to have non-empty intersections with each other, or all be the same even; some fundamental properties are given in Section 3.

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The typical reduction relation for free products is then defined (Sect. 4) on the set of all finite sequences of free atoms. Afterwards the free product naturally appears as the quotient of the finite sequences of free atoms and the equivalence closure of the reduction relation (Sect. 5) with corresponding injections and factorization given in Sect. 6 and 7. The final section of the article concludes with the definition of the attribute `free-abelian`.

1. Preliminaries

Let us consider a finite sequence p. Now we state the propositions:

(1) If $\operatorname{len} p \neq 0$, then $p{\restriction}1 = \langle p(1) \rangle$.

(2) If $\operatorname{len} p \neq 0$, then $p_{\lfloor \operatorname{len} p -' 1} = \langle p(\operatorname{len} p) \rangle$.

Let us consider a function f and an object x. Now we state the propositions:

(3) If $x \in \operatorname{dom} f$, then $(\operatorname{uncurry}\langle f \rangle)(1, x) = f(x)$.

(4) If $x \in \operatorname{dom} f$, then $(\operatorname{commute}(\langle f \rangle))(x) = \langle f(x) \rangle$.

Let X be a finite sequence-membered set and R be a binary relation on X. One can verify that every reduction sequence w.r.t. R which is non trivial is also finite sequence-yielding. Now we state the proposition:

(5) Let us consider a non empty set I, an element i of I, and a group family F of I. If I is trivial, then $F(i)$ and $\prod F$ are isomorphic.

Observe that $\langle \emptyset^*, ^\frown \rangle$ is non empty and trivial and $\langle \emptyset^*, ^\frown, \varepsilon \rangle$ is non empty and trivial.

2. The Set of Free Atoms

From now on x, y, z denote objects, X denotes a set, I denotes a non empty set, i, j denote elements of I, M_0 denotes a multiplicative magma yielding function, M denotes a non empty, multiplicative magma yielding function, M_1, M_2, M_3 denote non empty multiplicative magmas, G denotes a group-like multiplicative magma family of I, and H denotes a group-like, associative multiplicative magma family of I.

Let us consider M_0. The functor FreeAtoms(M_0) yielding a binary relation is defined by the term

(Def. 1) G_α, where α is the support of M_0.

Now we state the propositions:

(6) $\langle x, y \rangle \in \operatorname{FreeAtoms}(M_0)$ if and only if $x \in \operatorname{dom} M_0$ and $y \in$ (the support of $M_0)(x)$.

(7) Let us consider an element i of dom M. Then $\langle i, x\rangle \in \mathrm{FreeAtoms}(M)$ if and only if $x \in$ the carrier of $M(i)$. The theorem is a consequence of (6).

(8) Let us consider a multiplicative magma family N of I. Then $\langle i, x\rangle \in \mathrm{FreeAtoms}(N)$ if and only if $x \in$ the carrier of $N(i)$. The theorem is a consequence of (6).

(9) $M_0 = \emptyset$ if and only if $\mathrm{FreeAtoms}(M_0) = \emptyset$. The theorem is a consequence of (7).

Observe that FreeAtoms($\emptyset$) is empty. Let us consider M. One can verify that FreeAtoms(M) is non empty. Let us consider I and G. Let us observe that FreeAtoms(G) is non empty.

3. Properties of the Set of Free Atoms

Now we state the propositions:

(10) $\mathrm{FreeAtoms}(M) = \bigcup$ the set of all $\{i\} \times$ (the carrier of $M(i)$) where i is an element of dom M. The theorem is a consequence of (6) and (7).

(11) $\mathrm{FreeAtoms}(\langle M_1\rangle) = \{1\} \times$ (the carrier of M_1).

(12) $\mathrm{FreeAtoms}(\langle M_1, M_2\rangle) = \{1\} \times$ (the carrier of M_1) $\cup \{2\} \times$ (the carrier of M_2).

(13) $\mathrm{FreeAtoms}(\langle M_1, M_2, M_3\rangle) = (\{1\} \times$ (the carrier of M_1) $\cup \{2\} \times$ (the carrier of M_2)) $\cup \{3\} \times$ (the carrier of M_3).

(14) Let us consider an element x_1 of M_1. Then

- (i) $\langle 1, x_1\rangle \in \mathrm{FreeAtoms}(\langle M_1\rangle)$, and
- (ii) $\langle 1, x_1\rangle \in \mathrm{FreeAtoms}(\langle M_1, M_2\rangle)$, and
- (iii) $\langle 1, x_1\rangle \in \mathrm{FreeAtoms}(\langle M_1, M_2, M_3\rangle)$.

The theorem is a consequence of (11), (12), and (13).

(15) Let us consider an element x_2 of M_2. Then

- (i) $\langle 2, x_2\rangle \in \mathrm{FreeAtoms}(\langle M_1, M_2\rangle)$, and
- (ii) $\langle 2, x_2\rangle \in \mathrm{FreeAtoms}(\langle M_1, M_2, M_3\rangle)$.

The theorem is a consequence of (12) and (13).

(16) Let us consider an element x_3 of M_3. Then $\langle 3, x_3\rangle \in \mathrm{FreeAtoms}(\langle M_1, M_2, M_3\rangle)$. The theorem is a consequence of (13).

(17) $\mathrm{FreeAtoms}(X \longmapsto M_1) = X \times$ (the carrier of M_1).

Let us consider a multiplicative magma yielding function N_0. Now we state the propositions:

(18) FreeAtoms(M_0+·N_0) $\subseteq$ FreeAtoms(M_0) $\cup$ FreeAtoms(N_0). The theorem is a consequence of (6).

(19) If M_0 tolerates N_0, then FreeAtoms(M_0+·N_0) = FreeAtoms(M_0) $\cup$ FreeAtoms(N_0). The theorem is a consequence of (18) and (6).

(20) Let us consider a finite sequence p of elements of FreeAtoms(G). Then there exists a finite sequence q of elements of FreeAtoms(G) such that

(i) $\operatorname{len} p = \operatorname{len} q$, and

(ii) for every natural number k and for every element i of I and for every element g of $G(i)$ such that $p(k) = \langle i, g \rangle$ there exists an element h of $G(i)$ such that $g \cdot h = \mathbf{1}_{G(i)}$ and $(\mathrm{Rev}(q))(k) = \langle i, h \rangle$.

Proof: Define $\mathcal{P}$[object, object] $\equiv$ there exists an element i of I and there exist elements g, h of $G(i)$ such that $p(\$_1) = \langle i, g \rangle$ and $g \cdot h = \mathbf{1}_{G(i)}$ and $\$_2 = \langle i, h \rangle$. Consider q' being a finite sequence of elements of FreeAtoms(G) such that $\operatorname{dom} q' = \operatorname{Seg} \operatorname{len} p$ and for every natural number k such that $k \in \operatorname{Seg} \operatorname{len} p$ holds $\mathcal{P}[k, q'(k)]$. □

In the sequel p, q denote finite sequences of elements of FreeAtoms(H), g, h denote elements of $H(i)$, and k denotes a natural number. Now we state the propositions:

(21) There exists q such that

(i) $\operatorname{len} p = \operatorname{len} q$, and

(ii) for every k, i, and g such that $p(k) = \langle i, g \rangle$ holds $(\mathrm{Rev}(q))(k) = \langle i, g^{-1} \rangle$.

The theorem is a consequence of (20).

(22) Let us consider an element g of $G(i)$. Then $\langle \langle i, g \rangle \rangle$ is a finite sequence of elements of FreeAtoms(G). The theorem is a consequence of (8).

(23) Let us consider an element g of $G(i)$, and an element h of $G(j)$. Then $\langle \langle i, g \rangle, \langle j, h \rangle \rangle$ is a finite sequence of elements of FreeAtoms(G). The theorem is a consequence of (8).

4. Reduction Relation

Let I be a set and G be a group-like multiplicative magma family of I. The functor ReductionRel(G) yielding a binary relation on $\langle \mathrm{FreeAtoms}(G)^*, \frown, \varepsilon \rangle$ is defined by

(Def. 2) if I is empty, then $it = \emptyset$ and if I is not empty, then there exists a non empty set I' and there exists a group-like multiplicative magma family G' of I' such that $I = I'$ and $G = G'$ and for every finite sequences p, q of elements of FreeAtoms(G'), $\langle p, q \rangle \in it$ iff there exist finite sequences s,

t of elements of FreeAtoms(G') and there exists an element i of I' such that $p = (s \frown \langle\langle i, \mathbf{1}_{G'(i)}\rangle\rangle) \frown t$ and $q = s \frown t$ or there exist finite sequences s, t of elements of FreeAtoms(G') and there exists an element i of I' and there exist elements g, h of $G'(i)$ such that $p = (s \frown \langle\langle i, g\rangle, \langle i, h\rangle\rangle) \frown t$ and $q = (s \frown \langle\langle i, g \cdot h\rangle\rangle) \frown t$.

Let us consider I and G. One can verify that the functor ReductionRel(G) is defined by

(Def. 3) for every finite sequences p, q of elements of FreeAtoms(G), $\langle p, q\rangle \in it$ iff there exist finite sequences s, t of elements of FreeAtoms(G) and there exists an element i of I such that $p = (s \frown \langle\langle i, \mathbf{1}_{G(i)}\rangle\rangle) \frown t$ and $q = s \frown t$ or there exist finite sequences s, t of elements of FreeAtoms(G) and there exists an element i of I and there exist elements g, h of $G(i)$ such that $p = (s \frown \langle\langle i, g\rangle, \langle i, h\rangle\rangle) \frown t$ and $q = (s \frown \langle\langle i, g \cdot h\rangle\rangle) \frown t$.

Now we state the propositions:

(24) Let us consider finite sequences p, q, r of elements of FreeAtoms(G). Suppose $\langle p, q\rangle \in$ ReductionRel(G). Then $\langle p \frown r, q \frown r\rangle$, $\langle r \frown p, r \frown q\rangle \in$ ReductionRel(G).

(25) Let us consider finite sequences p, q of elements of FreeAtoms(G), and elements g, h of $G(i)$. Then $\langle (p \frown \langle\langle i, g\rangle, \langle i, h\rangle\rangle) \frown q, (p \frown \langle\langle i, g \cdot h\rangle\rangle) \frown q\rangle \in$ ReductionRel(G). The theorem is a consequence of (8).

(26) Let us consider elements g, h of $G(i)$. Then $\langle\langle\langle i, g\rangle, \langle i, h\rangle\rangle, \langle\langle i, g \cdot h\rangle\rangle\rangle \in$ ReductionRel(G). The theorem is a consequence of (25).

(27) Let us consider finite sequences p, q of elements of FreeAtoms(G). Then $\langle (p \frown \langle\langle i, \mathbf{1}_{G(i)}\rangle\rangle) \frown q, p \frown q\rangle \in$ ReductionRel(G). The theorem is a consequence of (8).

(28) $\langle\langle\langle i, \mathbf{1}_{G(i)}\rangle\rangle, \emptyset\rangle \in$ ReductionRel(G). The theorem is a consequence of (27).

(29) (i) dom(ReductionRel(G)) $\subseteq$ (FreeAtoms(G))*, and

(ii) rng ReductionRel(G) = (FreeAtoms(G))*, and

(iii) field ReductionRel(G) = (FreeAtoms(G))*.

The theorem is a consequence of (27).

(30) Let us consider objects x, y. Suppose $\langle x, y\rangle \in$ ReductionRel(G). Then

(i) x is a finite sequence of elements of FreeAtoms(G), and

(ii) y is a finite sequence of elements of FreeAtoms(G).

The theorem is a consequence of (29).

(31) Let us consider finite sequences p, q of elements of FreeAtoms(G), and elements g, h of $G(i)$. Suppose $g \cdot h = \mathbf{1}_{G(i)}$. Then ReductionRel($G$) reduces

$(p \frown \langle\langle i, g\rangle, \langle i, h\rangle\rangle) \frown q$ to $p \frown q$. The theorem is a consequence of (25) and (27).

(32) Let us consider finite sequences p, q of elements of FreeAtoms(G). Suppose $\operatorname{len} p = \operatorname{len} q$ and for every natural number k and for every element i of I and for every elements g, h of $G(i)$ such that $p(k) = \langle i, g\rangle$ and $g \cdot h = \mathbf{1}_{G(i)}$ holds $(\operatorname{Rev}(q))(k) = \langle i, h\rangle$. Then ReductionRel($G$) reduces $p \frown q$ to $\emptyset$.

PROOF: Define $\mathcal{S}$[finite sequence, finite sequence] $\equiv$ if $\operatorname{len} \$_1 = \operatorname{len} \$_2$ and for every natural number k and for every element i of I and for every elements g, h of $G(i)$ such that $\$_1(k) = \langle i, g\rangle$ and $g \cdot h = \mathbf{1}_{G(i)}$ holds $(\operatorname{Rev}(\$_2))(k) = \langle i, h\rangle$, then ReductionRel($G$) reduces $\$_1 \frown \$_2$ to $\emptyset$. Define $\mathcal{P}$[natural number] $\equiv$ for every finite sequences p, q of elements of FreeAtoms(G) such that $\operatorname{len} p = \$_1$ holds $\mathcal{S}[p, q]$. $\mathcal{P}[0]$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every natural number n, $\mathcal{P}[n]$. □

(33) Suppose $\operatorname{len} p = \operatorname{len} q$ and for every k, i, and g such that $p(k) = \langle i, g\rangle$ holds $(\operatorname{Rev}(q))(k) = \langle i, g^{-1}\rangle$. Then

(i) ReductionRel(H) reduces $p \frown q$ to $\emptyset$, and

(ii) ReductionRel(H) reduces $q \frown p$ to $\emptyset$.

PROOF: For every k, i, and h such that $q(k) = \langle i, h\rangle$ holds $(\operatorname{Rev}(p))(k) = \langle i, h^{-1}\rangle$. □

(34) Let us consider finite sequences p, q. Suppose $\langle p, q\rangle \in$ ReductionRel(G). Then $\operatorname{len} p = \operatorname{len} q + 1$. The theorem is a consequence of (30).

(35) Let us consider finite sequences p, q of elements of FreeAtoms(G). Suppose ReductionRel(G) reduces p to q. Then

(i) $p = q$, or

(ii) $\operatorname{len} q < \operatorname{len} p$.

PROOF: Consider r being a reduction sequence w.r.t. ReductionRel(G) such that $r(1) = p$ and $r(\operatorname{len} r) = q$. Define $\mathcal{P}$[natural number] $\equiv$ if $\$_1 < \operatorname{len} r$, then $\operatorname{len} r(\$_1 + 1) + \$_1 = \operatorname{len} p$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k + 1]$. For every natural number k, $\mathcal{P}[k]$. □

Let us consider I and G. One can check that ReductionRel(G) is strongly-normalizing. Now we state the propositions:

(36) $\emptyset$ is a normal form w.r.t. ReductionRel(G). The theorem is a consequence of (29) and (34).

(37) Let us consider an element g of $G(i)$. Suppose $g \neq \mathbf{1}_{G(i)}$. Then $\langle\langle i, g\rangle\rangle$ is a normal form w.r.t. ReductionRel(G). The theorem is a consequence of (29).

(38) Let us consider finite sequences p, q_1, q_2 of elements of FreeAtoms(G). Suppose $\langle p, q_1\rangle$, $\langle p, q_2\rangle \in$ ReductionRel(G) and $q_1 \neq q_2$. Then

(i) there exist finite sequences s, t of elements of FreeAtoms(G) and there exists an element i of I and there exist elements f, g, h of $G(i)$ such that $p = (s \frown \langle\langle i, f\rangle, \langle i, g\rangle, \langle i, h\rangle\rangle) \frown t$ and ($q_1 = (s \frown \langle\langle i, f \cdot g\rangle, \langle i, h\rangle\rangle) \frown t$ and $q_2 = (s \frown \langle\langle i, f\rangle, \langle i, g \cdot h\rangle\rangle) \frown t$ or $q_1 = (s \frown \langle\langle i, f\rangle, \langle i, g \cdot h\rangle\rangle) \frown t$ and $q_2 = (s \frown \langle\langle i, f \cdot g\rangle, \langle i, h\rangle\rangle) \frown t$), or

(ii) there exist finite sequences r, s, t of elements of FreeAtoms(G) and there exist elements i, j of I such that $p = (((r \frown \langle\langle i, \mathbf{1}_{G(i)}\rangle\rangle) \frown s) \frown \langle\langle j, \mathbf{1}_{G(j)}\rangle\rangle) \frown t$ and ($q_1 = ((r \frown s) \frown \langle\langle j, \mathbf{1}_{G(j)}\rangle\rangle) \frown t$ and $q_2 = ((r \frown \langle\langle i, \mathbf{1}_{G(i)}\rangle\rangle) \frown s) \frown t$ or $q_1 = ((r \frown \langle\langle i, \mathbf{1}_{G(i)}\rangle\rangle) \frown s) \frown t$ and $q_2 = ((r \frown s) \frown \langle\langle j, \mathbf{1}_{G(j)}\rangle\rangle) \frown t$) or there exist elements g, h of $G(i)$ such that $p = (((r \frown \langle\langle i, g\rangle, \langle i, h\rangle\rangle) \frown s) \frown \langle\langle j, \mathbf{1}_{G(j)}\rangle\rangle) \frown t$ and ($q_1 = (((r \frown \langle\langle i, g \cdot h\rangle\rangle) \frown s) \frown \langle\langle j, \mathbf{1}_{G(j)}\rangle\rangle) \frown t$ and $q_2 = ((r \frown \langle\langle i, g\rangle, \langle i, h\rangle\rangle) \frown s) \frown t$ or $q_1 = ((r \frown \langle\langle i, g\rangle, \langle i, h\rangle\rangle) \frown s) \frown t$ and $q_2 = (((r \frown \langle\langle i, g \cdot h\rangle\rangle) \frown s) \frown \langle\langle j, \mathbf{1}_{G(j)}\rangle\rangle) \frown t$) or $p = (((r \frown \langle\langle j, \mathbf{1}_{G(j)}\rangle\rangle) \frown s) \frown \langle\langle i, g\rangle, \langle i, h\rangle\rangle) \frown t$ and ($q_1 = (((r \frown \langle\langle j, \mathbf{1}_{G(j)}\rangle\rangle) \frown s) \frown \langle\langle i, g \cdot h\rangle\rangle) \frown t$ and $q_2 = ((r \frown s) \frown \langle\langle i, g\rangle, \langle i, h\rangle\rangle) \frown t$ or $q_1 = ((r \frown s) \frown \langle\langle i, g\rangle, \langle i, h\rangle\rangle) \frown t$ and $q_2 = (((r \frown \langle\langle j, \mathbf{1}_{G(j)}\rangle\rangle) \frown s) \frown \langle\langle i, g \cdot h\rangle\rangle) \frown t$) or there exist elements g', h' of $G(j)$ such that $p = (((r \frown \langle\langle i, g\rangle, \langle i, h\rangle\rangle) \frown s) \frown \langle\langle j, g'\rangle, \langle j, h'\rangle\rangle) \frown t$ and ($q_1 = (((r \frown \langle\langle i, g \cdot h\rangle\rangle) \frown s) \frown \langle\langle j, g'\rangle, \langle j, h'\rangle\rangle) \frown t$ and $q_2 = (((r \frown \langle\langle i, g\rangle, \langle i, h\rangle\rangle) \frown s) \frown \langle\langle j, g' \cdot h'\rangle\rangle) \frown t$ or $q_1 = (((r \frown \langle\langle i, g\rangle, \langle i, h\rangle\rangle) \frown s) \frown \langle\langle j, g' \cdot h'\rangle\rangle) \frown t$ and $q_2 = (((r \frown \langle\langle i, g \cdot h\rangle\rangle) \frown s) \frown \langle\langle j, g'\rangle, \langle j, h'\rangle\rangle) \frown t$).

Let us consider I and H. Observe that ReductionRel(H) is subcommutative and ReductionRel(H) is complete and has unique normal form property. Now we state the propositions:

(39) Let us consider an element g of $H(i)$, and an element h of $H(j)$. Then $\langle\langle i, g\rangle\rangle$ and $\langle\langle j, h\rangle\rangle$ are convertible w.r.t. ReductionRel(H) if and only if $g = \mathbf{1}_{H(i)}$ and $h = \mathbf{1}_{H(j)}$ or $i = j$ and $g = h$. The theorem is a consequence of (8), (35), (29), (37), and (28).

(40) Let us consider finite sequences p_1, p_2, q_1, q_2 of elements of FreeAtoms(G). Suppose ReductionRel(G) reduces p_1 to q_1 and ReductionRel(G) reduces p_2 to q_2. Then ReductionRel(G) reduces $p_1 \frown p_2$ to $q_1 \frown q_2$. The theorem is a consequence of (30) and (24).

(41) Suppose I is trivial. Let us consider a non empty finite sequence p of elements of FreeAtoms(G). Then there exists an element g of $G(i)$ such that ReductionRel(G) reduces p to $\langle\langle i, g\rangle\rangle$.

Proof: Define $\mathcal{P}$[natural number] $\equiv$ for every non empty finite sequence p of elements of FreeAtoms(G) such that $\operatorname{len} p = \$_1 + 1$ there exists an element g of $G(i)$ such that ReductionRel(G) reduces p to $\langle\langle i, g\rangle\rangle$. $\mathcal{P}[0]$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$. Consider k being a natural number such that $\operatorname{len} p = 1+k$. □

(42) Let us consider finite sequences p_1, p_2, q_1, q_2 of elements of FreeAtoms(H). Suppose p_1 and q_1 are convertible w.r.t. ReductionRel(H) and p_2 and q_2 are convertible w.r.t. ReductionRel(H). Then $p_1 \frown p_2$ and $q_1 \frown q_2$ are convertible w.r.t. ReductionRel(H). The theorem is a consequence of (29) and (40).

Let I be a set and H be a group-like, associative multiplicative magma family of I. Observe that EqCl(ReductionRel(H)) is compatible. Now we state the proposition:

(43) Suppose $p \frown q$ is a normal form w.r.t. ReductionRel(H) and $\operatorname{len} p \neq 0$ and $\operatorname{len} q \neq 0$. Then $(p(\operatorname{len} p))_{\mathbf{1}} \neq (q(1))_{\mathbf{1}}$. The theorem is a consequence of (6), (8), (2), and (1).

5. Free Product of Groups

Let I be a set and H be a group-like, associative multiplicative magma family of I. The functor $*H$ yielding a strict multiplicative magma is defined by the term

(Def. 4) $\langle \mathrm{FreeAtoms}(H)^*, \frown, \varepsilon\rangle / _{\mathrm{EqCl}(\mathrm{ReductionRel}(H))}$.

From now on s, t denote elements of $*H$. Now we state the propositions:

(44) Let us consider a set I, and a group-like, associative multiplicative magma family H of I. Then $\mathbf{1}_{*H} = [\emptyset]_{\mathrm{EqCl}(\mathrm{ReductionRel}(H))}$.

(45) Let us consider an empty set I, and a group-like, associative multiplicative magma family H of I. Then the carrier of $*H = \{\mathbf{1}_{*H}\}$. The theorem is a consequence of (44).

Let I be a set and H be a group-like, associative multiplicative magma family of I. Let us observe that $*H$ is group-like and non empty.

Observe that the functor $*H$ yields a strict group. Let I be an empty set. Let us note that $*H$ is trivial. Now we state the proposition:

(46) Suppose $s = [p]_{\mathrm{EqCl}(\mathrm{ReductionRel}(H))}$ and $t = [q]_{\mathrm{EqCl}(\mathrm{ReductionRel}(H))}$. Then $s \cdot t = [p \frown q]_{\mathrm{EqCl}(\mathrm{ReductionRel}(H))}$.

Let us consider I, H, i, and g. The functor $[i, g]$ yielding an element of $*H$ is defined by the term

(Def. 5) $[\langle\langle i, g\rangle\rangle]_{\mathrm{EqCl}(\mathrm{ReductionRel}(H))}$.

Now we state the propositions:

(47) $\langle\langle i,\, g\rangle\rangle \in [i, g]$. The theorem is a consequence of (8).

(48) $[i, \mathbf{1}_{H(i)}] = \mathbf{1}_{*H}$. The theorem is a consequence of (8), (28), and (44).

(49) Let us consider an element g of $H(i)$, and an element h of $H(j)$. Then $[i, g] = [j, h]$ if and only if $g = \mathbf{1}_{H(i)}$ and $h = \mathbf{1}_{H(j)}$ or $i = j$ and $g = h$. The theorem is a consequence of (8) and (39).

(50) $[i, g] \cdot [i, h] = [i, g \cdot h]$. The theorem is a consequence of (8), (26), and (46).

(51) $[i, g]^{-1} = [i, g^{-1}]$. The theorem is a consequence of (50) and (48).

(52) Let us consider many sorted sets f, g indexed by I. Then $\mathrm{dom}(\mathrm{commute}(\langle\langle f, g\rangle\rangle)) = I$.

(53) Let us consider an element g of $G(i)$. Then $\langle\langle i,\, g\rangle\rangle = (\mathrm{commute}(\langle\langle(\text{the carrier of } G(i)) \longmapsto i, \mathrm{id}_\alpha\rangle\rangle))(g)$, where α is the carrier of $G(i)$. The theorem is a consequence of (4).

(54) $\mathrm{rng}\,\mathrm{commute}(\langle\langle(\text{the carrier of } G(i)) \longmapsto i, \mathrm{id}_\alpha\rangle\rangle) = (\{i\} \times (\text{the carrier of } G(i)))^1$, where α is the carrier of $G(i)$. The theorem is a consequence of (52) and (53).

6. On the Injection

Let us consider I, H, and i. The functor $\mathrm{injection}(H, i)$ yielding a function from $H(i)$ into $*H$ is defined by the term

(Def. 6) (the projection onto Classes EqCl(ReductionRel(H))) $\cdot$ (commute($\langle\langle$(the carrier of $H(i)$) $\longmapsto i, \mathrm{id}_\alpha\rangle\rangle$)), where α is the carrier of $H(i)$.

Now we state the proposition:

(55) $(\mathrm{injection}(H, i))(g) = [i, g]$. The theorem is a consequence of (47), (52), and (53).

Let us consider I, H, and i. One can check that $\mathrm{injection}(H, i)$ is multiplicative and one-to-one. Now we state the propositions:

(56) If I is trivial, then $\mathrm{injection}(H, i)$ is bijective. The theorem is a consequence of (41), (8), (55), (44), and (48).

(57) If I is trivial, then $H(i)$ and $*H$ are isomorphic. The theorem is a consequence of (56).

Let us consider I, H, and s. The functor $\mathrm{nf}\, s$ yielding a finite sequence of elements of FreeAtoms(H) is defined by

(Def. 7) *it* $\in s$ and *it* is a normal form w.r.t. ReductionRel(H).

Now we state the propositions:

(58) If $s = [p]_{\mathrm{EqCl}(\mathrm{ReductionRel}(H))}$, then $\mathrm{nf}\, s = \mathrm{nf}_{\mathrm{ReductionRel}(H)}(p)$. The theorem is a consequence of (29).

(59) If $t = [\mathrm{nf}\, s \restriction k]_{\mathrm{EqCl}(\mathrm{ReductionRel}(H))}$, then $\mathrm{nf}\, t = \mathrm{nf}\, s \restriction k$.
PROOF: $\mathrm{nf}\, s \restriction k$ is a normal form w.r.t. ReductionRel(H). □

(60) If $t = [(\mathrm{nf}\, s)_{\lfloor k}]_{\mathrm{EqCl}(\mathrm{ReductionRel}(H))}$, then $\mathrm{nf}\, t = (\mathrm{nf}\, s)_{\lfloor k}$.
PROOF: $(\mathrm{nf}\, s)_{\lfloor k}$ is a normal form w.r.t. ReductionRel(H). □

(61) $\mathrm{nf}\, \mathbf{1}_{*H} = \emptyset$. The theorem is a consequence of (44), (58), and (36).

(62) If $\mathrm{len}\, \mathrm{nf}\, s = 0$, then $s = \mathbf{1}_{*H}$. The theorem is a consequence of (44).

(63) If $g \neq \mathbf{1}_{H(i)}$, then $\mathrm{nf}[i, g] = \langle\langle i, g\rangle\rangle$. The theorem is a consequence of (37), (8), and (47).

(64) If $\mathrm{len}\, \mathrm{nf}\, s = 1$, then there exists i and there exists g such that $g \neq \mathbf{1}_{H(i)}$ and $s = [i, g]$. The theorem is a consequence of (6), (8), and (28).

(65) Suppose $((\mathrm{nf}\, s)(\mathrm{len}\, \mathrm{nf}\, s))_{\mathbf{1}} \neq ((\mathrm{nf}\, t)(1))_{\mathbf{1}}$. Then $\mathrm{nf}\, s \cdot t = \mathrm{nf}\, s \frown \mathrm{nf}\, t$.
PROOF: Consider p being an element of $\langle \mathrm{FreeAtoms}(H)^*, \frown, \varepsilon\rangle$ such that $s = [p]_{\mathrm{EqCl}(\mathrm{ReductionRel}(H))}$. Consider q being an element of $\langle \mathrm{FreeAtoms}(H)^*, \frown, \varepsilon\rangle$ such that $t = [q]_{\mathrm{EqCl}(\mathrm{ReductionRel}(H))}$. $s \cdot t = [p \frown q]_{\mathrm{EqCl}(\mathrm{ReductionRel}(H))}$. $\mathrm{nf}\, s \frown \mathrm{nf}\, t \in [p \frown q]_{\mathrm{EqCl}(\mathrm{ReductionRel}(H))}$. $\mathrm{nf}\, s \frown \mathrm{nf}\, t$ is a normal form w.r.t. ReductionRel(H). □

(66) Suppose $k \leqslant \mathrm{len}\, \mathrm{nf}\, s$. Then there exist elements s_1, s_2 of $*H$ such that

(i) $s = s_1 \cdot s_2$, and

(ii) $\mathrm{nf}\, s = \mathrm{nf}\, s_1 \frown \mathrm{nf}\, s_2$, and

(iii) $\mathrm{len}\, \mathrm{nf}\, s_1 = k$.

The theorem is a consequence of (46), (59), and (60).

Let us consider I and H. Let G be a group.

A family of homomorphisms from H into G is a function yielding many sorted set indexed by I defined by

(Def. 8) for every element i of I, $it(i)$ is a homomorphism from $H(i)$ to G.

The functor injection(H) yielding a family of homomorphisms from H into $*H$ is defined by

(Def. 9) for every element i of I, $it(i) = \mathrm{injection}(H, i)$.

Let G be a group and F be a family of homomorphisms from H into G. Let us observe that the functor uncurry F yields a function from FreeAtoms(H) into G. Let p be a finite sequence of elements of FreeAtoms(H) and F be a function from FreeAtoms(H) into G. Let us observe that the functor $F \cdot p$ yields a finite sequence of elements of G.

7. On the Factorization

Let us consider I, H, and s. The functor factorization(s) yielding a finite sequence of elements of $*H$ is defined by the term

(Def. 10) $(\text{uncurry injection}(H)) \cdot (\text{nf}\, s)$.

Now we state the propositions:

(67) factorization$(\mathbf{1}_{*H}) = \emptyset$. The theorem is a consequence of (61).

(68) Let us consider an element g of $H(i)$. Suppose $g \neq \mathbf{1}_{H(i)}$. Then factorization$([i, g]) = \langle [i, g] \rangle$.
PROOF: $\langle i, g \rangle \in \text{dom}(\text{uncurry injection}(H))$ and $(\text{uncurry injection}(H))(\langle i, g \rangle) = [i, g]$. □

(69) Suppose $((\text{nf}\, s)(\text{len nf}\, s))_{\mathbf{1}} \neq ((\text{nf}\, t)(1))_{\mathbf{1}}$. Then factorization$(s \cdot t) =$ factorization$(s) \frown$ factorization(t). The theorem is a consequence of (65).

(70) Let us consider an element s of $*H$, and a natural number k. Suppose $1 \leqslant k \leqslant \text{len factorization}(s)$. Then there exists an element i of I and there exists an element g of $H(i)$ such that $(\text{factorization}(s))(k) = [i, g]$. The theorem is a consequence of (6) and (8).

(71) $\prod \text{factorization}(s) = s$.
PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every element s of $*H$ such that $\text{len nf}\, s = \$_1$ holds $\prod \text{factorization}(s) = s$. $\mathcal{P}[0]$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$. □

Let us consider I and H. Let s be an element of $*H$.

Note that $\prod \text{factorization}(s)$ reduces to s.

8. Free-Abelian Groups

Let G_1, G_2 be groups. One can check that $\langle G_1, G_2 \rangle$ is group-like and associative as a multiplicative magma family of 2.

The functor $(G_1) * (G_2)$ yielding a strict group is defined by

(Def. 11) there exists a group-like, associative multiplicative magma family H of 2 such that $H = \langle G_1, G_2 \rangle$ and $it = *H$.

Let G be a group. We say that G is free if and only if

(Def. 12) there exists a cardinal number c such that G and $*c \longmapsto (\mathbb{Z}^+)$ are isomorphic.

Note that every group which is trivial is also free and $\mathbb{Z}^+$ is free.

Let c be a cardinal number. Let us note that $*c \longmapsto (\mathbb{Z}^+)$ is free as a group.

Now we state the proposition:

(72) Let us consider groups G, H. If G and H are isomorphic, then G is free iff H is free.

One can verify that there exists a group which is free.

Let G be a group. We say that G is free-abelian if and only if

(Def. 13) there exists a cardinal number c such that G and $\mathrm{sum}(c \longmapsto (\mathbb{Z}^+))$ are isomorphic.

One can check that every group which is trivial is also free-abelian and $\mathbb{Z}^+$ is free-abelian.

Let c be a cardinal number. Note that $\mathrm{sum}(c \longmapsto (\mathbb{Z}^+))$ is free-abelian as a group. Now we state the proposition:

(73) Let us consider groups G, H. If G and H are isomorphic, then G is free-abelian iff H is free-abelian.

Let us observe that there exists a group which is free-abelian.

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