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Surreal Dyadic and Real Numbers: A Formal Construction

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Summary. The concept of surreal numbers, as postulated by John Conway, represents a complex and multifaceted structure that encompasses a multitude of familiar number systems, including the real numbers, as integral components. In this study, we undertake the construction of the real numbers, commencing with the integers and dyadic rationals as preliminary steps. We proceed to contrast the resulting set of real numbers derived from our construction with the axiomatically defined set of real numbers based on Conway's axiom. Our findings reveal that both approaches culminate in the same set.

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Introduction

In his seminal book [3], John Conway introduces an axiomatic definition of real numbers. Conway call a number x *real* number if $-n < x < n$ for some integer n and

$$x \approx \{x-1, x-\frac{1}{2}, x-\frac{1}{3}, \ldots | x+1, x+\frac{1}{2}, x+\frac{1}{3}, \ldots\}. \tag{I.1}$$

This property is self-contained within the context of the surreal number system [9], which is expressed using only the explicitly outlined conditions of the system itself, and it does not rely on the standard real numbers used in mathematical analysis [6, 7]. Note that all these real numbers appear in the Day ω which contains other numbers like infinitesimals and ω and the days formed previously contain only dyadic rationals surreal numbers. Conway indicates these dyadic numbers as exemplars of the *reals*, yet does not formally establish a connection between the concepts of the reals or dyadic numbers and their counterparts in mathematical analysis. The map that converts dyadic rationals into their surreal counterparts, called as *Dali* function by Tøndering [20], has been analyzed in [11, 17, 20].

In our formalization, we introduce the *Dali* function in two steps. First, we define the recursive integer function $\mathrm{s}_{\mathbb{Z}}$, as follows: the base step is given as $\mathrm{s}_{\mathbb{Z}}(0) = 0$, while $\mathrm{s}_{\mathbb{Z}}(n+1) = \{\mathrm{s}_{\mathbb{Z}}(n) \mid \}$, $\mathrm{s}_{\mathbb{Z}}(-n-1) = \{ \mid \mathrm{s}_{\mathbb{Z}}(-n)\}$ for all $n > 0$ (see Def. 1). Then, $\mathrm{s}_{\mathbb{Z}}$ is used to define the base step of $\mathrm{s}_{\mathbb{D}}$ as follows: $\mathrm{s}_{\mathbb{D}}(d) = \mathrm{s}_{\mathbb{Z}}(d)$ for all $d \in \mathbb{Z}$ and $\{\mathrm{s}_{\mathbb{D}}(\frac{j}{2^p}) \mid \mathrm{s}_{\mathbb{D}}(\frac{j+1}{2^p})\}$ if $d = \frac{2j+1}{2^{p+1}}$ for some $j \in \mathbb{Z}$, $p \in \mathbb{N}$ (see Def. 5). We prove that the values of the function $\mathrm{s}_{\mathbb{D}}$ have `uniq-surreal`, i.e. $\mathrm{s}_{\mathbb{D}}(d) = \mathrm{Unique}_{\mathbf{No}}\mathrm{s}_{\mathbb{D}}(d)$ for every dyadic rational d, or more formally, $\mathrm{s}_{\mathbb{D}}(d)$ is equal to our construction of the $\approx$ equivalence class representative of $\mathrm{s}_{\mathbb{D}}(d)$. This property is important for the next stage of our construction.

We subsequently employ the function $\mathrm{s}_{\mathbb{S}}$ to establish a homeomorphism between the real numbers and their Conway representations. The fundamental premise of this construction is that the sequences of dyadic rational numbers $\{\frac{\lceil r\cdot 2^n-1\rceil}{2^n}\}_{n>0}$ and $\{\frac{\lfloor r\cdot 2^n+1\rfloor}{2^n}\}_{n>0}$ represent successive approximations of a given real number r. Moreover, these sequences are non-decreasing and non-increasing, respectively, and the relation the inequality $\frac{\lceil r\cdot 2^n-1\rceil}{2^n} < r < \frac{\lfloor r\cdot 2^n+1\rfloor}{2^n}$ is satisfied for all values of $n > 0$. This allows us to associate any real number r with the Conway number $\mathrm{s}_{\mathbb{R}}(r)$ (see Def. 6, Def. 7), which is equal to:

$$\mathrm{Unique}_{\mathbf{No}}\left\{\left\{\mathrm{s}_{\mathbb{D}}\left(\frac{\lceil r\cdot 2^n-1\rceil}{2^n}\right) \mid n\in\mathbb{N}\right\} \mid \left\{\mathrm{s}_{\mathbb{D}}\left(\frac{\lfloor r\cdot 2^n+1\rfloor}{2^n}\right) \mid n\in\mathbb{N}\right\}\right\} \tag{I.2}$$

Note that we apply additionally $\mathrm{Unique}_{\mathbf{No}}$ to obtain $\mathrm{s}_{\mathbb{R}}(d) = \mathrm{s}_{\mathbb{D}}(r)$ for each dyadic number d.

We prove that that the function $\mathrm{s}_{\mathbb{R}}$ preserves the identity elements for both addition (see Th47) and multiplication (see Th48). Furthermore, it is shown that it respects the operations of addition (see Th55) and multiplication (see Th57). We conduct also a comparison between the set of values of function $\mathrm{s}_{\mathbb{R}}$, and the set of *real* numbers that fulfils the Conway property. We prove that $\mathrm{s}_{\mathbb{R}}(r)$ satisfies Conway's property for all $r \in \mathbb{R}$ and that for each *real* number x, there exists a real number r such that $x \approx \mathrm{s}_{\mathbb{R}}(r)$.

As in our earlier Mizar formalizations of Conway numbers [16], a detailed exposition of the corresponding informal background can be found in [1] (see also the Coq [10] and Isabelle [12], [22] developments). Within the Mizar framework, we are naturally bound to set theory [2] (where cross-dependencies between formal notions can be explored more effectively using recent graph representation [19]), rather than to the inductive-inductive HoTT approach [5], which arguably provides a more natural foundation (cf. Sect. 11.6 of [21]). Having a formalization of real surreal numbers at hand, we may then follow the path of Conway, Kruskal, and Norton [8], with the goal of developing a surreal analysis, in which integration plays a central role [4], [18].

1. Mappings between Integers and Surreal Integers

From now on A, B, O denote ordinal numbers, o denotes an object, x, y, z denote surreal numbers, and n, m denote natural numbers.

The functor $\mathrm{s}_{\mathbb{Z}}$ yielding a many sorted set indexed by $\mathbb{Z}$ is defined by

(Def. 1) $it(0) = \mathbf{0_{No}}$ and $it(n+1) = \langle\{it(n)\}, \emptyset\rangle$ and $it(-(n+1)) = \langle\emptyset, \{it(-n)\}\rangle$.

Now we state the proposition:

(1) $\mathrm{s}_{\mathbb{Z}}(n), \mathrm{s}_{\mathbb{Z}}(-n) \in \mathrm{Day}\,n$.

PROOF: Define $\mathcal{P}$[natural number] $\equiv \mathrm{s}_{\mathbb{Z}}(\$_1), \mathrm{s}_{\mathbb{Z}}(-\$_1) \in \mathrm{Day}\,\$_1$. For every n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every n, $\mathcal{P}[n]$. □

Let i be an integer. Let us observe that $\mathrm{s}_{\mathbb{Z}}(i)$ is surreal. Now we state the propositions:

(2) If $x \in \mathrm{Day}\,n$, then $\mathrm{s}_{\mathbb{Z}}(-n) \leqslant x \leqslant \mathrm{s}_{\mathbb{Z}}(n)$.

PROOF: Define $\mathcal{P}$[natural number] $\equiv$ for every x such that $x \in \mathrm{Day}\,\$_1$ holds $\mathrm{s}_{\mathbb{Z}}(-\$_1) \leqslant x \leqslant \mathrm{s}_{\mathbb{Z}}(\$_1)$. $\mathcal{P}[0]$. For every n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every n, $\mathcal{P}[n]$. □

(3) Let us consider integers i, j. If $i < j$, then $\mathrm{s}_{\mathbb{Z}}(i) < \mathrm{s}_{\mathbb{Z}}(j)$.

PROOF: For every natural number k such that $k \geqslant 1$ holds $\mathrm{s}_{\mathbb{Z}}(n) < \mathrm{s}_{\mathbb{Z}}(n+k)$. For every natural number k such that $k \geqslant 1$ holds $\mathrm{s}_{\mathbb{Z}}(-(n+k)) < \mathrm{s}_{\mathbb{Z}}(-n)$. Consider I being a natural number such that $i = I$ or $i = -I$. Consider J being a natural number such that $j = J$ or $j = -J$. □

Let n be a positive natural number. Let us observe that $\mathrm{s}_{\mathbb{Z}}(n)$ is positive.

Now we state the propositions:

(4) (i) $n = \mathfrak{born}\,\mathrm{s}_{\mathbb{Z}}(n)$, and

(ii) $n = \mathfrak{born}\,\mathrm{s}_{\mathbb{Z}}(-n)$.

PROOF: $\mathrm{s}_{\mathbb{Z}}(n) \in \mathrm{Day}\,n$. For every O such that $\mathrm{s}_{\mathbb{Z}}(n) \in \mathrm{Day}\,O$ holds $n \subseteq O$. $\mathrm{s}_{\mathbb{Z}}(-n) \in \mathrm{Day}\,n$. For every O such that $\mathrm{s}_{\mathbb{Z}}(-n) \in \mathrm{Day}\,O$ holds $n \subseteq O$. □

(5) (i) $\mathfrak{born}_{\approx}\,\mathrm{s}_{\mathbb{Z}}(n) = n$, and

(ii) $\mathfrak{born}_{\approx}\,\mathrm{s}_{\mathbb{Z}}(-n) = n$.

PROOF: $\mathfrak{born}\,\mathrm{s}_{\mathbb{Z}}(n) = n$. For every surreal number y such that $y \approx \mathrm{s}_{\mathbb{Z}}(n)$ holds $\mathfrak{born}\,\mathrm{s}_{\mathbb{Z}}(n) \subseteq \mathfrak{born}\,y$. $\mathfrak{born}\,\mathrm{s}_{\mathbb{Z}}(-n) = n$. For every surreal number y such that $y \approx \mathrm{s}_{\mathbb{Z}}(-n)$ holds $\mathfrak{born}\,\mathrm{s}_{\mathbb{Z}}(-n) \subseteq \mathfrak{born}\,y$. □

(6) $\mathbf{0}_{\mathbf{No}} \leqslant \mathrm{s}_{\mathbb{Z}}(n)$. The theorem is a consequence of (3).

(7) $\mathrm{L}_{\mathrm{s}_{\mathbb{Z}}(-n)} = \emptyset = \mathrm{R}_{\mathrm{s}_{\mathbb{Z}}(n)}$.

PROOF: $\mathrm{L}_{\mathrm{s}_{\mathbb{Z}}(-n)} = \emptyset$. □

Let i be an integer. Note that $\mathrm{s}_{\mathbb{Z}}(i)$ is unique surreal.

Let us consider integers i, j. Now we state the propositions:

(8) If $\mathrm{s}_{\mathbb{Z}}(i) = \mathrm{s}_{\mathbb{Z}}(j)$, then $i = j$.

(9) $i < j$ if and only if $\mathrm{s}_{\mathbb{Z}}(i) < \mathrm{s}_{\mathbb{Z}}(j)$.

(10) Let us consider an integer i, and x. Then

(i) $\langle\{\mathrm{s}_{\mathbb{Z}}(i-1)\}, \{\mathrm{s}_{\mathbb{Z}}(i+1)\}\rangle$ is a surreal number, and

(ii) if $x = \langle\{\mathrm{s}_{\mathbb{Z}}(i-1)\}, \{\mathrm{s}_{\mathbb{Z}}(i+1)\}\rangle$, then $x \approx \mathrm{s}_{\mathbb{Z}}(i)$.

PROOF: Set $S = \mathrm{s}_{\mathbb{Z}}(i)$. $\mathrm{s}_{\mathbb{Z}}(i-1) < S$. $\mathrm{L}_S \ll \{x\} \ll \mathrm{R}_S$ by [14, (21)], [13, (43)]. $S < \mathrm{s}_{\mathbb{Z}}(i+1)$. □

(11) $\mathrm{s}_{\mathbb{Z}}(1) = \mathbf{1}_{\mathbf{No}}$.

(12) Let us consider an integer i. Then $-\mathrm{s}_{\mathbb{Z}}(i) = \mathrm{s}_{\mathbb{Z}}(-i)$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv -\mathrm{s}_{\mathbb{Z}}(\$_1) = \mathrm{s}_{\mathbb{Z}}(-\$_1)$. If $\mathcal{P}[n]$, then $\mathcal{P}[n+1]$ by [15, (22),(7),(21)]. $\mathcal{P}[n]$. Consider o being a natural number such that $i = o$ or $i = -o$. □

(13) $\mathrm{s}_{\mathbb{Z}}(n) + \mathrm{s}_{\mathbb{Z}}(m) = \mathrm{s}_{\mathbb{Z}}(n+m)$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv \mathrm{s}_{\mathbb{Z}}(\$_1) + \mathbf{1}_{\mathbf{No}} = \mathrm{s}_{\mathbb{Z}}(\$_1 + 1)$. $\mathrm{s}_{\mathbb{Z}}(0) = \mathbf{0}_{\mathbf{No}}$ and $\mathrm{s}_{\mathbb{Z}}(1) = \mathbf{1}_{\mathbf{No}}$. For every n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every n, $\mathcal{P}[n]$. Define $\mathcal{Q}[\text{natural number}] \equiv \mathrm{s}_{\mathbb{Z}}(n) + \mathrm{s}_{\mathbb{Z}}(\$_1) = \mathrm{s}_{\mathbb{Z}}(n + \$_1)$. For every m such that $\mathcal{Q}[m]$ holds $\mathcal{Q}[m+1]$. For every m, $\mathcal{Q}[m]$. □

Let us consider integers i, j. Now we state the propositions:

(14) $\mathrm{s}_{\mathbb{Z}}(i) + \mathrm{s}_{\mathbb{Z}}(j) \approx \mathrm{s}_{\mathbb{Z}}(i+j)$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every n and m such that $n+m = \$_1$ holds $\mathrm{s}_{\mathbb{Z}}(n) + \mathrm{s}_{\mathbb{Z}}(-m) \approx \mathrm{s}_{\mathbb{Z}}(n-m)$. $\mathcal{P}[0]$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$. Consider k being a natural number such that $i = k$ or $i = -k$. Consider n being a natural number such that $j = n$ or $j = -n$. □

(15) $\mathrm{s}_{\mathbb{Z}}(i) \cdot \mathrm{s}_{\mathbb{Z}}(j) \approx \mathrm{s}_{\mathbb{Z}}(i \cdot j)$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every n and m such that $n+m = \$_1$ holds $\mathrm{s}_{\mathbb{Z}}(n) \cdot \mathrm{s}_{\mathbb{Z}}(m) \approx \mathrm{s}_{\mathbb{Z}}(n \cdot m)$. For every natural number k such that for every n such that $n < k$ holds $\mathcal{P}[n]$ holds $\mathcal{P}[k]$. For every natural number k, $\mathcal{P}[k]$. Consider k being a natural number such that $i = k$ or $i = -k$. Consider n being a natural number such that $j = n$ or $j = -n$. □

(16) If $x = \langle\{y\}, \emptyset\rangle$ and $y < \mathbf{0}_{\mathbf{No}}$, then $x \approx \mathbf{0}_{\mathbf{No}}$.

(17) Suppose $x = \langle\{y\}, \emptyset\rangle$ and $\mathfrak{b}\mathrm{orn}\, x$ is finite and $\mathbf{0}_{\mathbf{No}} \leqslant y$. Then there exists a natural number n such that

(i) $x \approx s_{\mathbb{Z}}(n+1)$, and

(ii) $s_{\mathbb{Z}}(n) \leqslant y < s_{\mathbb{Z}}(n+1)$, and

(iii) $n \in \mathfrak{born}\, x$.

PROOF: Reconsider $a = \mathfrak{born}\, x$ as a natural number. Define $\mathcal{O}$[natural number] $\equiv \mathrm{L}_x \ll \{s_{\mathbb{Z}}(\$_1)\}$. $\mathcal{O}[a]$. Consider k being a natural number such that $\mathcal{O}[k]$ and for every natural number n such that $\mathcal{O}[n]$ holds $k \leqslant n$. $k \neq 0$. Reconsider $k_1 = k - 1$ as a natural number. For every z such that $\mathrm{L}_x \ll \{z\} \ll \mathrm{R}_x$ holds $\mathfrak{born}\, s_{\mathbb{Z}}(k) \subseteq \mathfrak{born}\, z$. $s_{\mathbb{Z}}(k_1) \leqslant y$. $k_1 \subseteq \mathfrak{born}\, y$. □

2. DYADIC NUMBERS

Let r be a rational number. We say that r is dyadic-like if and only if

(Def. 2) there exists a natural number n such that $\operatorname{den} r = 2^n$.

Now we state the proposition:

(18) Let us consider a rational number r. Then r is dyadic-like if and only if there exists an integer i and there exists a natural number n such that $r = \frac{i}{2^n}$.

PROOF: If r is dyadic-like, then there exists an integer i and there exists a natural number n such that $r = \frac{i}{2^n}$. Consider w being a natural number such that $i = (\operatorname{num} r) \cdot w$ and $2^n = (\operatorname{den} r) \cdot w$. Consider t being an element of $\mathbb{N}$ such that $w = 2^t$ and $t \leqslant n$. □

Let i be an integer and n be a natural number. Let us observe that $\frac{i}{2^n}$ is dyadic-like and every integer is dyadic-like. Let x be a dyadic-like rational number. Note that $-x$ is dyadic-like. Let y be a dyadic-like rational number. One can check that $x + y$ is dyadic-like and $x + y$ is dyadic-like and $x \cdot y$ is dyadic-like.

The functor $\mathbb{D}$ yielding a set is defined by

(Def. 3) $o \in it$ iff o is a dyadic-like rational number.

Let us observe that $\mathbb{D}$ is rational-membered and non empty and every element of $\mathbb{D}$ is dyadic-like. A $\mathbb{D}$yadic is a dyadic-like rational number. From now on d, d_1, d_2 denote $\mathbb{D}$yadics. Let n be a natural number. The functor $\mathbb{D}(n)$ yielding a subset of $\mathbb{D}$ is defined by

(Def. 4) $d \in it$ iff there exists an integer i such that $d = \frac{i}{2^n}$.

In the sequel i, j denote integers and n, m, p denote natural numbers.

Now we state the propositions:

(19) If $n \leqslant m$, then $\mathbb{D}(n) \subseteq \mathbb{D}(m)$.

(20) $d \in (\mathbb{D}(n+1)) \setminus (\mathbb{D}(n))$ if and only if there exists an integer i such that $d = \frac{2 \cdot i+1}{2^{n+1}}$.
PROOF: If $d \in (\mathbb{D}(n+1)) \setminus (\mathbb{D}(n))$, then there exists an integer i such that $d = \frac{2 \cdot i+1}{2^{n+1}}$. $d \notin \mathbb{D}(n)$. □

(21) $\mathbb{Z} = \mathbb{D}(0)$.

(22) $\operatorname{rng} s_{\mathbb{Z}} \subseteq \mathrm{Day}\mathbb{N}$. The theorem is a consequence of (1).

(23) (i) d is an integer, or

(ii) there exists p and there exists i such that $d = \frac{2 \cdot i+1}{2^{p+1}}$.

PROOF: Consider i being an integer, n being a natural number such that $d = \frac{i}{2^n}$. Define $\mathcal{M}[\text{natural number}] \equiv d \in \mathbb{D}(\$_1 + 1)$. $n \neq 0$. Consider m being a natural number such that $\mathcal{M}[m]$ and for every natural number n such that $\mathcal{M}[n]$ holds $m \leqslant n$. $d \notin \mathbb{D}(m)$. There exists an integer i such that $d = \frac{2 \cdot i+1}{2^{m+1}}$. □

3. Mappings between Dyadic Numbers and Surreal Dyadic Numbers

The functor $s_{\mathbb{D}}$ yielding a many sorted set indexed by $\mathbb{D}$ is defined by

(Def. 5) $it(i) = s_{\mathbb{Z}}(i)$ and $it(\frac{2 \cdot j+1}{2^{p+1}}) = \langle \{it(\frac{j}{2^p})\}, \{it(\frac{j+1}{2^p})\} \rangle$.

Let us consider d. Note that $s_{\mathbb{D}}(d)$ is surreal. Now we state the propositions:

(24) $d_1 < d_2$ if and only if $s_{\mathbb{D}}(d_1) < s_{\mathbb{D}}(d_2)$. The theorem is a consequence of (18).

(25) (i) if $\mathbf{0_{No}} \leqslant z$ and $z \in \mathrm{Day}n$ and $z \not\approx s_{\mathbb{D}}(n)$, then there exist natural numbers x, y, p such that $z \approx s_{\mathbb{D}}(x + \frac{y}{2^p})$ and $y < 2^p$ and $x + p < n$, and

(ii) for every natural numbers x, y, p such that $y < 2^p$ and $x + p < n$ holds $\mathbf{0_{No}} \leqslant s_{\mathbb{D}}(x + \frac{y}{2^p}) \in \mathrm{Day}n$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every surreal number s such that $s \in \mathrm{Day}\$_1$ and $\mathbf{0_{No}} \leqslant s$ holds $s \approx s_{\mathbb{D}}(\$_1)$ or there exists a $\mathbb{D}$yadic d and there exist natural numbers x, y, p such that $s \approx s_{\mathbb{D}}(d)$ and $y < 2^p$ and $d = x + \frac{y}{2^p}$ and $x + p < \$_1$ and for every natural numbers x, y, p such that $y < 2^p$ and $x + p < \$_1$ holds $\mathbf{0_{No}} \leqslant s_{\mathbb{D}}(x + \frac{y}{2^p}) \in \mathrm{Day}\$_1$. $\mathcal{P}[0]$. For every n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every n, $\mathcal{P}[n]$. If $\mathbf{0_{No}} \leqslant z$ and $z \in \mathrm{Day}n$ and $z \not\approx s_{\mathbb{D}}(n)$, then there exist natural numbers x, y, p such that $z \approx s_{\mathbb{D}}(x + \frac{y}{2^p})$ and $y < 2^p$ and $x + p < n$. □

(26) If $2 \cdot m + 1 < 2^p$, then $\mathfrak{b}\mathrm{orn}\, s_{\mathbb{D}}(n + \frac{2 \cdot m+1}{2^p}) = n + p + 1$.
PROOF: Set $d = n + \frac{2 \cdot m+1}{2^p}$. $s_{\mathbb{D}}(d) \not\approx s_{\mathbb{D}}(n+p)$. $\mathbf{0_{No}} \leqslant s_{\mathbb{D}}(d) \in \mathrm{Day}(n+p+1)$. For every O such that $s_{\mathbb{D}}(d) \in \mathrm{Day}O$ holds $n + p + 1 \subseteq O$. □

(27) $\mathrm{s}_{\mathbb{D}}(-d) = -\mathrm{s}_{\mathbb{D}}(d)$.
PROOF: Define $\mathcal{P}$[natural number] $\equiv$ for every d such that $d \in \mathbb{D}(\$_1)$ holds $\mathrm{s}_{\mathbb{D}}(-d) = -\mathrm{s}_{\mathbb{D}}(d)$. $\mathcal{P}[0]$. If $\mathcal{P}[n]$, then $\mathcal{P}[n+1]$. $\mathcal{P}[n]$. Consider i being an integer, n being a natural number such that $d = \frac{i}{2^n}$. □

(28) If $0 \leqslant d$ and d is not an integer, then there exist natural numbers n, m, p such that $d = n + \frac{2\cdot m+1}{2^{p+1}}$ and $2 \cdot m + 1 < 2^{p+1}$.
PROOF: Consider p, i such that $d = \frac{2\cdot i+1}{2^{p+1}}$. $i \geqslant 0$. □

(29) $0 \leqslant d$ if and only if $\mathbf{0_{No}} \leqslant \mathrm{s}_{\mathbb{D}}(d)$. The theorem is a consequence of (24).

(30) $\mathrm{s}_{\mathbb{D}}(d) \in \mathfrak{B}\mathrm{orn}_{\approx}\, \mathrm{s}_{\mathbb{D}}(d)$. The theorem is a consequence of (28), (29), (26), (27), (24), and (25).

(31) Suppose $\mathfrak{b}\mathrm{orn}\, x$ is finite and $\overline{\overline{\mathrm{L}_x}} \oplus \overline{\overline{\mathrm{R}_x}} \subseteq 1$. Then there exists an integer i such that $x \approx \mathrm{s}_{\mathbb{Z}}(i)$. The theorem is a consequence of (16), (17), and (12).

Let us consider natural numbers x_1, x_2, y_1, y_2, p_1, p_2. Now we state the propositions:

(32) If $x_1 + \frac{y_1}{2^{p_1}} = x_2 + \frac{y_2}{2^{p_2}}$ and $y_1 < 2^{p_1}$ and $y_2 < 2^{p_2}$, then $x_1 = x_2$.

(33) If $x_1 + \frac{y_1}{2^{p_1}} < x_2 + \frac{y_2}{2^{p_2}}$ and $y_1 < 2^{p_1}$ and $y_2 < 2^{p_2}$, then $x_1 \leqslant x_2$.

(34) Let us consider natural numbers x_1, x_2, p_1, p_2. If $\frac{2\cdot x_1+1}{2^{p_1}} = \frac{x_2}{2^{p_2}}$, then $p_1 \leqslant p_2$.

(35) If $x \in \mathrm{Day}\, n$, then there exists a $\mathbb{D}$yadic d such that $x \approx \mathrm{s}_{\mathbb{D}}(d)$ and $\mathrm{s}_{\mathbb{D}}(d) \in \mathrm{Day}\, n$. The theorem is a consequence of (30), (25), (28), (32), (34), (26), and (27).

(36) There exists n such that $\mathrm{s}_{\mathbb{D}}(d) \in \mathrm{Day}\, n$. The theorem is a consequence of (27).

Let us consider d. One can verify that $\mathrm{s}_{\mathbb{D}}(d)$ is unique surreal. Now we state the propositions:

(37) x is a unique surreal number and $\mathfrak{b}\mathrm{orn}\, x$ is finite if and only if there exists a $\mathbb{D}$yadic d such that $x = \mathrm{s}_{\mathbb{D}}(d)$. The theorem is a consequence of (35) and (36).

(38) Let us consider an integer i, a natural number p, and a surreal number x. Then

(i) $\langle\{\mathrm{s}_{\mathbb{D}}(\frac{i}{2^p})\}, \{\mathrm{s}_{\mathbb{D}}(\frac{i+2}{2^p})\}\rangle$ is a surreal number, and

(ii) if $x = \langle\{\mathrm{s}_{\mathbb{D}}(\frac{i}{2^p})\}, \{\mathrm{s}_{\mathbb{D}}(\frac{i+2}{2^p})\}\rangle$, then $x \approx \mathrm{s}_{\mathbb{D}}(\frac{i+1}{2^p})$.

The theorem is a consequence of (24), (10), and (27).

(39) $\mathrm{s}_{\mathbb{D}}(d_1) + \mathrm{s}_{\mathbb{D}}(d_2) \approx \mathrm{s}_{\mathbb{D}}(d_1 + d_2)$.
PROOF: Define $\mathcal{P}$[natural number] $\equiv$ for every natural numbers n_1, n_2 such that $n_1 + n_2 \leqslant \$_1$ and $n_1 \leqslant n_2$ for every d_1 and d_2 such that $d_1 \in \mathbb{D}(n_1)$ and $d_2 \in \mathbb{D}(n_2)$ holds $\mathrm{s}_{\mathbb{D}}(d_1) + \mathrm{s}_{\mathbb{D}}(d_2) \approx \mathrm{s}_{\mathbb{D}}(d_1 + d_2)$. $\mathcal{P}[0]$. If $\mathcal{P}[m]$, then

$\mathcal{P}[m+1]$. $\mathcal{P}[m]$. Consider i_1 being an integer, n_1 being a natural number such that $d_1 = \frac{i_1}{2^{n_1}}$. Consider i_2 being an integer, n_2 being a natural number such that $d_2 = \frac{i_2}{2^{n_2}}$. $d_2 \in \mathbb{D}(n_2) \subseteq \mathbb{D}(n_1 + n_2)$. □

(40) $\mathrm{s}_{\mathbb{D}}(d_1) \cdot \mathrm{s}_{\mathbb{D}}(d_2) \approx \mathrm{s}_{\mathbb{D}}(d_1 \cdot d_2)$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every natural numbers n_1, n_2 such that $n_1+n_2 \leqslant \$_1$ and $n_1 \leqslant n_2$ for every d_1 and d_2 such that $d_1 \in \mathbb{D}(n_1)$ and $d_2 \in \mathbb{D}(n_2)$ holds $\mathrm{s}_{\mathbb{D}}(d_1) \cdot \mathrm{s}_{\mathbb{D}}(d_2) \approx \mathrm{s}_{\mathbb{D}}(d_1 \cdot d_2)$. $\mathcal{P}[0]$. If $\mathcal{P}[m]$, then $\mathcal{P}[m+1]$. $\mathcal{P}[m]$. Consider i_1 being an integer, n_1 being a natural number such that $d_1 = \frac{i_1}{2^{n_1}}$. Consider i_2 being an integer, n_2 being a natural number such that $d_2 = \frac{i_2}{2^{n_2}}$. $d_2 \in \mathbb{D}(n_2) \subseteq \mathbb{D}(n_1 + n_2)$. □

4. Mappings between Real Numbers and Surreal Real Numbers

In the sequel r, r_1, r_2 denote real numbers.

The functor $\mathrm{s}'_{\mathbb{R}}$ yielding a many sorted set indexed by $\mathbb{R}$ is defined by

(Def. 6) $it(r) = \langle$the set of all $\mathrm{s}_{\mathbb{D}}(\frac{\lceil r\cdot 2^n - 1\rceil}{2^n})$, the set of all $\mathrm{s}_{\mathbb{D}}(\frac{\lfloor r\cdot 2^m + 1\rfloor}{2^m})\rangle$.

Now we state the proposition:

(41) $\frac{\lceil r\cdot 2^n - 1\rceil}{2^n} < r < \frac{\lfloor r\cdot 2^n + 1\rfloor}{2^n}$.

Let us consider r. Note that $\mathrm{s}'_{\mathbb{R}}(r)$ is surreal.

The functor $\mathrm{s}_{\mathbb{R}}$ yielding a many sorted set indexed by $\mathbb{R}$ is defined by

(Def. 7) $it(r) = \mathrm{Unique}_{\mathbf{No}}(\mathrm{s}'_{\mathbb{R}}(r))$.

Let us consider r. Note that $\mathrm{s}_{\mathbb{R}}(r)$ is surreal and $\mathrm{s}_{\mathbb{R}}(r)$ is unique surreal. Now we state the propositions:

(42) $x \in \mathrm{L}_{\mathrm{s}'_{\mathbb{R}}(r)}$ if and only if there exists n such that $x = \mathrm{s}_{\mathbb{D}}(\frac{\lceil r\cdot 2^n - 1\rceil}{2^n})$.

(43) $x \in \mathrm{R}_{\mathrm{s}'_{\mathbb{R}}(r)}$ if and only if there exists n such that $x = \mathrm{s}_{\mathbb{D}}(\frac{\lfloor r\cdot 2^n + 1\rfloor}{2^n})$.

(44) $\mathrm{s}_{\mathbb{D}}(\frac{\lceil r\cdot 2^n - 1\rceil}{2^n}) < \mathrm{s}'_{\mathbb{R}}(r) < \mathrm{s}_{\mathbb{D}}(\frac{\lfloor r\cdot 2^n + 1\rfloor}{2^n})$. The theorem is a consequence of (42) and (43).

(45) Let us consider integers i_1, i_2, and natural numbers n_1, n_2. Suppose $\frac{i_1}{2^{n_1}} < \frac{i_2}{2^{n_2}}$. Then $\frac{i_1}{2^{n_1}} < \frac{i_1\cdot 2^{n_2}\cdot 2+1}{2^{n_1+n_2+1}} \leqslant \frac{i_2\cdot 2^{n_1}\cdot 2-1}{2^{n_1+n_2+1}} < \frac{i_2}{2^{n_2}}$.

(46) $\mathrm{s}'_{\mathbb{R}}(d) \approx \mathrm{s}_{\mathbb{D}}(d) = \mathrm{s}_{\mathbb{R}}(d)$.

PROOF: Set $R_3 = \mathrm{s}'_{\mathbb{R}}(d)$. Set $D_2 = \mathrm{s}_{\mathbb{D}}(d)$. Consider i being an integer, k being a natural number such that $d = \frac{i}{2^k}$. $\mathrm{L}_{R_3} \ll \{D_2\} \ll \mathrm{R}_{R_3}$. For every z such that $\mathrm{L}_{R_3} \ll \{z\} \ll \mathrm{R}_{R_3}$ holds $\mathfrak{born}\, D_2 \subseteq \mathfrak{born}\, z$. □

(47) $\mathrm{s}_{\mathbb{R}}(0) = \mathbf{0}_{\mathbf{No}}$. The theorem is a consequence of (46).

(48) $\mathrm{s}_{\mathbb{R}}(1) = \mathbf{1}_{\mathbf{No}}$. The theorem is a consequence of (46) and (11).

(49) $\mathfrak{born}\, \mathrm{s}'_{\mathbb{R}}(r) \subseteq \omega$.

(50) $\mathrm{s}'_{\mathbb{R}}(r_1) < \mathrm{s}'_{\mathbb{R}}(r_2)$ if and only if $r_1 < r_2$.
PROOF: Set $R_1 = \mathrm{s}'_{\mathbb{R}}(r_1)$. Set $R_2 = \mathrm{s}'_{\mathbb{R}}(r_2)$. If $R_1 < R_2$, then $r_1 < r_2$. Consider k being a natural number such that $\frac{1}{2^k} \leqslant r_2 - r_1$. Set $K_2 = 2^{k+1}$. $\mathrm{s}_{\mathbb{D}}(\frac{\lfloor K_2 \cdot r_1 + 1 \rfloor}{K_2}) \leqslant \mathrm{s}_{\mathbb{D}}(\frac{\lceil r_2 \cdot K_2 - 1 \rceil}{K_2})$. $R_1 < \mathrm{s}_{\mathbb{D}}(\frac{\lfloor K_2 \cdot r_1 + 1 \rfloor}{K_2})$. $\mathrm{s}_{\mathbb{D}}(\frac{\lceil r_2 \cdot K_2 - 1 \rceil}{K_2}) \leqslant R_2$. □

(51) $\mathrm{s}_{\mathbb{R}}(r_1) < \mathrm{s}_{\mathbb{R}}(r_2)$ if and only if $r_1 < r_2$.
PROOF: If $\mathrm{s}_{\mathbb{R}}(r_1) < \mathrm{s}_{\mathbb{R}}(r_2)$, then $r_1 < r_2$. $\mathrm{s}_{\mathbb{R}}(r_1) < \mathrm{s}'_{\mathbb{R}}(r_2)$. □

Let r be a positive real number. One can check that $\mathrm{s}_{\mathbb{R}}(r)$ is positive. Now we state the propositions:

(52) $\mathfrak{b}\mathrm{orn}\, \mathrm{s}_{\mathbb{R}}(r) = \omega$ if and only if r is not a $\mathbb{D}$yadic. The theorem is a consequence of (37), (46), (49), (35), and (51).

(53) If $r_1 < r_2$, then there exists n such that $\frac{\lfloor r_1 \cdot 2^n + 1 \rfloor}{2^n} < r_2$.

(54) If $r_1 < r_2$, then there exists n such that $r_1 < \frac{\lceil r_2 \cdot 2^n - 1 \rceil}{2^n}$.

(55) $\mathrm{s}_{\mathbb{R}}(r_1) + \mathrm{s}_{\mathbb{R}}(r_2) \approx \mathrm{s}_{\mathbb{R}}(r_1 + r_2)$.

(56) $-\mathrm{s}_{\mathbb{R}}(r) \approx \mathrm{s}_{\mathbb{R}}(-r)$.

(57) $\mathrm{s}_{\mathbb{R}}(r_1) \cdot \mathrm{s}_{\mathbb{R}}(r_2) \approx \mathrm{s}_{\mathbb{R}}(r_1 \cdot r_2)$.

(58) If $n > 0$, then $\mathrm{s}_{\mathbb{Z}}(n)^{-1} \approx \mathrm{s}_{\mathbb{R}}(\frac{1}{n})$. The theorem is a consequence of (9), (46), (57), and (48).

5. *Real Surreal Numbers

Let x be a surreal number. The functor $\mathrm{real}_{\approx}(x)$ yielding a surreal number is defined by

(Def. 8) $\mathrm{L}_{it} =$ the set of all $x - \mathrm{s}_{\mathbb{Z}}(n)^{-1}$ where n is a positive natural number and $\mathrm{R}_{it} =$ the set of all $x + \mathrm{s}_{\mathbb{Z}}(n)^{-1}$ where n is a positive natural number.

We say that x is *real if and only if

(Def. 9) $x \approx \mathrm{real}_{\approx}(x)$ and there exists a natural number n such that $\mathrm{s}_{\mathbb{Z}}(-n) < x < \mathrm{s}_{\mathbb{Z}}(n)$.

Now we state the propositions:

(59) Let us consider a positive natural number n.
Then $x - \mathrm{s}_{\mathbb{Z}}(n)^{-1} < \mathrm{real}_{\approx}(x) < x + \mathrm{s}_{\mathbb{Z}}(n)^{-1}$.

(60) If $x \approx y$, then $\mathrm{real}_{\approx}(x) \approx \mathrm{real}_{\approx}(y)$.

(61) If $x \approx y$ and x is *real, then y is *real.

Let r be a real number. One can check that $\mathrm{s}'_{\mathbb{R}}(r)$ is *real and $\mathrm{s}_{\mathbb{R}}(r)$ is *real and there exists a unique surreal number which is *real. Now we state the proposition:

(62) x is *real if and only if there exists r such that $x \approx \mathrm{s}_{\mathbb{R}}(r)$.
PROOF: If x is *real, then there exists r such that $x \approx \mathrm{s}_{\mathbb{R}}(r)$. □

Let x be a *real surreal number. One can check that $-x$ is *real. Let y be a *real surreal number. Let us note that $x + y$ is *real and $x \cdot y$ is *real.

6. Surreal Ordinals

Let x be a surreal number. We say that x is **On** if and only if

(Def. 10) $\mathrm{R}_x = \emptyset$.

Let us observe that $\mathbf{0}_{\mathbf{No}}$ is **On**. Let us consider n. One can check that $\mathrm{s}_{\mathbb{Z}}(n)$ is **On** and there exists a unique surreal number which is **On**. Let A be an ordinal number. The functor $\mathrm{ordinal}_{\mathbf{On}}(A)$ yielding a set is defined by

(Def. 11) there exists a transfinite sequence S such that $it = S(A)$ and $\mathrm{dom}\, S = \mathrm{succ}\, A$ and for every O such that $\mathrm{succ}\, O \in \mathrm{succ}\, A$ holds $S(\mathrm{succ}\, O) = \langle \{S(O)\}, \emptyset \rangle$ and for every O such that $O \in \mathrm{succ}\, A$ and O is limit ordinal holds $S(O) = \langle \mathrm{rng}(S \restriction O), \emptyset \rangle$.

Now we state the propositions:

(63) Let us consider a transfinite sequence S. Suppose $\mathrm{dom}\, S = \mathrm{succ}\, A$ and for every O such that $\mathrm{succ}\, O \in \mathrm{succ}\, A$ holds $S(\mathrm{succ}\, O) = \langle \{S(O)\}, \emptyset \rangle$ and for every O such that $O \in \mathrm{succ}\, A$ and O is limit ordinal holds $S(O) = \langle \mathrm{rng}(S \restriction O), \emptyset \rangle$. If $O \in \mathrm{succ}\, A$, then $S(O) = \mathrm{ordinal}_{\mathbf{On}}(O)$.

PROOF: Consider S_1 being a transfinite sequence such that $\mathrm{ordinal}_{\mathbf{On}}(O) = S_1(O)$ and $\mathrm{dom}\, S_1 = \mathrm{succ}\, O$ and for every B such that $\mathrm{succ}\, B \in \mathrm{succ}\, O$ holds $S_1(\mathrm{succ}\, B) = \langle \{S_1(B)\}, \emptyset \rangle$ and for every B such that $B \in \mathrm{succ}\, O$ and B is limit ordinal holds $S_1(B) = \langle \mathrm{rng}(S_1 \restriction B), \emptyset \rangle$. Define $\mathcal{P}[\text{ordinal number}] \equiv$ if $\$_1 \subseteq O$, then $S_1(\$_1) = S(\$_1)$. For every ordinal number B such that for every ordinal number C such that $C \in B$ holds $\mathcal{P}[C]$ holds $\mathcal{P}[B]$. For every ordinal number B, $\mathcal{P}[B]$. $\square$

(64) $\mathrm{ordinal}_{\mathbf{On}}(0) = \mathbf{0}_{\mathbf{No}}$.

(65) $\mathrm{ordinal}_{\mathbf{On}}(\mathrm{succ}\, A) = \langle \{\mathrm{ordinal}_{\mathbf{On}}(A)\}, \emptyset \rangle$. The theorem is a consequence of (63).

(66) Suppose A is limit ordinal. Then there exists a set X such that

(i) $\mathrm{ordinal}_{\mathbf{On}}(A) = \langle X, \emptyset \rangle$, and

(ii) for every o, $o \in X$ iff there exists B such that $B \in A$ and $o = \mathrm{ordinal}_{\mathbf{On}}(B)$.

PROOF: Set $B = \mathrm{succ}\, A$. Consider S being a transfinite sequence such that $\mathrm{ordinal}_{\mathbf{On}}(A) = S(A)$ and $\mathrm{dom}\, S = B$ and for every O such that $\mathrm{succ}\, O \in B$ holds $S(\mathrm{succ}\, O) = \langle \{S(O)\}, \emptyset \rangle$ and for every O such that $O \in B$ and O is limit ordinal holds $S(O) = \langle \mathrm{rng}(S \restriction O), \emptyset \rangle$. If $o \in X$, then there exists B such that $B \in A$ and $o = \text{ordinal}_{\mathbf{On}}(B)$. $\text{ordinal}_{\mathbf{On}}(C) = S(C) = (S \restriction A)(C)$. □

(67) $\text{ordinal}_{\mathbf{On}}(A) \in \text{Day}\,A$.

PROOF: Define $\mathcal{P}[\text{ordinal number}] \equiv \text{ordinal}_{\mathbf{On}}(\$_1) \in \text{Day}\$_1$. For every ordinal number D such that for every ordinal number C such that $C \in D$ holds $\mathcal{P}[C]$ holds $\mathcal{P}[D]$. For every ordinal number D, $\mathcal{P}[D]$. □

Let us consider A. One can check that $\text{ordinal}_{\mathbf{On}}(A)$ is surreal and $\text{ordinal}_{\mathbf{On}}(A)$ is **On**. Now we state the propositions:

(68) $\text{ordinal}_{\mathbf{On}}(A) < \text{ordinal}_{\mathbf{On}}(B)$ if and only if $A \in B$.

PROOF: If $\text{ordinal}_{\mathbf{On}}(A) < \text{ordinal}_{\mathbf{On}}(B)$, then $A \in B$. □

(69) If $x \in \text{Day}\,A$, then $x \leqslant \text{ordinal}_{\mathbf{On}}(A)$.

PROOF: Define $\mathcal{P}[\text{ordinal number}] \equiv$ for every x such that $x \in \text{Day}\$_1$ holds $x \leqslant \text{ordinal}_{\mathbf{On}}(\$_1)$. For every ordinal number D such that for every ordinal number C such that $C \in D$ holds $\mathcal{P}[C]$ holds $\mathcal{P}[D]$. For every ordinal number D, $\mathcal{P}[D]$. □

(70) $\mathfrak{born}\,\text{ordinal}_{\mathbf{On}}(A) = A$.

PROOF: $\text{ordinal}_{\mathbf{On}}(A) \in \text{Day}\,A$. For every O such that $\text{ordinal}_{\mathbf{On}}(A) \in \text{Day}\,O$ holds $A \subseteq O$. □

(71) If $x \in \text{L}_{\text{ordinal}_{\mathbf{On}}(A)}$, then there exists B such that $B \in A$ and $x = \text{ordinal}_{\mathbf{On}}(B)$. The theorem is a consequence of (66) and (65).

(72) $\text{s}_{\mathbb{Z}}(n) = \text{ordinal}_{\mathbf{On}}(n)$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv \text{s}_{\mathbb{Z}}(\$_1) = \text{ordinal}_{\mathbf{On}}(\$_1)$. $\mathcal{P}[0]$. If $\mathcal{P}[m]$, then $\mathcal{P}[m+1]$. $\mathcal{P}[m]$. □

Let O be a **On** surreal number. One can verify that $\text{Unique}_{\mathbf{No}}(O)$ is **On**.

Let A be an ordinal number. The functor $\text{Ordinal}_{\mathbf{On}}(A)$ yielding a **On** unique surreal number is defined by the term

(Def. 12) $\text{Unique}_{\mathbf{No}}(\text{ordinal}_{\mathbf{On}}(A))$.

Now we state the propositions:

(73) (i) $\text{Ordinal}_{\mathbf{On}}(A) \approx \text{ordinal}_{\mathbf{On}}(A)$, and

(ii) $\mathfrak{born}\,\text{Ordinal}_{\mathbf{On}}(A) = A$.

PROOF: $\mathfrak{born}_{\approx}\,\text{Ordinal}_{\mathbf{On}}(A) = \mathfrak{born}_{\approx}\,\text{ordinal}_{\mathbf{On}}(A) \subseteq \mathfrak{born}\,\text{ordinal}_{\mathbf{On}}(A) = A$. $A \subseteq \mathfrak{born}\,\text{Ordinal}_{\mathbf{On}}(A)$. □

(74) $\text{Ordinal}_{\mathbf{On}}(A) \in \text{Day}\,A$. The theorem is a consequence of (73).

(75) $\text{Ordinal}_{\mathbf{On}}(A) < \text{Ordinal}_{\mathbf{On}}(B)$ if and only if $A \in B$.

PROOF: $\text{Ordinal}_{\mathbf{On}}(A) \approx \text{ordinal}_{\mathbf{On}}(A)$ and $\text{Ordinal}_{\mathbf{On}}(B) \approx \text{ordinal}_{\mathbf{On}}(B)$. If $\text{Ordinal}_{\mathbf{On}}(A) < \text{Ordinal}_{\mathbf{On}}(B)$, then $A \in B$. $\text{Ordinal}_{\mathbf{On}}(A) < \text{ordinal}_{\mathbf{On}}(B)$. □

(76) If $x \in \mathrm{Day}A$, then $x \leqslant \mathrm{Ordinal}_{\mathbf{On}}(A)$. The theorem is a consequence of (69) and (73).

(77) If x is **On**, then there exists A such that $x \approx \mathrm{Ordinal}_{\mathbf{On}}(A)$. The theorem is a consequence of (73).

(78) $\mathrm{s}_{\mathbb{Z}}(n) = \mathrm{Ordinal}_{\mathbf{On}}(n)$. The theorem is a consequence of (72) and (73).

(79) $\mathrm{Ordinal}_{\mathbf{On}}(\mathrm{succ}\, A) = \langle\{\mathrm{Ordinal}_{\mathbf{On}}(A)\}, \emptyset\rangle$.

Proof: Set $O_1 = \mathrm{Ordinal}_{\mathbf{On}}(A)$. Set $x = \langle\{O_1\}, \emptyset\rangle$. $\mathfrak{b}\mathrm{orn}\, O_1 = A$. If $o \in \{O_1\} \cup \emptyset$, then there exists O such that $O \in \mathrm{succ}\, A$ and $o \in \mathrm{Day}O$. $\mathrm{ordinal}_{\mathbf{On}}(\mathrm{succ}\, A) = \langle\{\mathrm{ordinal}_{\mathbf{On}}(A)\}, \emptyset\rangle$. $O_1 \approx \mathrm{ordinal}_{\mathbf{On}}(A)$. For every surreal number y such that $y \approx x$ holds $\mathrm{succ}\, A \subseteq \mathfrak{b}\mathrm{orn}\, y$. For every z such that $z \in \mathfrak{Born}_{\approx} x$ and $\mathrm{L}_z \cup \mathrm{R}_z$ is unique surreal-membered and $x \neq z$ holds $\overline{\overline{\mathrm{L}_x}} \oplus \overline{\overline{\mathrm{R}_x}} \in \overline{\overline{\mathrm{L}_z}} \oplus \overline{\overline{\mathrm{R}_z}}$. $\mathrm{ordinal}_{\mathbf{On}}(\mathrm{succ}\, A) \approx \mathrm{Ordinal}_{\mathbf{On}}(\mathrm{succ}\, A)$. □

(80) There exists a **On** surreal number x such that

(i) $\mathfrak{b}\mathrm{orn}\, x = A$, and

(ii) $\mathrm{Ordinal}_{\mathbf{On}}(A) \approx x$, and

(iii) for every o, $o \in \mathrm{L}_x$ iff there exists B such that $B \in A$ and $o = \mathrm{Ordinal}_{\mathbf{On}}(B)$.

Proof: Define $\mathcal{P}[\text{object}] \equiv$ there exists B such that $B \in A$ and $\$_1 = \mathrm{Ordinal}_{\mathbf{On}}(B)$. Consider X being a set such that $o \in X$ iff $o \in \mathrm{Day}A$ and $\mathcal{P}[o]$. If $o \in X \cup \emptyset$, then there exists O such that $O \in A$ and $o \in \mathrm{Day}O$. Reconsider $x = \langle X, \emptyset\rangle$ as a surreal number. For every O such that $x \in \mathrm{Day}O$ holds $A \subseteq O$. $\mathrm{L}_{\mathrm{ordinal}_{\mathbf{On}}(A)} \ll \{x\}$. $\mathrm{L}_x \ll \{\mathrm{ordinal}_{\mathbf{On}}(A)\}$. $\mathrm{Ordinal}_{\mathbf{On}}(A) \approx \mathrm{ordinal}_{\mathbf{On}}(A) \approx x$. $o \in \mathrm{Day}B \subseteq \mathrm{Day}A$. □

Let α, β be **On** surreal numbers. Observe that $\alpha + \beta$ is **On** and $\alpha \cdot \beta$ is **On**.

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