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Surreal Numbers: A Study of Square Roots

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Summary. This paper sets out to formalize the concept of the square root as proposed by Clive Bach in the section entitled *Properties of Division* in Conway's book. The proposed construction extends the classical approach to the square root of real numbers to include both infinitely large and infinitely small numbers.

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Introduction

In the chapter *The class **No** is a field* [6], Conway also quotes the definition of a square root of a surreal number [10] proposed by Clive Bach. This definition is formulated in Conway's typical way, using double recursion and the concept and typical options as follows:

$$\sqrt{x} = y = \{\sqrt{x^L}, \frac{x + y^L \cdot y^R}{y^L + y^R} \mid \sqrt{x^R}, \frac{x + y^L \cdot y^{L\bullet}}{y^L + y^{L\bullet}}, \frac{x + y^R \cdot y^{R\bullet}}{y^R + y^{R\bullet}}\} \tag{I.1}$$

where x^L, x^R represent non-negative options of x, while y^L, $y^{L\bullet}$, y^R, $y^{R\bullet}$ denote the options for y such that no denominator is zero. In addition, the construction of this number is entirely absent, and the veracity of this definition is left to the reader to demonstrate through an *easy inductive proof*.

In our formalization, we adapt the idea presented for the inverse element proposed by Schleicher and Stoll [18], which was previously employed in our earlier formalization [14] in the Mizar system [4]. We first introduce a restriction that limits the members of the sets L_x, R_x to those that are non-negative. Let x be a surreal number. We define the function $\texttt{NNPart}(x)$ (see Def. 1) to be

$$\{0, \{x^L \in L_x \mid x^L \geqslant 0\} \mid \{x^R \in R_x \mid x^R \geqslant 0\}\} \tag{I.2}$$

and we prove that NNPart is born no later than x (see Th3) and $\approx x$ (see Th5) for any non-negative x.

Then two sequences of sets of surreal numbers are introduced: $\{\sqrt[L]{n, x_0, x}\}_{n\in\mathbb{N}}$, $\{\sqrt[R]{n, x_0, x}\}_{n\in\mathbb{N}}$ for a given surreal number x and an initial pair of of surreal number sets x_0. These sequences are defined (see Def. 3, Def. 4, Def. 5) recursively as follows:

$$\begin{aligned}
\sqrt[L]{0, x_0, x} &= L_{x_0},\\
\sqrt[R]{0, x_0, x} &= R_{x_0},\\
\sqrt[L]{n+1, x_0, x} &= \sqrt[L]{n, x_0, x} \cup \texttt{S}(x,\ \sqrt[L]{n, x_0, x},\ \sqrt[R]{n, x_0, x})\\
\sqrt[R]{n+1, x_0, x} &= \sqrt[R]{n, x_0, x} \cup \texttt{S}(x,\ \sqrt[L]{n, x_0, x},\ \sqrt[L]{n, x_0, x})\\
&\qquad \cup \texttt{S}(x,\ \sqrt[R]{n, x_0, x},\ \sqrt[R]{n, x_0, x}),
\end{aligned} \tag{I.3}$$

where $\texttt{S}(x, A, B) = \{\frac{x+a\cdot b}{a+b} \mid a \in X \wedge b \in Y \wedge a + b \neq 0\}$ and A and B represent arbitrary sets of surreal numbers (see Def. 2).

The condition (I.1) can now be expressed in a more formal, but still recursive way, as follows:

$$\sqrt{x} = \langle \bigcup_{n\in\mathbb{N}} \sqrt[L]{n, x_0, x}, \bigcup_{n\in\mathbb{N}} \sqrt[R]{n, x_0, x} \rangle \tag{I.4}$$

where $x_0 = \langle\{\sqrt{x_L} \mid x_L \in L_{\texttt{NNPart}(x)}\}, \{\sqrt{x_R} \mid x_R \in R_{\texttt{NNPart}(x)}\}\rangle$.

To implement this kind of recursion in the Mizar system we use a sequence $\sqrt[\alpha]{\cdot}$, where $\sqrt[\alpha]{\cdot}$ is a function defined on day α for each ordinal α, with the following definition:

$$\sqrt[\alpha]{x} = \langle \bigcup_{n\in\mathbb{N}} \sqrt[L]{n, (\bigcup_{\beta<\alpha} \sqrt[\beta]{\cdot})[L_{\texttt{NNPart}(x)}], x} \bigcup_{n\in\mathbb{N}} \sqrt[R]{n, (\bigcup_{\beta<\alpha} \sqrt[\beta]{\cdot})[R_{\texttt{NNPart}(x)}], x} \rangle \tag{I.5}$$

where x represents an element of day α. It is important for understanding the correctness of the definition that the constructed sequence is a $\subseteq$-monotone in the set-theoretic sense, so that we can treat $\bigcup_{\beta<\alpha} \sqrt[\beta]{\cdot}$ as a function. We may now define $\sqrt{(x)}$ as $\sqrt[\alpha]{(x)}$, where α represents the day on which a given positive x is born (see Def. 7) and satisfies the fundamental properties of square root for non-negative real numbers such as: $0 \leqslant \sqrt{x} < \sqrt{y}$ for all surreal numbers $0 < x < y$ (see Th27), $\sqrt{x}^{-1} \approx \sqrt{x^{-1}}$ for positive surreal (see Th30).

The concept proposed by Clive Bach was initially introduced for non-negative numbers; however, there are no inherent limitations to its application beyond the natural domain. We have shown that, outside the domain, the fundamental property $x \approx y \Rightarrow \sqrt{x} \approx \sqrt{y}$ is lost. Indeed, we prove that $\sqrt{-1} = -1$ (although obviously, this is not connected in a straightforward way with surcomplex numbers [1]) and for any positive x we can construct a surreal number $y \approx -1$ such that $\sqrt{y} < -x$ (see Th31).

For a detailed exposition of the formalized topic, see [2] (for developments in another systems – see [11] and [12], [21]). The development of the square root provides a foundation for further advances, notably the integration of surreal numbers [7] (cf. the discussion in [17]). Our formalization is oriented more towards set theory [5], building on the Mizar framework (with recently improved possibility of finding interconnections [19]), rather than the inductive-inductive [8] HoTT approach, which however seems to be more natural (Sect. 11.6 of [20]). This may be viewed as the first step in a longstanding program, initiated by Conway, Kruskal, and Norton [9], aiming to develop analysis on ***No***, beginning with a genetic definition of integration [7].

1. Surreal Numbers Without Negative Options

From now on n, m denote natural numbers, o denotes an object, p denotes a pair object, and x, y, z denote surreal numbers.

Let x be an object. The functor $\mathrm{Part}_{\geqslant \mathbf{0_{No}}}(x)$ yielding a pair set is defined by

(Def. 1) ($o \in \mathrm{L}_{it}$ iff there exists a surreal number l such that $o = l$ and $l \in \mathrm{L}_x$ and $\mathbf{0_{No}} \leqslant l$) and ($o \in \mathrm{R}_{it}$ iff there exists a surreal number r such that $o = r$ and $r \in \mathrm{R}_x$ and $\mathbf{0_{No}} \leqslant r$).

One can check that $\mathrm{L}_{\mathrm{Part}_{\geqslant \mathbf{0_{No}}}(x)}$ is surreal-membered as a set and $\mathrm{R}_{\mathrm{Part}_{\geqslant \mathbf{0_{No}}}(x)}$ is surreal-membered as a set. Now we state the proposition:

(1) (i) $\mathrm{L}_{\mathrm{Part}_{\geqslant \mathbf{0_{No}}}(o)} \subseteq \mathrm{L}_o$, and

(ii) $\mathrm{R}_{\mathrm{Part}_{\geqslant \mathbf{0_{No}}}(o)} \subseteq \mathrm{R}_o$.

Let x be a surreal number. One can check that $\mathrm{Part}_{\geqslant \mathbf{0_{No}}}(x)$ is surreal. Now we state the propositions:

(2) (i) $x \in \mathrm{L}_{\mathrm{Part}_{\geqslant \mathbf{0_{No}}}(o)}$ iff $x \in \mathrm{L}_o$ and $\mathbf{0_{No}} \leqslant x$, and

(ii) $x \in \mathrm{R}_{\mathrm{Part}_{\geqslant \mathbf{0_{No}}}(o)}$ iff $x \in \mathrm{R}_o$ and $\mathbf{0_{No}} \leqslant x$.

(3) $\mathfrak{born}\,\mathrm{Part}_{\geqslant \mathbf{0_{No}}}(x) \subseteq \mathfrak{born}\,x$.

PROOF: Set $N = \mathrm{Part}_{\geqslant \mathbf{0_{No}}}(x)$. For every object o such that $o \in \mathrm{L}_N \cup \mathrm{R}_N$ there exists an ordinal number O such that $O \in \mathfrak{born}\,x$ and $o \in \mathrm{Day}O$. □

(4) If $\mathbf{0_{No}} \leqslant x$, then $\mathbf{0_{No}} \leqslant \mathrm{Part}_{\geqslant \mathbf{0_{No}}}(x)$.

PROOF: Set $N = \mathrm{Part}_{\geqslant \mathbf{0_{No}}}(x)$. $\{\mathbf{0_{No}}\} \ll \mathrm{R}_N$. □

(5) If $\mathbf{0}_{\mathbf{No}} \leqslant x$, then $\mathrm{Part}_{\geqslant \mathbf{0}_{\mathbf{No}}}(x) \approx x$.
PROOF: Set $N = \mathrm{Part}_{\geqslant \mathbf{0}_{\mathbf{No}}}(x)$. $\mathbf{0}_{\mathbf{No}} \leqslant N$. $\mathrm{L}_N \ll \{x\}$. $\{N\} \ll \mathrm{R}_x$ by [15, (11), (4)], [13, (43)]. $\mathrm{L}_x \ll \{N\}$. $\{x\} \ll \mathrm{R}_N$. □

2. Square Root Construction

Let l_1 be an object and X, Y be sets. The functor $\sqrt{l_1, X, Y}$ yielding a surreal-membered set is defined by

(Def. 2) $o \in it$ iff there exists x and there exists y such that $x \in X$ and $y \in Y$ and $x + y \not\approx \mathbf{0}_{\mathbf{No}}$ and $o = (l_1 +' x \cdot y) \cdot ((x + y)^{-1})$.

Let x_0 be a pair object and x be an object. The functor $\mathrm{Transitions}(x_0, x)$ yielding a function is defined by

(Def. 3) $\mathrm{dom}\, it = \mathbb{N}$ and $it(0) = x_0$ and for every n, $it(n)$ is pair and $(it(n+1))_{\mathbf{1}} = \mathrm{L}_{it(n)} \cup \sqrt{x, \mathrm{L}_{it(n)}, \mathrm{R}_{it(n)}}$ and $(it(n+1))_{\mathbf{2}} = (\mathrm{R}_{it(n)} \cup \sqrt{x, \mathrm{L}_{it(n)}, \mathrm{L}_{it(n)}}) \cup \sqrt{x, \mathrm{R}_{it(n)}, \mathrm{R}_{it(n)}}$.

The functor $\sqrt[L]{x_0, x}$ yielding a function is defined by

(Def. 4) $\mathrm{dom}\, it = \mathbb{N}$ and for every natural number k, $it(k) = ((\mathrm{Transitions}(x_0, x))(k))_{\mathbf{1}}$.

The functor $\sqrt[R]{x_0, x}$ yielding a function is defined by

(Def. 5) $\mathrm{dom}\, it = \mathbb{N}$ and for every natural number k, $it(k) = ((\mathrm{Transitions}(x_0, x))(k))_{\mathbf{2}}$.

Now we state the propositions:

(6) (i) $(\sqrt[L]{p, o})(0) = \mathrm{L}_p$, and

(ii) $(\sqrt[R]{p, o})(0) = \mathrm{R}_p$.

(7) If $n \leqslant m$, then $(\sqrt[L]{p, o})(n) \subseteq (\sqrt[L]{p, o})(m)$ and $(\sqrt[R]{p, o})(n) \subseteq (\sqrt[R]{p, o})(m)$.
PROOF: Define $\mathcal{P}[\text{natural number}] \equiv (\sqrt[L]{p, o})(n) \subseteq (\sqrt[L]{p, o})(n + \$_1)$ and $(\sqrt[R]{p, o})(n) \subseteq (\sqrt[R]{p, o})(n + \$_1)$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$. □

(8) (i) $(\sqrt[L]{p, o})(n+1) = (\sqrt[L]{p, o})(n) \cup \sqrt{o, (\sqrt[L]{p, o})(n), (\sqrt[R]{p, o})(n)}$, and

(ii) $(\sqrt[R]{p, o})(n+1) = ((\sqrt[R]{p, o})(n) \cup \sqrt{o, (\sqrt[L]{p, o})(n), (\sqrt[L]{p, o})(n)}) \cup \sqrt{o, (\sqrt[R]{p, o})(n), (\sqrt[R]{p, o})(n)}$.

(9) Suppose L_p is surreal-membered and R_p is surreal-membered. Then

(i) $(\sqrt[L]{p, o})(n)$ is surreal-membered, and

(ii) $(\sqrt[R]{p, o})(n)$ is surreal-membered.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv (\sqrt[L]{p,o})(\$_1)$ is surreal-membered and $(\sqrt[R]{p,o})(\$_1)$ is surreal-membered. $\mathcal{P}[0]$. For every n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every n, $\mathcal{P}[n]$. $\square$

(10) Suppose L_p is surreal-membered and R_p is surreal-membered. Then

(i) $\bigcup \sqrt[L]{p,o}$ is surreal-membered, and

(ii) $\bigcup \sqrt[R]{p,o}$ is surreal-membered.

PROOF: $\bigcup \sqrt[L]{p,o}$ is surreal-membered. Consider n being an object such that $n \in \mathrm{dom}(\sqrt[R]{p,o})$ and $a \in (\sqrt[R]{p,o})(n)$. $(\sqrt[R]{p,o})(n)$ is surreal-membered. $\square$

(11) Let us consider sets X_1, X_2, Y_1, Y_2. Suppose $X_1 \subseteq X_2$ and $Y_1 \subseteq Y_2$. Then $\sqrt{o, X_1, Y_1} \subseteq \sqrt{o, X_2, Y_2}$.

(12) $\bigcup \sqrt[L]{p,o} = \mathrm{L}_p \cup \sqrt{o, \bigcup \sqrt[L]{p,o}, \bigcup \sqrt[R]{p,o}}$.
PROOF: Define $\mathcal{P}[\text{natural number}] \equiv (\sqrt[L]{p,o})(\$_1) \subseteq \mathrm{L}_p \cup \sqrt{o, \bigcup \sqrt[L]{p,o}, \bigcup \sqrt[R]{p,o}}$. $(\sqrt[L]{p,o})(0) = \mathrm{L}_p$. If $\mathcal{P}[n]$, then $\mathcal{P}[n+1]$. $\mathcal{P}[n]$. $\bigcup \sqrt[L]{p,o} \subseteq \mathrm{L}_p \cup \sqrt{o, \bigcup \sqrt[L]{p,o}, \bigcup \sqrt[R]{p,o}}$. $\sqrt{o, \bigcup \sqrt[L]{p,o}, \bigcup \sqrt[R]{p,o}} \subseteq \bigcup \sqrt[L]{p,o}$. $\mathrm{L}_p = (\sqrt[L]{p,o})(0)$. $\square$

(13) $\bigcup \sqrt[R]{p,o} = (\mathrm{R}_p \cup \sqrt{o, \bigcup \sqrt[L]{p,o}, \bigcup \sqrt[L]{p,o}}) \cup \sqrt{o, \bigcup \sqrt[R]{p,o}, \bigcup \sqrt[R]{p,o}}$.
PROOF: Define $\mathcal{P}[\text{natural number}] \equiv (\sqrt[R]{p,o})(\$_1) \subseteq (\mathrm{R}_p \cup \sqrt{o, \bigcup \sqrt[L]{p,o}, \bigcup \sqrt[L]{p,o}}) \cup \sqrt{o, \bigcup \sqrt[R]{p,o}, \bigcup \sqrt[R]{p,o}}$. $(\sqrt[R]{p,o})(0) = \mathrm{R}_p$. If $\mathcal{P}[n]$, then $\mathcal{P}[n+1]$. $\mathcal{P}[n]$. $\bigcup \sqrt[R]{p,o} \subseteq (\mathrm{R}_p \cup \sqrt{o, \bigcup \sqrt[L]{p,o}, \bigcup \sqrt[L]{p,o}}) \cup \sqrt{o, \bigcup \sqrt[R]{p,o}, \bigcup \sqrt[R]{p,o}}$. $\sqrt{o, \bigcup \sqrt[L]{p,o}, \bigcup \sqrt[L]{p,o}} \subseteq \bigcup \sqrt[R]{p,o}$. $\sqrt{o, \bigcup \sqrt[R]{p,o}, \bigcup \sqrt[R]{p,o}} \subseteq \bigcup \sqrt[R]{p,o}$. $\mathrm{R}_p = (\sqrt[R]{p,o})(0)$. $\square$

3. The Square Root of a Surreal Number

Let A be an ordinal number. The functor $\mathrm{sqrt}_{\mathbf{No}}(A)$ yielding a many sorted set indexed by $\mathrm{Day}A$ is defined by

(Def. 6) there exists a $\subseteq$-monotone, function yielding transfinite sequence S such that $\mathrm{dom}\, S = \mathrm{succ}\, A$ and $it = S(A)$ and for every ordinal number B such that $B \in \mathrm{succ}\, A$ there exists a many sorted set S_4 indexed by $\mathrm{Day}B$ such that $S(B) = S_4$ and for every object x such that $x \in \mathrm{Day}B$ holds $S_4(x) = \langle \bigcup \sqrt[L]{\langle (\bigcup \mathrm{rng}(S \restriction B))^\circ(\mathrm{L}_{\mathrm{Part}_{\geqslant \mathbf{0_{No}}}(x)}),\ (\bigcup \mathrm{rng}(S \restriction B))^\circ(\mathrm{R}_{\mathrm{Part}_{\geqslant \mathbf{0_{No}}}(x)}) \rangle, x}, \bigcup \sqrt[R]{\langle (\bigcup \mathrm{rng}(S \restriction B))^\circ(\mathrm{L}_{\mathrm{Part}_{\geqslant \mathbf{0_{No}}}(x)}),\ (\bigcup \mathrm{rng}(S \restriction B))^\circ(\mathrm{R}_{\mathrm{Part}_{\geqslant \mathbf{0_{No}}}(x)}) \rangle, x} \rangle$.

Now we state the proposition:

(14) Let us consider a $\subseteq$-monotone, function yielding transfinite sequence S. Suppose for every ordinal number B such that $B \in \operatorname{dom} S$ there exists a many sorted set S_4 indexed by Day$B$ such that $S(B) = S_4$ and for every o such that $o \in \text{Day}B$ holds $S_4(o) =$ $\langle \bigcup \sqrt[L]{\langle (\bigcup \operatorname{rng}(S\restriction B))^\circ(\mathrm{L}_{\mathrm{Part}_{\geqslant \mathbf{0_{No}}}(o)}), (\bigcup \operatorname{rng}(S\restriction B))^\circ(\mathrm{R}_{\mathrm{Part}_{\geqslant \mathbf{0_{No}}}(o)})\rangle, o}$, $\bigcup \sqrt[R]{\langle (\bigcup \operatorname{rng}(S\restriction B))^\circ(\mathrm{L}_{\mathrm{Part}_{\geqslant \mathbf{0_{No}}}(o)}), (\bigcup \operatorname{rng}(S\restriction B))^\circ(\mathrm{R}_{\mathrm{Part}_{\geqslant \mathbf{0_{No}}}(o)})\rangle, o}\rangle$. Let us consider an ordinal number A. If $A \in \operatorname{dom} S$, then $\operatorname{sqrt}_{\mathbf{No}}(A) = S(A)$.
PROOF: Define $\mathcal{D}$(ordinal number) $=$ Day$\$_1$. Define $\mathcal{H}$(object, $\subseteq$-monotone, function yielding transfinite sequence) $=$ $\langle \bigcup \sqrt[L]{\langle (\bigcup \operatorname{rng} \$_2)^\circ(\mathrm{L}_{\mathrm{Part}_{\geqslant \mathbf{0_{No}}}(\$_1)}), (\bigcup \operatorname{rng} \$_2)^\circ(\mathrm{R}_{\mathrm{Part}_{\geqslant \mathbf{0_{No}}}(\$_1)})\rangle, \$_1}$, $\bigcup \sqrt[R]{\langle (\bigcup \operatorname{rng} \$_2)^\circ(\mathrm{L}_{\mathrm{Part}_{\geqslant \mathbf{0_{No}}}(\$_1)}), (\bigcup \operatorname{rng} \$_2)^\circ(\mathrm{R}_{\mathrm{Part}_{\geqslant \mathbf{0_{No}}}(\$_1)})\rangle, \$_1}\rangle$.

Consider S_2 being a $\subseteq$-monotone, function yielding transfinite sequence such that $\operatorname{dom} S_2 = \operatorname{succ} A$ and $S_2(A) = \operatorname{sqrt}_{\mathbf{No}}(A)$ and for every ordinal number B such that $B \in \operatorname{succ} A$ there exists a many sorted set S_4 indexed by $\mathcal{D}(B)$ such that $S_2(B) = S_4$ and for every object x such that $x \in \mathcal{D}(B)$ holds $S_4(x) = \mathcal{H}(x, S_2\restriction B)$. $S1\restriction \operatorname{succ} A = S_2\restriction \operatorname{succ} A$. □

Let o be an object. Assume o is a surreal number. The functor $\sqrt{o}$ yielding a set is defined by

(Def. 7) for every x such that $x = o$ holds $\mathit{it} = (\operatorname{sqrt}_{\mathbf{No}}(\mathfrak{born}\, x))(x)$.

Let x be a surreal number. Observe that the functor $\sqrt{x}$ yields a set and is defined by the term

(Def. 8) $(\operatorname{sqrt}_{\mathbf{No}}(\mathfrak{born}\, x))(x)$.

Let x be an object. The functor $\operatorname{sqrt}_{\mathbf{0_{No}}}(x)$ yielding a pair set is defined by

(Def. 9) ($o \in \mathrm{L}_{\mathit{it}}$ iff there exists a surreal number l such that $o = \sqrt{l}$ and $l \in \mathrm{L}_{\mathrm{Part}_{\geqslant \mathbf{0_{No}}}(x)}$) and ($o \in \mathrm{R}_{\mathit{it}}$ iff there exists a surreal number r such that $o = \sqrt{r}$ and $r \in \mathrm{R}_{\mathrm{Part}_{\geqslant \mathbf{0_{No}}}(x)}$).

Now we state the propositions:

(15) $\sqrt{x} = \langle \bigcup \sqrt[L]{\operatorname{sqrt}_{\mathbf{0_{No}}}(x), x}, \bigcup \sqrt[R]{\operatorname{sqrt}_{\mathbf{0_{No}}}(x), x}\rangle$.
PROOF: Set $A = \mathfrak{born}\, x$. Set $N_1 = \mathrm{Part}_{\geqslant \mathbf{0_{No}}}(x)$. Consider S being a $\subseteq$-monotone, function yielding transfinite sequence such that $\operatorname{dom} S = \operatorname{succ} A$ and $\operatorname{sqrt}_{\mathbf{No}}(A) = S(A)$ and for every ordinal number B such that $B \in \operatorname{succ} A$ there exists a many sorted set S_4 indexed by Day$B$ such that $S(B) = S_4$ and for every object o such that $o \in \text{Day}B$ holds $S_4(o) =$ $\langle \bigcup \sqrt[L]{\langle (\bigcup \operatorname{rng}(S\restriction B))^\circ(\mathrm{L}_{\mathrm{Part}_{\geqslant \mathbf{0_{No}}}(o)}), (\bigcup \operatorname{rng}(S\restriction B))^\circ(\mathrm{R}_{\mathrm{Part}_{\geqslant \mathbf{0_{No}}}(o)})\rangle, o}$, $\bigcup \sqrt[R]{\langle (\bigcup \operatorname{rng}(S\restriction B))^\circ(\mathrm{L}_{\mathrm{Part}_{\geqslant \mathbf{0_{No}}}(o)}), (\bigcup \operatorname{rng}(S\restriction B))^\circ(\mathrm{R}_{\mathrm{Part}_{\geqslant \mathbf{0_{No}}}(o)})\rangle, o}\rangle$.

Set $U = \bigcup \operatorname{rng}(S\restriction A)$. Consider S_4 being a many sorted set indexed by Day$A$ such that $S(A) = S_4$ and for every o such that $o \in \text{Day}A$ holds

$S_4(o) = \langle \bigcup \sqrt[L]{\langle U^\circ(\mathrm{L}_{\mathrm{Part}_{\geqslant \mathbf{0}_{\mathbf{No}}}(o)}), U^\circ(\mathrm{R}_{\mathrm{Part}_{\geqslant \mathbf{0}_{\mathbf{No}}}(o)})\rangle, o},$
$\bigcup \sqrt[R]{\langle U^\circ(\mathrm{L}_{\mathrm{Part}_{\geqslant \mathbf{0}_{\mathbf{No}}}(o)}), U^\circ(\mathrm{R}_{\mathrm{Part}_{\geqslant \mathbf{0}_{\mathbf{No}}}(o)})\rangle, o}\rangle$. $U^\circ(\mathrm{L}_{N_1})$
$\subseteq \mathrm{L}_{\mathrm{sqrt}_{\mathbf{0}_{\mathbf{No}}}(x)}$. $\mathrm{L}_{\mathrm{sqrt}_{\mathbf{0}_{\mathbf{No}}}(x)} \subseteq U^\circ(\mathrm{L}_{N_1})$. $U^\circ(\mathrm{R}_{N_1}) \subseteq \mathrm{R}_{\mathrm{sqrt}_{\mathbf{0}_{\mathbf{No}}}(x)}$. $\mathrm{R}_{\mathrm{sqrt}_{\mathbf{0}_{\mathbf{No}}}(x)} \subseteq$ $U^\circ(\mathrm{R}_{N_1})$. □

(16) If $\bigcup \sqrt[L]{p, o} = \emptyset$, then $\mathrm{L}_p = \emptyset$. The theorem is a consequence of (6).

(17) Let us consider surreal numbers x_1, x_2, y, z. Suppose $x_2 \not\approx \mathbf{0}_{\mathbf{No}}$ and $y = x_1 \cdot (x_2{}^{-1})$. Then

(i) $y \cdot y < z$ iff $x_1 \cdot x_1 < z \cdot (x_2 \cdot x_2)$, and

(ii) $z < y \cdot y$ iff $z \cdot (x_2 \cdot x_2) < x_1 \cdot x_1$.

PROOF: If $y \cdot y < z$, then $x_1 \cdot x_1 < z \cdot (x_2 \cdot x_2)$. If $x_1 \cdot x_1 < z \cdot (x_2 \cdot x_2)$, then $y \cdot y < z$. If $z < y \cdot y$, then $z \cdot (x_2 \cdot x_2) < x_1 \cdot x_1$. □

(18) If $x \leqslant \mathbf{0}_{\mathbf{No}}$, then $\bigcup \sqrt[L]{\mathrm{sqrt}_{\mathbf{0}_{\mathbf{No}}}(x), o} = \emptyset$ by [16, (70)].

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv (\sqrt[L]{\mathrm{sqrt}_{\mathbf{0}_{\mathbf{No}}}(x), o})(\$_1) = \emptyset$. $\mathcal{P}[0]$. If $\mathcal{P}[n]$, then $\mathcal{P}[n+1]$. $\mathcal{P}[n]$.

Consider a being an object such that $a \in \bigcup \sqrt[L]{\mathrm{sqrt}_{\mathbf{0}_{\mathbf{No}}}(x), o}$. Consider n being an object such that $n \in \mathrm{dom}(\sqrt[L]{\mathrm{sqrt}_{\mathbf{0}_{\mathbf{No}}}(x), o})$ and $a \in (\sqrt[L]{\mathrm{sqrt}_{\mathbf{0}_{\mathbf{No}}}(x), o})(n)$. □

(19) Suppose $\mathbf{0}_{\mathbf{No}} \leqslant x$. Then

(i) if $y = \sqrt{x}$, then $\mathbf{0}_{\mathbf{No}} \leqslant y$ and $y \cdot y \approx x$ and if $x \approx \mathbf{0}_{\mathbf{No}}$, then $y \approx \mathbf{0}_{\mathbf{No}}$ and if $\mathbf{0}_{\mathbf{No}} < x$, then $\mathbf{0}_{\mathbf{No}} < y$, and

(ii) if $y \in \mathrm{L}_{\sqrt{x}}$, then $\mathbf{0}_{\mathbf{No}} \leqslant y$ and $y \cdot y < x$, and

(iii) if $y \in \mathrm{R}_{\sqrt{x}}$, then $\mathbf{0}_{\mathbf{No}} < y$ and $x < y \cdot y$, and

(iv) $\sqrt{x}$ is surreal.

PROOF: Define $\mathcal{O}[\text{ordinal number}] \equiv$ for every x such that $\mathfrak{b}\mathrm{orn}\, x = \$_1$ and $\mathbf{0}_{\mathbf{No}} \leqslant x$ holds $\sqrt{x}$ is surreal and for every y such that $y = \sqrt{x}$ holds $\mathbf{0}_{\mathbf{No}} \leqslant y$ and $y \cdot y \approx x$ and if $x \approx \mathbf{0}_{\mathbf{No}}$, then $y \approx \mathbf{0}_{\mathbf{No}}$ and if $\mathbf{0}_{\mathbf{No}} < x$, then $\mathbf{0}_{\mathbf{No}} < y$ and for every y such that $y \in \mathrm{L}_{\sqrt{x}}$ holds $\mathbf{0}_{\mathbf{No}} \leqslant y$ and $y \cdot y < x$ and for every y such that $y \in \mathrm{R}_{\sqrt{x}}$ holds $\mathbf{0}_{\mathbf{No}} < y$ and $x < y \cdot y$. For every ordinal number D such that for every ordinal number C such that $C \in D$ holds $\mathcal{O}[C]$ holds $\mathcal{O}[D]$. For every ordinal number D, $\mathcal{O}[D]$. □

(20) If $x \leqslant \mathbf{0}_{\mathbf{No}}$, then $\sqrt{x}$ is surreal. The theorem is a consequence of (2), (19), (10), (18), and (15).

Let us consider x. One can check that $\sqrt{x}$ is surreal and $\mathrm{sqrt}_{\mathbf{0}_{\mathbf{No}}}(x)$ is surreal.

4. Selected Square Root Properties

Now we state the propositions:

(21) If $\mathbf{0_{No}} \leqslant x$, then $\mathbf{0_{No}} \leqslant \sqrt{x}$ and $\sqrt{x} \cdot \sqrt{x} \approx x$.

(22) If $\mathbf{0_{No}} \approx x$, then $\sqrt{x} \approx \mathbf{0_{No}}$.

(23) Suppose $\mathbf{0_{No}} \leqslant x$. Then

(i) if $y \in \mathrm{L}_{\sqrt{x}}$, then $\mathbf{0_{No}} \leqslant y$ and $y \cdot y < x$, and

(ii) if $y \in \mathrm{R}_{\sqrt{x}}$, then $\mathbf{0_{No}} < y$ and $x < y \cdot y$.

(24) If $x < \mathbf{0_{No}}$ and for every y such that $y \in \mathrm{R}_x$ holds $y < \mathbf{0_{No}}$, then $\sqrt{x} = \mathbf{0_{No}}$.

Proof: Define $\mathcal{P}[\text{natural number}] \equiv (\sqrt[R]{\mathrm{sqrt}_{\mathbf{0_{No}}}(x), x})(\$_1) = \emptyset$. $\mathcal{P}[0]$. If $\mathcal{P}[n]$, then $\mathcal{P}[n+1]$. $\mathcal{P}[n]$. $\bigcup \sqrt[R]{\mathrm{sqrt}_{\mathbf{0_{No}}}(x), x} = \emptyset$. $\bigcup \sqrt[L]{\mathrm{sqrt}_{\mathbf{0_{No}}}(x), x} = \emptyset$. □

(25) Suppose for every y such that $y \in \mathrm{L}_{\mathrm{Part}_{\geqslant \mathbf{0_{No}}}(x)} \cup \mathrm{R}_{\mathrm{Part}_{\geqslant \mathbf{0_{No}}}(x)}$ holds $y \approx \mathbf{0_{No}}$. Then $\sqrt{x} = \mathrm{sqrt}_{\mathbf{0_{No}}}(x)$.

Proof: Define $\mathcal{P}[\text{natural number}] \equiv (\sqrt[L]{\mathrm{sqrt}_{\mathbf{0_{No}}}(x), x})(\$_1) = \mathrm{L}_{\mathrm{sqrt}_{\mathbf{0_{No}}}(x)}$ and $(\sqrt[R]{\mathrm{sqrt}_{\mathbf{0_{No}}}(x), x})(\$_1) = \mathrm{R}_{\mathrm{sqrt}_{\mathbf{0_{No}}}(x)}$. $\mathcal{P}[0]$. If $\mathcal{P}[n]$, then $\mathcal{P}[n+1]$. $\mathcal{P}[n]$. $\sqrt{x} = \langle \bigcup \sqrt[L]{\mathrm{sqrt}_{\mathbf{0_{No}}}(x), x}, \bigcup \sqrt[R]{\mathrm{sqrt}_{\mathbf{0_{No}}}(x), x} \rangle$. $\mathrm{L}_{\sqrt{x}} = \mathrm{L}_{\mathrm{sqrt}_{\mathbf{0_{No}}}(x)}$. $\mathrm{R}_{\sqrt{x}} = \mathrm{R}_{\mathrm{sqrt}_{\mathbf{0_{No}}}(x)}$ by [3, (1)]. □

One can verify that $\sqrt{\mathbf{0_{No}}}$ reduces to $\mathbf{0_{No}}$ and $\sqrt{\mathbf{1_{No}}}$ reduces to $\mathbf{1_{No}}$ and $\sqrt{-\mathbf{1_{No}}}$ reduces to $-\mathbf{1_{No}}$.

Now we state the propositions:

(26) If $\mathbf{0_{No}} \leqslant x \leqslant y$, then $\sqrt{x} \leqslant \sqrt{y}$. The theorem is a consequence of (19).

(27) If $\mathbf{0_{No}} \leqslant x < y$, then $\sqrt{x} < \sqrt{y}$. The theorem is a consequence of (19).

(28) If $\mathbf{0_{No}} \leqslant x \approx y \cdot y$, then $y \approx \sqrt{x}$ or $y \approx -\sqrt{x}$. The theorem is a consequence of (19).

Let x be a positive surreal number. Let us observe that $\sqrt{x}$ is positive.

Now we state the propositions:

(29) If $\mathbf{0_{No}} \leqslant x$, then $\sqrt{x \cdot x} \approx x$. The theorem is a consequence of (28) and (19).

(30) If $\mathbf{0_{No}} < x$, then $\sqrt{x}^{-1} \approx \sqrt{x^{-1}}$. The theorem is a consequence of (21) and (28).

5. Square Root of Negative Surreal Numbers – Outside the Defined Range

Now we state the propositions:

(31) Let us consider a surreal number x. Suppose $\mathbf{0}_{\mathbf{No}} < x$. Then there exists a surreal number y such that

(i) $-\mathbf{1}_{\mathbf{No}} \approx y$, and

(ii) $\sqrt{y} < -x$, and

(iii) $y = \langle \emptyset, \{\mathbf{0}_{\mathbf{No}}, (\sqrt{x \cdot x + \mathbf{1}_{\mathbf{No}}} - x) \cdot (\sqrt{x \cdot x + \mathbf{1}_{\mathbf{No}}} - x)\} \rangle$.

The theorem is a consequence of (27), (29), (6), (15), (8), and (19).

(32) There exist surreal numbers x, y such that

(i) $x \approx y < \mathbf{0}_{\mathbf{No}}$, and

(ii) $\sqrt{x} < \sqrt{y}$.

The theorem is a consequence of (31).

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