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Some Number Relations

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Summary. In this article some number relations are formalized in the Mizar system.

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Introduction

Binary relations are interesting for the study of simple (di)graphs and in their own right (cf. [3], [8], [9]). The relationship between binary relations and graphs [7] has been formalized in [6]. Here some simple binary relations are introduced to the Mizar Mathematical Library (cf. [1], [2]): the successor relation, an additive and multiplicative relation and the modulo relation. These can be used in the future for e.g. canonical (directed) path or cycle graphs (where the vertices are just numbers) including the ray and double-ray graph [5]. More complicated structures [4] can be obtained by combining some of these binary relations, although that is out of scope for this article.

1. Preliminaries

One can verify that every natural number is natural-membered.

From now on X, Y denote sets and A denotes an ordinal number.

Let us consider X. The functor succRel(X) yielding a binary relation on X is defined by

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(Def. 1) for every sets x, y, $\langle x, y\rangle \in it$ iff $x, y \in X$ and $y = \operatorname{succ} x$.

One can verify that $\operatorname{succRel}(X)$ is asymmetric and $\operatorname{succRel}(\emptyset)$ is empty.

Let X be a trivial set. Note that $\operatorname{succRel}(X)$ is empty.

Now we state the propositions:

(1) If $X \subseteq Y$, then $\operatorname{succRel}(X) \subseteq \operatorname{succRel}(Y)$.

(2) $\operatorname{succRel}(X \cap Y) = \operatorname{succRel}(X) \cap \operatorname{succRel}(Y)$.

(3) $\operatorname{succRel}(X) \cup \operatorname{succRel}(Y) \subseteq \operatorname{succRel}(X \cup Y)$. The theorem is a consequence of (1).

(4) Let us consider ordinal numbers B, C. Then $\langle B, C\rangle \in \operatorname{succRel}(A)$ if and only if $\operatorname{succ} B \in A$ and $C = \operatorname{succ} B$.

(5) Let us consider an ordinal number B. Suppose $\operatorname{succ} B \in A$. Then $\langle B, \operatorname{succ} B\rangle \in \operatorname{succRel}(A)$.

(6) $\operatorname{succRel}(2) = \{\langle 0, 1\rangle\}$. The theorem is a consequence of (4).

(7) $\operatorname{succRel}(3) = \{\langle 0, 1\rangle, \langle 1, 2\rangle\}$. The theorem is a consequence of (5).

(8) $\operatorname{succRel}(4) = \{\langle 0, 1\rangle, \langle 1, 2\rangle, \langle 2, 3\rangle\}$. The theorem is a consequence of (5).

(9) $\operatorname{succRel}(5) = \{\langle 0, 1\rangle, \langle 1, 2\rangle, \langle 2, 3\rangle, \langle 3, 4\rangle\}$. The theorem is a consequence of (4).

(10) $\operatorname{succRel}(\omega) =$ the set of all $\langle n, n+1\rangle$ where n is a natural number.

2. Successor Relation

From now on z, z_1, z_2 denote complex numbers, r, r_1, r_2 denote real numbers, q, q_1, q_2 denote rational numbers, i, i_1, i_2 denote integers, and n, n_1, n_2 denote natural numbers.

Let us consider z. Let us note that $(\operatorname{curry} +_{\mathbb{C}})(z)$ is function-like and relation-like.

Let X be a complex-membered set. The functor $\operatorname{addRel}(X, z)$ yielding a binary relation on X is defined by the term

(Def. 2) $(\operatorname{curry} +_{\mathbb{C}})(z) \restriction^2 X$.

Let us observe that $\operatorname{addRel}(\emptyset, z)$ is empty.

Let us consider a complex-membered set X. Now we state the propositions:

(11) $\langle z_1, z_2\rangle \in \operatorname{addRel}(X, z)$ if and only if $z_1, z_2 \in X$ and $z_2 = z + z_1$.

(12) $\operatorname{addRel}(X, 0) = \mathrm{id}_X$. The theorem is a consequence of (11).

Let X be a complex-membered set and z be a non zero complex number. Observe that $\operatorname{addRel}(X, z)$ is asymmetric.

Let us consider complex-membered sets X, Y. Now we state the propositions:

(13) If $X \subseteq Y$, then $\mathrm{addRel}(X, z) \subseteq \mathrm{addRel}(Y, z)$. The theorem is a consequence of (11).

(14) $\mathrm{addRel}(X \cap Y, z) = \mathrm{addRel}(X, z) \cap \mathrm{addRel}(Y, z)$. The theorem is a consequence of (11).

(15) $\mathrm{addRel}(X, z) \cup \mathrm{addRel}(Y, z) \subseteq \mathrm{addRel}(X \cup Y, z)$. The theorem is a consequence of (13).

Let us consider a complex-membered set X. Now we state the propositions:

(16) $(\mathrm{addRel}(X, z))^{\smile} = \mathrm{addRel}(X, -z)$. The theorem is a consequence of (11).

(17) $(\mathrm{addRel}(X, z_1)) \cdot (\mathrm{addRel}(X, z_2)) \subseteq \mathrm{addRel}(X, z_1 + z_2)$. The theorem is a consequence of (11).

(18) $(\mathrm{addRel}(\mathbb{C}, z_1)) \cdot (\mathrm{addRel}(\mathbb{C}, z_2)) = \mathrm{addRel}(\mathbb{C}, z_1 + z_2)$. The theorem is a consequence of (17) and (11).

(19) $\langle z_1, z_1 + z\rangle \in \mathrm{addRel}(\mathbb{C}, z)$. The theorem is a consequence of (11).

(20) $\mathrm{addRel}(\mathbb{C}, z) =$ the set of all $\langle z_1, z_1 + z\rangle$ where z_1 is a complex number. The theorem is a consequence of (11).

Let us consider r. Let us note that $(\mathrm{curry} +_{\mathbb{R}})(r)$ is function-like and relation-like. Let us consider a real-membered set X. Now we state the propositions:

(21) $\mathrm{addRel}(X, r) = (\mathrm{curry} +_{\mathbb{R}})(r)|^2 X$. The theorem is a consequence of (11).

(22) $\langle r_1, r_2\rangle \in \mathrm{addRel}(X, r)$ if and only if $r_1, r_2 \in X$ and $r_2 = r + r_1$.

(23) $(\mathrm{addRel}(\mathbb{R}, r_1)) \cdot (\mathrm{addRel}(\mathbb{R}, r_2)) = \mathrm{addRel}(\mathbb{R}, r_1 + r_2)$. The theorem is a consequence of (17) and (11).

(24) $\langle r_1, r_1 + r\rangle \in \mathrm{addRel}(\mathbb{R}, r)$. The theorem is a consequence of (11).

(25) $\mathrm{addRel}(\mathbb{R}, r) =$ the set of all $\langle r_1, r_1 + r\rangle$ where r_1 is a real number. The theorem is a consequence of (11).

Let us consider q. Observe that $(\mathrm{curry} +_{\mathbb{Q}})(q)$ is function-like and relation-like. Let us consider a rational-membered set X. Now we state the propositions:

(26) $\mathrm{addRel}(X, q) = (\mathrm{curry} +_{\mathbb{Q}})(q)|^2 X$. The theorem is a consequence of (11).

(27) $\langle q_1, q_2\rangle \in \mathrm{addRel}(X, q)$ if and only if $q_1, q_2 \in X$ and $q_2 = q + q_1$.

(28) $(\mathrm{addRel}(\mathbb{Q}, q_1)) \cdot (\mathrm{addRel}(\mathbb{Q}, q_2)) = \mathrm{addRel}(\mathbb{Q}, q_1 + q_2)$. The theorem is a consequence of (17) and (11).

(29) $\langle q_1, q_1 + q\rangle \in \mathrm{addRel}(\mathbb{Q}, q)$. The theorem is a consequence of (11).

(30) $\mathrm{addRel}(\mathbb{Q}, q) =$ the set of all $\langle q_1, q_1 + q\rangle$ where q_1 is a rational number. The theorem is a consequence of (11).

Let us consider i. Let us observe that $(\mathrm{curry}(+_{\mathbb{Z}}))(i)$ is function-like and relation-like.

Let us consider an integer-membered set X. Now we state the propositions:

(31) $\mathrm{addRel}(X, i) = (\mathrm{curry}(+_{\mathbb{Z}}))(i) \mid^2 X$. The theorem is a consequence of (11).

(32) $\langle i_1, i_2 \rangle \in \mathrm{addRel}(X, i)$ if and only if $i_1, i_2 \in X$ and $i_2 = i + i_1$.

(33) $(\mathrm{addRel}(\mathbb{Z}, i_1)) \cdot (\mathrm{addRel}(\mathbb{Z}, i_2)) = \mathrm{addRel}(\mathbb{Z}, i_1 + i_2)$. The theorem is a consequence of (17) and (11).

(34) $\langle i_1, i_1 + i \rangle \in \mathrm{addRel}(\mathbb{Z}, i)$. The theorem is a consequence of (11).

(35) $\mathrm{addRel}(\mathbb{Z}, i) =$ the set of all $\langle i_1, i_1 + i \rangle$ where i_1 is an integer. The theorem is a consequence of (11).

Let us consider n. One can verify that $(\mathrm{curry} +_{\mathbb{N}})(n)$ is function-like and relation-like.

Let us consider a natural-membered set X. Now we state the propositions:

(36) $\mathrm{addRel}(X, n) = (\mathrm{curry} +_{\mathbb{N}})(n) \mid^2 X$. The theorem is a consequence of (11).

(37) $\langle n_1, n_2 \rangle \in \mathrm{addRel}(X, n)$ if and only if $n_1, n_2 \in X$ and $n_2 = n + n_1$.

(38) $(\mathrm{addRel}(\mathbb{N}, n_1)) \cdot (\mathrm{addRel}(\mathbb{N}, n_2)) = \mathrm{addRel}(\mathbb{N}, n_1 + n_2)$. The theorem is a consequence of (17) and (11).

(39) $\langle n_1, n_1 + n \rangle \in \mathrm{addRel}(\mathbb{N}, n)$. The theorem is a consequence of (11).

(40) $\mathrm{addRel}(\mathbb{N}, n) =$ the set of all $\langle n_1, n_1 + n \rangle$ where n_1 is a natural number. The theorem is a consequence of (11).

(41) Let us consider a natural-membered set X.
Then $\mathrm{addRel}(X, 1) = \mathrm{succRel}(X)$. The theorem is a consequence of (11).

3. Additive Relation

Let us consider z. Note that $(\mathrm{curry} \cdot_{\mathbb{C}})(z)$ is function-like and relation-like.

Let X be a complex-membered set. The functor $\mathrm{multRel}(X, z)$ yielding a binary relation on X is defined by the term

(Def. 3) $(\mathrm{curry} \cdot_{\mathbb{C}})(z) \mid^2 X$.

One can verify that $\mathrm{multRel}(\emptyset, z)$ is empty.

Let us consider a complex-membered set X. Now we state the propositions:

(42) $\langle z_1, z_2 \rangle \in \mathrm{multRel}(X, z)$ if and only if $z_1, z_2 \in X$ and $z_2 = z \cdot z_1$.

(43) If $0 \in X$, then $\mathrm{multRel}(X, 0) = X \times \{0\}$. The theorem is a consequence of (42).

(44) $\mathrm{multRel}(X, 1) = \mathrm{id}_X$. The theorem is a consequence of (42).

(45) If $z \neq 1$ and $z \neq -1$ and $0 \notin X$, then $\mathrm{multRel}(X, z)$ is asymmetric. The theorem is a consequence of (42).

Let us consider complex-membered sets X, Y. Now we state the propositions:

(46) If $X \subseteq Y$, then $\mathrm{multRel}(X, z) \subseteq \mathrm{multRel}(Y, z)$. The theorem is a consequence of (42).

(47) $\mathrm{multRel}(X \cap Y, z) = \mathrm{multRel}(X, z) \cap \mathrm{multRel}(Y, z)$. The theorem is a consequence of (42).

(48) $\mathrm{multRel}(X, z) \cup \mathrm{multRel}(Y, z) \subseteq \mathrm{multRel}(X \cup Y, z)$. The theorem is a consequence of (46).

Let us consider a complex-membered set X. Now we state the propositions:

(49) If $z \neq 0$, then $(\mathrm{multRel}(X, z))^{\smile} = \mathrm{multRel}(X, z^{-1})$. The theorem is a consequence of (42).

(50) $(\mathrm{multRel}(X, z_1)) \cdot (\mathrm{multRel}(X, z_2)) \subseteq \mathrm{multRel}(X, z_1 \cdot z_2)$. The theorem is a consequence of (42).

(51) $(\mathrm{multRel}(\mathbb{C}, z_1)) \cdot (\mathrm{multRel}(\mathbb{C}, z_2)) = \mathrm{multRel}(\mathbb{C}, z_1 \cdot z_2)$. The theorem is a consequence of (50) and (42).

(52) $\langle z_1, z_1 \cdot z \rangle \in \mathrm{multRel}(\mathbb{C}, z)$. The theorem is a consequence of (42).

(53) $\mathrm{multRel}(\mathbb{C}, z) =$ the set of all $\langle z_1, z_1 \cdot z \rangle$ where z_1 is a complex number. The theorem is a consequence of (42).

Let us consider r. One can check that $(\mathrm{curry} \cdot_{\mathbb{R}})(r)$ is function-like and relation-like. Let us consider a real-membered set X. Now we state the propositions:

(54) $\mathrm{multRel}(X, r) = (\mathrm{curry} \cdot_{\mathbb{R}})(r) |^2 X$. The theorem is a consequence of (42).

(55) $\langle r_1, r_2 \rangle \in \mathrm{multRel}(X, r)$ if and only if $r_1, r_2 \in X$ and $r_2 = r \cdot r_1$.

(56) $(\mathrm{multRel}(\mathbb{R}, r_1)) \cdot (\mathrm{multRel}(\mathbb{R}, r_2)) = \mathrm{multRel}(\mathbb{R}, r_1 \cdot r_2)$. The theorem is a consequence of (50) and (42).

(57) $\langle r_1, r_1 \cdot r \rangle \in \mathrm{multRel}(\mathbb{R}, r)$. The theorem is a consequence of (42).

(58) $\mathrm{multRel}(\mathbb{R}, r) =$ the set of all $\langle r_1, r_1 \cdot r \rangle$ where r_1 is a real number. The theorem is a consequence of (42).

Let us consider q. Note that $(\mathrm{curry} \cdot_{\mathbb{Q}})(q)$ is function-like and relation-like.

Let us consider a rational-membered set X. Now we state the propositions:

(59) $\mathrm{multRel}(X, q) = (\mathrm{curry} \cdot_{\mathbb{Q}})(q) |^2 X$. The theorem is a consequence of (42).

(60) $\langle q_1, q_2 \rangle \in \mathrm{multRel}(X, q)$ if and only if $q_1, q_2 \in X$ and $q_2 = q \cdot q_1$.

(61) $(\mathrm{multRel}(\mathbb{Q}, q_1)) \cdot (\mathrm{multRel}(\mathbb{Q}, q_2)) = \mathrm{multRel}(\mathbb{Q}, q_1 \cdot q_2)$. The theorem is a consequence of (50) and (42).

(62) $\langle q_1, q_1 \cdot q \rangle \in \mathrm{multRel}(\mathbb{Q}, q)$. The theorem is a consequence of (42).

(63) $\mathrm{multRel}(\mathbb{Q}, q) =$ the set of all $\langle q_1, q_1 \cdot q \rangle$ where q_1 is a rational number. The theorem is a consequence of (42).

Let us consider i. Let us note that $(\mathrm{curry} \cdot_{\mathbb{Z}})(i)$ is function-like and relation-like. Let us consider an integer-membered set X. Now we state the propositions:

(64) $\mathrm{multRel}(X, i) = (\mathrm{curry} \cdot_{\mathbb{Z}})(i) \mid^2 X$. The theorem is a consequence of (42).

(65) $\langle i_1, i_2 \rangle \in \mathrm{multRel}(X, i)$ if and only if $i_1, i_2 \in X$ and $i_2 = i \cdot i_1$.

(66) $(\mathrm{multRel}(\mathbb{Z}, i_1)) \cdot (\mathrm{multRel}(\mathbb{Z}, i_2)) = \mathrm{multRel}(\mathbb{Z}, i_1 \cdot i_2)$. The theorem is a consequence of (50) and (42).

(67) $\langle i_1, i_1 \cdot i \rangle \in \mathrm{multRel}(\mathbb{Z}, i)$. The theorem is a consequence of (42).

(68) $\mathrm{multRel}(\mathbb{Z}, i) =$ the set of all $\langle i_1, i_1 \cdot i \rangle$ where i_1 is an integer. The theorem is a consequence of (42).

Let us consider n. Observe that $(\mathrm{curry} \cdot_{\mathbb{N}})(n)$ is function-like and relation-like. Let us consider a natural-membered set X. Now we state the propositions:

(69) $\mathrm{multRel}(X, n) = (\mathrm{curry} \cdot_{\mathbb{N}})(n) \mid^2 X$. The theorem is a consequence of (42).

(70) $\langle n_1, n_2 \rangle \in \mathrm{multRel}(X, n)$ if and only if $n_1, n_2 \in X$ and $n_2 = n \cdot n_1$.

(71) $(\mathrm{multRel}(\mathbb{N}, n_1)) \cdot (\mathrm{multRel}(\mathbb{N}, n_2)) = \mathrm{multRel}(\mathbb{N}, n_1 \cdot n_2)$. The theorem is a consequence of (50) and (42).

(72) $\langle n_1, n_1 \cdot n \rangle \in \mathrm{multRel}(\mathbb{N}, n)$. The theorem is a consequence of (42).

(73) $\mathrm{multRel}(\mathbb{N}, n) =$ the set of all $\langle n_1, n_1 \cdot n \rangle$ where n_1 is a natural number. The theorem is a consequence of (42).

4. Multiplicative Relation

Let n be a non zero natural number. The functor $\mathrm{modRel}(n)$ yielding a binary relation on n is defined by the term

(Def. 4) $\mathrm{addRel}(n, 1) \cup \{\langle n - 1, 0 \rangle\}$.

Now we state the propositions:

(74) $\mathrm{modRel}(1) = \{\langle 0, 0 \rangle\}$. The theorem is a consequence of (41).

(75) $\mathrm{modRel}(2) = \{\langle 0, 1 \rangle, \langle 1, 0 \rangle\}$. The theorem is a consequence of (41) and (6).

(76) $\mathrm{modRel}(3) = \{\langle 0, 1 \rangle, \langle 1, 2 \rangle, \langle 2, 0 \rangle\}$. The theorem is a consequence of (41) and (7).

(77) $\mathrm{modRel}(4) = \{\langle 0, 1 \rangle, \langle 1, 2 \rangle, \langle 2, 3 \rangle, \langle 3, 0 \rangle\}$. The theorem is a consequence of (41) and (8).

(78) $\mathrm{modRel}(5) = \{\langle 0, 1 \rangle, \langle 1, 2 \rangle, \langle 2, 3 \rangle, \langle 3, 4 \rangle, \langle 4, 0 \rangle\}$. The theorem is a consequence of (41) and (9).

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