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Formalization of Trellises and Tolerance Relations

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Summary. The main aim of this article is to construct two non-trivial examples of weakly associative lattices (also known as trellises). These are generalizations of lattices, not assuming associativity of the lattice operations. We show some connections between trellises and tolerance relations according to the paper of Chajda and Zelinka.

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Introduction

Lattice theory is widely represented in the Mizar Mathematical Library, encoding the lines of Birkhoff [1] and Grätzer [13], [14]. In parallel, the theory of relational structures, with some additional assumptions, as partial ordering [3], was developed. From informal point of view, these notions can be easily unified, but using computerized proof-assistant, we have to cope with some important issues [10], [11]. The survey on the mechanization of lattice theory in Mizar, with the example of the solution of the Robbins problem, is contained in [6]. We made significant progress in translating [16] and [17] into Mizar formalism and reuse it within rough set theory [7].

The class of trellises [20], [19], or weakly associative lattices [4] (WA-lattices, WAL) can be characterized by the standard set of axioms for lattices (with idempotence for the join and meet operations included), where the ordinary associative laws are replaced by the so-called part-preservation laws (and these properties are equationally [12] defined as W3 and W3').

The main aim of this article is to show some connections between trellises and tolerance relations according to the paper of Chajda and Zelinka [2] and to construct two non-trivial examples of weakly associative lattices as defined in [18]. Some of the results were proven in the Mizar system with the help of Prover9 proof assistant.

The outline of this paper is as follows: Section 1 contains the set of usual properties of these lattices (some of them are automatically understood by the Mizar checker, but we keep them to have this presentation possibly self-contained) as WA-lattices are in fact pseudo-orders and the assumption of transitivity makes them partial ordering structures. Theorems 13 and 14 are part-preservation laws.

Section 2 contains introduces tolerance structures, with additional binary relation. It was already available in the MML, so it was reused here, and in this setting we defined notions of compatibility, SymmetricHull, Segment, tournament etc. from [2]. In Sect. 3, Theorem 1 from this paper is proven as (25). To express that a tolerance relation is reflexive or symmetric, we use prefix "$\beta-$" (analogous properties of the ordering relation are not prefixed). The next section recalls interconnections between lattice operations and orderings; we also prove that previously defined example of a near lattice, ExNearLattice, also satisfies W3 and W3', and it is the only fact proven in Sect. 5. Two next sections (6 and 7) continue the creation of technical environment for pseudo-orderings. Section 8 deals with cycles and tournaments.

Section 9 is devoted to the construction of `ExWALLattice` (see Def. 12 and Def. 13), where the lattice operations are given by the following two tables:

$\sqcup$	0	1	2	3	4
0	0	1	2	3	4
1	1	1	2	1	4
2	2	2	2	3	4
3	3	1	3	3	4
4	4	4	4	4	4

$\sqcap$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	1	3	1
2	0	1	2	2	2
3	0	3	2	3	3
4	0	1	2	3	4

The article concludes with the set of cluster registrations showing that associativity laws do not hold for this example, so this is not a lattice (elements 1, 2, 3 make a cycle). ExWALLattice satisfies however part-preservation laws (W3 and W3'), so it is an important non-trivial example of a bounded trellis. This structure is used in the literature to show some characteristic properties of triangular norms based on bounded trellises [15], [21], which will be an interesting future work; some encoding of the area of fuzzy logic was already done in [5], [8], and [9].

1. Preliminaries

One can verify that there exists a non empty relational structure which is reflexive and antisymmetric and has l.u.b.'s and g.l.b.'s. An RT-Lattice is an antisymmetric, non empty relational structure with l.u.b.'s and g.l.b.'s. A WA-lattice is a reflexive, antisymmetric, non empty relational structure with l.u.b.'s and g.l.b.'s. From now on W denotes a WA-lattice and a, b, c denote elements of W.

Now we state the propositions:

(1) $a \leqslant b$ if and only if $a \sqcup b = b$.

(2) $a \leqslant b$ if and only if $a \sqcap b = a$.

(3) $a \sqcap a = a$.

(4) $a \sqcup a = a$.

(5) $a \sqcap b = b \sqcap a$.

(6) $a \sqcup b = b \sqcup a$.

(7) $(a \sqcap b) \sqcup a = a$.

(8) $(a \sqcup b) \sqcap a = a$.

(9) If W is transitive, then $(a \sqcup b) \sqcup c = a \sqcup (b \sqcup c)$.

(10) If W is transitive, then $(a \sqcap b) \sqcap c = a \sqcap (b \sqcap c)$.

(11) If $a \leqslant c$ and $b \leqslant c$, then $a \sqcup b \leqslant c$.

(12) If $c \leqslant a$ and $c \leqslant b$, then $c \leqslant a \sqcap b$.

(13) $((a \sqcap c) \sqcup (b \sqcap c)) \sqcup c = c$. The theorem is a consequence of (7), (1), and (11).

(14) $((a \sqcup c) \sqcap (b \sqcup c)) \sqcap c = c$. The theorem is a consequence of (8), (2), and (12).

2. Tolerance Structures

Let D be a trivial, non empty set. Let us note that TotalPCS D is antisymmetric and there exists a pcs structure which is reflexive, antisymmetric, and non empty and has l.u.b.'s and g.l.b.'s.

A WAP-lattice is a reflexive, antisymmetric, non empty pcs structure with l.u.b.'s and g.l.b.'s. Let P be a pcs structure. We say that P is pcs-compatible if and only if

(Def. 1) for every elements a_1, a_2, b_1, b_2 of P such that $a_1 \sim b_1$ and $a_2 \sim b_2$ holds $a_1 \sqcup a_2 \sim b_1 \sqcup b_2$ and $a_1 \sqcap a_2 \sim b_1 \sqcap b_2$.

Let us observe that there exists a WAP-lattice which is trivial, total, and pcs-compatible.

Let W be a WA-lattice and a, b be elements of W. The functor Segment(a, b) yielding a subset of W is defined by the term

(Def. 2) $\{x$, where x is an element of $W : a \leqslant x \leqslant b$ or $b \leqslant x \leqslant a\}$.

Let W be a non empty relational structure. The functor SymmetricHull(W) yielding a binary relation on the carrier of W is defined by

(Def. 3) for every elements x, y of W, $\langle x, y\rangle \in it$ iff $x \leqslant y$ or $y \leqslant x$.

Let W be a WA-lattice. Let us observe that SymmetricHull(W) is total and SymmetricHull(W) is reflexive and symmetric. Let L be a WA-lattice. Observe that there exists a binary relation on the carrier of L which is total, reflexive, and symmetric. Now we state the proposition:

(15) Let us consider a WA-lattice W. Then SymmetricHull(W) is a tolerance of the carrier of W.

Let S be a non empty relational structure. We say that S is tournament if and only if

(Def. 4) for every elements x, y of S, $x \leqslant y$ or $y \leqslant x$.

Now we state the proposition:

(16) Let us consider a non empty relational structure S. Then S is tournament if and only if SymmetricHull$(S) = \nabla_\alpha$, where α is the carrier of S.
PROOF: If S is tournament, then SymmetricHull$(S) = \nabla_\alpha$, where α is the carrier of S. For every elements x, y of S, $x \leqslant y$ or $y \leqslant x$. $\square$

3. Compatibility Relation Revisited

Let us consider a non empty pcs structure P. Now we state the propositions:

(17) If the alternative relation of $P = \nabla_\alpha$, then P is pcs-compatible, where α is the carrier of P.

(18) If the alternative relation of $P = \mathrm{id}_\alpha$, then P is pcs-compatible, where α is the carrier of P.

One can verify that there exists a WAP-lattice which is pcs-compatible, β-reflexive, and β-symmetric. From now on W denotes a pcs-compatible, β-reflexive, β-symmetric WAP-lattice and a, b denote elements of W. Now we state the propositions:

(19) Let us consider an element x of W. Then $x \sim x$.

(20) Let us consider elements x, y of W. If $x \sim y$, then $y \sim x$.

Let us consider elements a, b of W. Now we state the propositions:

(21) If $a \sim b$, then $a \sqcap b \sim b \sqcap b$. The theorem is a consequence of (19).

(22) If $a \sim b$, then $a \sqcap b \sim b$. The theorem is a consequence of (19) and (3).

(23) If $a \sim b$, then $a \sqcap b \sim a$.

(24) If $a \sim b$, then $a \sqcup b \sim a$. The theorem is a consequence of (19) and (4).

(25) Let us consider a pcs-compatible, β-reflexive, β-symmetric WAP-lattice W, and elements a, b of W. Suppose $a \sim b$. Let us consider elements x, y of W. If x, $y \in \mathrm{Segment}(a \sqcap b, a \sqcup b)$, then $x \sim y$. The theorem is a consequence of (3), (19), (4), (2), and (1).

4. WA-Lattices and Pseudo-Orders

From now on L denotes a WA-lattice. Now we state the propositions:

(26) Let us consider a WA-lattice L, and elements x, y of L. Then $\langle x, y\rangle \in \leqslant_L$ if and only if $x \sqsubseteq y$.

(27) (i) $\mathrm{dom}(\leqslant_L) =$ the carrier of L, and

(ii) rng $\leqslant_L =$ the carrier of L, and

(iii) field $\leqslant_L =$ the carrier of L.

The theorem is a consequence of (26).

Let us consider L. One can verify that the functor $\leqslant_L$ yields a binary relation on the carrier of L. Let W be a non empty WA-lattice. Let us note that $\leqslant_W$ is (the carrier of W)-defined and $\leqslant_W$ is (the carrier of W)-valued.

Let us consider L. One can verify that $\leqslant_L$ is total as a binary relation on the carrier of L.

Let W be a non empty WA-lattice. One can verify that $\leqslant_W$ is reflexive and $\leqslant_W$ is antisymmetric.

5. ExNearLattice as an Example of WA-Lattice

One can check that ExNearLattice is satisfying W3 and satisfying W3'.

6. WA-Lattices as Lattices vs. Relational Structures

Let L be a WA-lattice. The functor $\langle L, \leqslant_L\rangle$ yielding a strict relational structure is defined by the term

(Def. 5) $\langle$the carrier of $L, \leqslant_L\rangle$.

Now we state the proposition:

(28) Let us consider a WA-lattice L, elements x, y of L, and elements x_3, y_3 of $\langle L, \leqslant_L \rangle$. If $x = x_3$ and $y = y_3$, then $x \sqsubseteq y$ iff $x_3 \leqslant y_3$. The theorem is a consequence of (26).

Let L be a WA-lattice and p be an element of L. The functor $p^{\bullet}$ yielding an element of $\langle L, \leqslant_L \rangle$ is defined by the term

(Def. 6) p.

Let p be an element of $\langle L, \leqslant_L \rangle$. The functor ${}^{\bullet}p$ yielding an element of L is defined by the term

(Def. 7) p.

One can check that $\langle L, \leqslant_L \rangle$ is reflexive and antisymmetric and $\langle L, \leqslant_L \rangle$ has l.u.b.'s and g.l.b.'s and there exists a WA-lattice which is strict.

7. On Pseudo-ordering Relations

Now we state the propositions:

(29) Let us consider WA-lattices L_1, L_2. Suppose $\langle L_1, \leqslant_{L_1} \rangle = \langle L_2, \leqslant_{L_2} \rangle$. Then the lattice structure of L_1 = the lattice structure of L_2. The theorem is a consequence of (26).

(30) Let us consider a WA-lattice A with l.u.b.'s g.l.b.'s. Then there exists a strict WA-lattice L such that the relational structure of $A = \langle L, \leqslant_L \rangle$.
Proof: Define $\mathcal{X}[\text{element of } A, \text{element of } A, \text{set}] \equiv$ for every elements x', y' of A such that $x' = \$_1$ and $y' = \$_2$ holds $\$_3 = x' \sqcup y'$. For every elements x, y of A, there exists an element u of A such that $\mathcal{X}[x, y, u]$. Consider j being a binary operation on the carrier of A such that for every elements x, y of A, $\mathcal{X}[x, y, j(x, y)]$. Define $\mathcal{Y}[\text{element of } A, \text{element of } A, \text{set}] \equiv$ for every elements x_1, y_1 of A such that $x_1 = \$_1$ and $y_1 = \$_2$ holds $\$_3 = x_1 \sqcap y_1$. For every elements x, y of A, there exists an element u of A such that $\mathcal{Y}[x, y, u]$. Consider m being a binary operation on the carrier of A such that for every elements x, y of A, $\mathcal{Y}[x, y, m(x, y)]$. Set $L = \langle \text{the carrier of } A, j, m \rangle$. L is join-commutative. L is meet-commutative. L is meet-idempotent. L is join-idempotent. For every elements v_2, v_1, v_0 of L, $((v_0 \sqcap v_1) \sqcup (v_2 \sqcap v_1)) \sqcup v_1 = v_1$. For every elements v_2, v_1, v_0 of L, $((v_0 \sqcup v_1) \sqcap (v_2 \sqcup v_1)) \sqcap v_1 = v_1$. $\leqslant_L =$ the internal relation of A. $\square$

(31) $\langle \text{ExNearLattice}, \leqslant_{\text{ExNearLattice}} \rangle$ is a WA-lattice.

Let A be a relational structure. Assume A is reflexive and antisymmetric and has l.u.b.'s and g.l.b.'s. The functor wlatt(A) yielding a strict WA-lattice is defined by

(Def. 8) the relational structure of $A = \langle it, \leqslant_{it} \rangle$.

Now we state the proposition:

(32) wlatt($\langle$ExNearLattice, $\leqslant_{\text{ExNearLattice}}\rangle$) is a WA-lattice.

One can verify that every meet-associative, join-absorbing lattice is satisfying W3 and every lattice is join-idempotent and meet-idempotent. Now we state the propositions:

(33) Let us consider a lattice L. Then LattRel$(L) = \leqslant_L$.

(34) Let us consider a poset L with l.u.b.'s g.l.b.'s, elements a, b of L, and elements a_3, b_3 of $\mathbb{L}_L$. If $a = a_3$ and $b = b_3$, then $a_3 \sqsubseteq b_3$ iff $a \leqslant b$. The theorem is a consequence of (33).

(35) Let us consider a WA-lattice L, elements a, b of L, and elements a_3, b_3 of wlatt(L). If $a = a_3$ and $b = b_3$, then $a_3 \sqsubseteq b_3$ iff $a \leqslant b$.

(36) Let us consider a WA-lattice L. Suppose the internal relation of L is transitive. Then wlatt(L) is a lattice. The theorem is a consequence of (34), (35), and (29).

8. Cycles and Tournaments

Let L be a WA-lattice and A be a set. We say that A is a cycle of L if and only if

(Def. 9) for every elements a, b of L such that $a \neq b$ and a, $b \in A$ holds $A =$ Segment(a, b).

Now we state the proposition:

(37) Let us consider a WA-lattice L, and an element a of L. Then $\{a\}$ is a cycle of L.

Let L be a WA-lattice and A be a subset of L. We say that A is cyclic if and only if

(Def. 10) A is a cycle of L.

One can check that there exists a subset of L which is cyclic.

A cycle of L is a cyclic subset of L. Now we state the proposition:

(38) Let us consider a WA-lattice L, and elements a, b, x of L. If $x \in$ Segment(a, b), then $a \leqslant x \leqslant b$ or $b \leqslant x \leqslant a$.

Let L be a WA-lattice and A be a subset of L. We say that A is tournament if and only if

(Def. 11) for every elements a, b of L such that a, $b \in A$ holds $a \leqslant b$ or $b \leqslant a$.

We introduce the notation A is acyclic as an antonym for A is cyclic.

9. Another Example of Weakly-Associative Lattice

Observe that $\{0,1,2,3,4\}$ is non empty and every element of $\{0,1,2,3,4\}$ is natural.

Let x, y be elements of $\{0,1,2,3,4\}$. The functors: $x \sqcup_{\rm WAL} y$ and $x \sqcap_{\rm WAL} y$ yielding elements of $\{0,1,2,3,4\}$ are defined by terms

(Def. 12) $\begin{cases} 1, & \textbf{if } x = 1 \text{ and } y = 3 \text{ or } x = 3 \text{ and } y = 1, \\ \max(x,y), & \textbf{otherwise}, \end{cases}$

(Def. 13) $\begin{cases} 3, & \textbf{if } x = 1 \text{ and } y = 3 \text{ or } x = 3 \text{ and } y = 1, \\ \min(x,y), & \textbf{otherwise}, \end{cases}$

respectively. The functors: $\sqcup_{\rm WAL}$ and $\sqcap_{\rm WAL}$ yielding binary operations on $\{0,1,2,3,4\}$ are defined by conditions

(Def. 14) for every elements x, y of $\{0,1,2,3,4\}$, $\sqcup_{\rm WAL}(x,y) = x \sqcup_{\rm WAL} y$,

(Def. 15) for every elements x, y of $\{0,1,2,3,4\}$, $\sqcap_{\rm WAL}(x,y) = x \sqcap_{\rm WAL} y$,

respectively. The functor ExWALattice yielding a non empty lattice structure is defined by the term

(Def. 16) $\langle\{0,1,2,3,4\}, \sqcup_{\rm WAL}, \sqcap_{\rm WAL}\rangle$.

One can verify that ExWALattice is non join-associative and non meet-associative and ExWALattice is join-commutative, meet-commutative, join-absorbing, and meet-absorbing and ExWALattice is satisfying W3 and satisfying W3'.

References

[1] Garrett Birkhoff. *Lattice Theory*. Providence, Rhode Island, New York, 1967.

[2] Ivan Chajda and Bohdan Zelinka. Weakly associative lattices and tolerance relations. *Czechoslovak Mathematical Journal*, 26(2):259–269, 1976.

[3] B.A. Davey and H.A. Priestley. *Introduction to Lattices and Order*. Cambridge University Press, 2002.

[4] Ervin Fried and George Grätzer. Some examples of weakly associative lattices. *Colloquium Mathematicum*, 27:215–221, 1973. doi:10.4064/cm-27-2-215-221.

[5] Adam Grabowski. On fuzzy negations and laws of contraposition. Lattice of fuzzy negations. *Formalized Mathematics*, 31(1):151–159, 2023. doi:10.2478/forma-2023-0014.

[6] Adam Grabowski. Mechanizing complemented lattices within Mizar system. *Journal of Automated Reasoning*, 55:211–221, 2015. doi:10.1007/s10817-015-9333-5.

[7] Adam Grabowski. Lattice theory for rough sets – a case study with Mizar. *Fundamenta Informaticae*, 147(2–3):223–240, 2016. doi:10.3233/FI-2016-1406.

[8] Adam Grabowski and Takashi Mitsuishi. *Extending Formal Fuzzy Sets with Triangular Norms and Conorms*, volume 642: *Advances in Intelligent Systems and Computing*, pages 176–187. Springer International Publishing, Cham, 2018. doi:10.1007/978-3-319-66824-6_16.

[9] Adam Grabowski and Takashi Mitsuishi. Formalizing lattice-theoretical aspects of rough and fuzzy sets. In D. Ciucci, G. Wang, S. Mitra, and W.Z. Wu, editors, *Rough Sets and Knowledge Technology – 10th International Conference held as part of the International Joint Conference on Rough Sets (IJCRS), Tianjin, PR China, November 20–23,*

2015, Proceedings, volume 9436 of *Lecture Notes in Artificial Intelligence*, pages 347–356. Springer, 2015. doi:10.1007/978-3-319-25754-9_31.

[10] Adam Grabowski and Markus Moschner. Managing heterogeneous theories within a mathematical knowledge repository. In Andrea Asperti, Grzegorz Bancerek, and Andrzej Trybulec, editors, *Mathematical Knowledge Management Proceedings*, volume 3119 of *Lecture Notes in Computer Science*, pages 116–129. Springer, 2004. doi:10.1007/978-3-540-27818-4_9. 3rd International Conference on Mathematical Knowledge Management, Bialowieza, Poland, Sep. 19–21, 2004.

[11] Adam Grabowski and Christoph Schwarzweller. Translating mathematical vernacular into knowledge repositories. In Michael Kohlhase, editor, *Mathematical Knowledge Management*, volume 3863 of *Lecture Notes in Computer Science*, pages 49–64. Springer, 2006. doi:10.1007/11618027_4. 4th International Conference on Mathematical Knowledge Management, Bremen, Germany, MKM 2005, July 15–17, 2005, Revised Selected Papers.

[12] Adam Grabowski, Artur Korniłowicz, and Christoph Schwarzweller. Equality in computer proof-assistants. In Ganzha, Maria and Maciaszek, Leszek and Paprzycki, Marcin, editor, *Proceedings of the 2015 Federated Conference on Computer Science and Information Systems*, volume 5 of *ACSIS-Annals of Computer Science and Information Systems*, pages 45–54. IEEE, 2015. doi:10.15439/2015F229.

[13] George Grätzer. *General Lattice Theory*. Academic Press, New York, 1978.

[14] George Grätzer. *Lattice Theory: Foundation*. Birkhäuser, 2011.

[15] Yu Kong and Bin Zhao. Uninorms on bounded trellises. *Fuzzy Sets and Systems*, 481: 108898, 2024. doi:10.1016/j.fss.2024.108898.

[16] William McCune and Ranganathan Padmanabhan. *Automated Deduction in Equational Logic and Cubic Curves*. Springer-Verlag, Berlin, 1996.

[17] Ranganathan Padmanabhan and Sergiu Rudeanu. *Axioms for Lattices and Boolean Algebras*. World Scientific Publishers, 2008.

[18] Damian Sawicki and Adam Grabowski. On weakly associative lattices and near lattices. *Formalized Mathematics*, 29(**2**):77–85, 2021. doi:10.2478/forma-2021-0008.

[19] Helen Skala. *Trellis Theory*. Providence, R.I.: American Mathematical Society, 1972.

[20] Helen L. Skala. Trellis theory. *Algebra Universalis*, 1:218–233, 1971. doi:10.1007/BF02944982.

[21] Lemnaouar Zedam and Bernard De Baets. Triangular norms on bounded trellises. *Fuzzy Sets and Systems*, 462:108468, 2023. doi:10.1016/j.fss.2023.01.003.

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