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Higher-Order Differentiation and Inverse Function Theorem in Real Normed Spaces[1]

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Summary. This article extends the formalization of the theory of differentiation in real normed spaces in the Mizar system. The focus is on higher-order derivatives and the inverse function theorem. Additionally, we encode the differentiability of the inversion operator on invertible linear operators.

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Introduction

This article, using the Mizar system [1], [2], extends the theory of differentiation in real normed spaces [5], focusing on higher-order derivatives and the inverse function theorem [7]. The work presents a comprehensive treatment of higher-order derivatives for vector-valued functions and develops theorems on the composition of differentiable functions and their higher-order derivatives [10], [11]. It provides an analysis of partial derivatives for multivariable vector-valued functions and extends the inverse function theorem to higher-order differentiability, including continuity properties [12]. Additionally, the paper formalizes the differentiability of the inversion operator on invertible linear operators [3].

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1. Foundations of Differentiation in Real Normed Spaces

From now on E, F, G, S, T, W, Y denote real normed spaces, f, f_1, f_2 denote partial functions from S to T, Z denotes a subset of S, and i, n denote natural numbers. Now we state the proposition:

(1) Let us consider real normed spaces S, T, a partial function f from S to T, a subset Z of S, and a point x of S. Suppose Z is open and $x \in Z$ and $Z \subseteq \operatorname{dom} f$. Then $f{\restriction}Z$ is differentiable in x if and only if f is differentiable in x.

Let us consider real normed spaces S, T, a partial function f from S to T, and a subset Z of S. Now we state the propositions:

(2) $f{\restriction}Z$ is differentiable on Z if and only if f is differentiable on Z.

(3) If f is differentiable on Z, then $(f{\restriction}Z)'_{\restriction Z} = f'_{\restriction Z}$. The theorem is a consequence of (2).

Let us consider real normed spaces S, T, a partial function f from S to T, and subsets X, Z of S. Now we state the propositions:

(4) If Z is open and $Z \subseteq X$ and f is differentiable on X, then $f'_{\restriction Z} = f'_{\restriction X}{\restriction}Z$.
Proof: For every object x such that $x \in \operatorname{dom}(f'_{\restriction X}{\restriction}Z)$ holds $(f'_{\restriction X}{\restriction}Z)(x) = f'_{\restriction Z}(x)$. □

(5) If Z is open and $Z \subseteq X$ and f is differentiable on X and $f'_{\restriction X}$ is continuous on X, then $f'_{\restriction Z}$ is continuous on Z. The theorem is a consequence of (4).

Let us consider real normed spaces S, T, a partial function f from S to T, a subset Z of S, and a natural number k. Now we state the propositions:

(6) Suppose f is differentiable k times on Z. Then

(i) $f{\restriction}Z$ is differentiable k times on Z, and

(ii) $\operatorname{diff}_Z(f{\restriction}Z, k) = \operatorname{diff}_Z(f, k)$.

Proof: Define $\mathcal{P}[\text{natural number}] \equiv$ if f is differentiable $\$_1$ times on Z, then $f{\restriction}Z$ is differentiable $\$_1$ times on Z and $\operatorname{diff}_Z(f{\restriction}Z, \$_1) = \operatorname{diff}_Z(f, \$_1)$. $\mathcal{P}[0]$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$. □

(7) Suppose f is differentiable k times on Z and $\operatorname{diff}_Z(f, k)$ is continuous on Z. Then

(i) $f{\restriction}Z$ is differentiable k times on Z, and

(ii) $\operatorname{diff}_Z(f{\restriction}Z, k)$ is continuous on Z.

Let us consider real normed spaces S, T, a partial function f from S to T, subsets X, Z of S, and a natural number i. Now we state the propositions:

(8) Suppose Z is open and $Z \subseteq X$. Then if f is differentiable i times on X, then f is differentiable i times on Z and $\mathrm{diff}_Z(f, i) = \mathrm{diff}_X(f, i) \restriction Z$.
PROOF: Define $\mathcal{P}$[natural number] $\equiv$ if f is differentiable $\$_1$ times on X, then f is differentiable $\$_1$ times on Z and $\mathrm{diff}_Z(f, \$_1) = \mathrm{diff}_X(f, \$_1) \restriction Z$. $\mathcal{P}[0]$. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$. For every natural number i, $\mathcal{P}[i]$. □

(9) Suppose Z is open and $Z \subseteq X$. Then suppose f is differentiable i times on X and $\mathrm{diff}_X(f, i)$ is continuous on X. Then

(i) f is differentiable i times on Z, and

(ii) $\mathrm{diff}_Z(f, i)$ is continuous on Z.

The theorem is a consequence of (8).

(10) Let us consider real normed spaces X, Y, a real number a, Lipschitzian linear operators v_1, v_2 from X into Y, and points w_1, w_2 of the real norm space of bounded linear operators from X into Y. Suppose $v_1 = w_1$ and $v_2 = w_2$. Then

(i) $v_1 + v_2 = w_1 + w_2$, and

(ii) $a \cdot v_1 = a \cdot w_1$.

PROOF: Reconsider $w_{12} = w_1 + w_2$ as a point of the real norm space of bounded linear operators from X into Y. For every object s such that $s \in \mathrm{dom}(v_1 + v_2)$ holds $(v_1 + v_2)(s) = w_{12}(s)$. Reconsider $w_{12} = a \cdot w_1$ as a point of the real norm space of bounded linear operators from X into Y. For every object s such that $s \in \mathrm{dom}(a \cdot v_1)$ holds $(a \cdot v_1)(s) = w_{12}(s)$. □

(11) Let us consider real normed spaces X, Y, Lipschitzian linear operators v_1, v_2, v_3 from X into Y, and real numbers a, b. Then

(i) $v_1 + v_2 = v_2 + v_1$, and

(ii) $(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$, and

(iii) $a \cdot (v_1 + v_2) = a \cdot v_1 + a \cdot v_2$, and

(iv) $(a + b) \cdot v_1 = a \cdot v_1 + b \cdot v_1$, and

(v) $a \cdot b \cdot v_1 = a \cdot (b \cdot v_1)$.

The theorem is a consequence of (10).

(12) Let us consider real normed spaces X, Y, Z, a Lipschitzian linear operator v from X into Y, a Lipschitzian linear operator s from Y into Z, a point p_6 of the real norm space of bounded linear operators from X into Y, and a point p_5 of the real norm space of bounded linear operators from Y into Z. If $v = p_6$ and $s = p_5$, then $s \cdot v = p_5 \cdot p_6$.

(13) Let us consider real normed spaces X, Y, Z, Lipschitzian linear operators v_1, v_2 from X into Y, Lipschitzian linear operators s_1, s_2 from Y into Z, and a real number a. Then

(i) $s_1 \cdot (v_1 + v_2) = s_1 \cdot v_1 + s_1 \cdot v_2$, and

(ii) $(s_1 + s_2) \cdot v_1 = s_1 \cdot v_1 + s_2 \cdot v_1$, and

(iii) $s_1 \cdot (a \cdot v_1) = a \cdot s_1 \cdot v_1$.

The theorem is a consequence of (10) and (12).

(14) Let us consider real normed spaces S, T, U, a partial function f_1 from S to T, a partial function f_2 from T to U, and a point x_0 of S. Suppose $x_0 \in \mathrm{dom}(f_2 \cdot f_1)$ and f_1 is continuous in x_0 and f_2 is continuous in f_{1/x_0}. Then $f_2 \cdot f_1$ is continuous in x_0.

PROOF: Set $f = f_2 \cdot f_1$. For every real number r such that $0 < r$ there exists a real number s such that $0 < s$ and for every point x_1 of S such that $x_1 \in \mathrm{dom}\, f$ and $\|x_1 - x_0\| < s$ holds $\|f_{/x_1} - f_{/x_0}\| < r$. □

(15) Let us consider real normed spaces E, F, G, a subset Z of E, a subset T of F, a partial function u from E to F, and a partial function v from F to G. Suppose $u^\circ Z \subseteq T$ and u is continuous on Z and v is continuous on T. Then $v \cdot u$ is continuous on Z.

PROOF: Set $f = v \cdot u$. For every point x_0 of E and for every real number r such that $x_0 \in Z$ and $0 < r$ there exists a real number s such that $0 < s$ and for every point x_1 of E such that $x_1 \in Z$ and $\|x_1 - x_0\| < s$ holds $\|f_{/x_1} - f_{/x_0}\| < r$. □

(16) Let us consider real normed spaces X, Y, a point x of X, a point y of Y, and a point z of $X \times Y$. Suppose $z = \langle x, y \rangle$. Then $\|z\| \leqslant \|x\| + \|y\|$.

(17) Let us consider real normed spaces E, F, G, a partial function u from E to F, a Lipschitzian linear operator L from F into G, and a point x of E. Suppose u is differentiable in x. Then

(i) $L \cdot u$ is differentiable in x, and

(ii) $(L \cdot u)'(x) = L \cdot u'(x)$.

(18) Let us consider real normed spaces E, F, G, a subset Z of E, a subset T of F, a partial function u from E to F, and a partial function v from F to G. Suppose $u^\circ Z \subseteq T$ and u is differentiable on Z and v is differentiable on T. Then

(i) $v \cdot u$ is differentiable on Z, and

(ii) for every point x of E such that $x \in Z$ holds $(v \cdot u)'_{\restriction Z/x} = v'_{\restriction T/u_{/x}} \cdot (u'_{\restriction Z/x})$.

Proof: For every point x of E such that $x \in Z$ holds $v \cdot u$ is differentiable in x and $(v \cdot u)'(x) = v'(u_{/x}) \cdot u'(x)$. For every point x of E such that $x \in Z$ holds $(v \cdot u)'_{\restriction Z/x} = v'_{\restriction T/u_{/x}} \cdot (u'_{\restriction Z/x})$. □

(19) Let us consider real normed spaces F, G, and a Lipschitzian linear operator L from F into G. Then

(i) L is differentiable on Ω_F, and

(ii) $L'_{\restriction \Omega_F}$ is continuous on Ω_F, and

(iii) for every point x of F, $L'_{\restriction \Omega_F/x} = L$.

(20) Let us consider real normed spaces E, F, G, a partial function u from E to F, a subset Z of E, and a Lipschitzian linear operator L from F into G. Suppose u is differentiable on Z. Then

(i) $L \cdot u$ is differentiable on Z, and

(ii) for every point x of E such that $x \in Z$ holds $(L \cdot u)'_{\restriction Z/x} = L \cdot (u'_{\restriction Z/x})$.

The theorem is a consequence of (19) and (18).

Let E, F, G be real normed spaces and L be a Lipschitzian linear operator from F into G. The functor $\mathrm{LTRN}(L, E)$ yielding a function is defined by

(Def. 1) $\mathrm{dom}\, it = \mathbb{N}$ and $it(0) = L$ and for every natural number i, there exists a Lipschitzian linear operator K from $\mathrm{diff}_{\mathrm{SP}}(E^{(i+1)}, F)$ into $\mathrm{diff}_{\mathrm{SP}}(E^{(i+1)}, G)$ and there exists a Lipschitzian linear operator M from $\mathrm{diff}_{\mathrm{SP}}(E^i, F)$ into $\mathrm{diff}_{\mathrm{SP}}(E^i, G)$ such that $it(i+1) = K$ and $it(i)(\in$ the real norm space of bounded linear operators from $\mathrm{diff}_{\mathrm{SP}}(E^i, F)$ into $\mathrm{diff}_{\mathrm{SP}}(E^i, G)) = M$ and for every Lipschitzian linear operator V from E into $\mathrm{diff}_{\mathrm{SP}}(E^i, F)$, $K(V) = M \cdot V$.

Let i be a natural number. The functor $\mathrm{LTRN}(i, L, E)$ yielding a Lipschitzian linear operator from $\mathrm{diff}_{\mathrm{SP}}(E^i, F)$ into $\mathrm{diff}_{\mathrm{SP}}(E^i, G)$ is defined by the term

(Def. 2) $(\mathrm{LTRN}(L, E))(i)$.

2. Higher-Order Differentiation and Function Composition

Now we state the propositions:

(21) Let us consider real normed spaces E, F, G, and a Lipschitzian linear operator L from F into G. Then

(i) $\mathrm{LTRN}(0, L, E) = L$, and

(ii) for every natural number i and for every Lipschitzian linear operator V from E into $\mathrm{diff}_{\mathrm{SP}}(E^i, F)$, $(\mathrm{LTRN}(i+1, L, E))(V) = (\mathrm{LTRN}(i, L, E)) \cdot V$.

(22) Let us consider real normed spaces E, F, G, a subset Z of E, a subset T of F, a partial function u from E to F, and a partial function v from F to G. Suppose $u^{\circ}Z \subseteq T$ and u is differentiable on Z and $u'_{\restriction Z}$ is continuous on Z and v is differentiable on T and $v'_{\restriction T}$ is continuous on T. Then

(i) $v \cdot u$ is differentiable on Z, and

(ii) $(v \cdot u)'_{\restriction Z}$ is continuous on Z.

PROOF: $v \cdot u$ is differentiable on Z and for every point x of E such that $x \in Z$ holds $(v \cdot u)'_{\restriction Z/x} = v'_{\restriction T/u_{/x}} \cdot (u'_{\restriction Z/x})$. Set $f = (v \cdot u)'_{\restriction Z}$. For every point x_0 of E and for every real number r such that $x_0 \in Z$ and $0 < r$ there exists a real number s such that $0 < s$ and for every point x_1 of E such that $x_1 \in Z$ and $\|x_1 - x_0\| < s$ holds $\|f_{/x_1} - f_{/x_0}\| < r$. □

(23) Let us consider real normed spaces E, F, G, a subset Z of E, a partial function u from E to F, and a Lipschitzian linear operator L from F into G. Suppose u is differentiable on Z and $u'_{\restriction Z}$ is continuous on Z. Then

(i) $L \cdot u$ is differentiable on Z, and

(ii) $(L \cdot u)'_{\restriction Z}$ is continuous on Z.

The theorem is a consequence of (19) and (22).

Let us consider real normed spaces E, F, G, a subset Z of E, a partial function u from E to F, a Lipschitzian linear operator L from F into G, and a natural number i. Now we state the propositions:

(24) Suppose u is differentiable i times on Z. Then

(i) $L \cdot u$ is differentiable i times on Z, and

(ii) $\mathrm{diff}_Z(L \cdot u, i) = (\mathrm{LTRN}(i, L, E)) \cdot \mathrm{diff}_Z(u, i)$.

PROOF: Define $\mathcal{P}$[natural number] $\equiv$ if u is differentiable $\$_1$ times on Z, then $L \cdot u$ is differentiable $\$_1$ times on Z and $\mathrm{diff}_Z(L \cdot u, \$_1) = (\mathrm{LTRN}(\$_1, L, E)) \cdot \mathrm{diff}_Z(u, \$_1)$. $\mathcal{P}[0]$. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$. For every natural number i, $\mathcal{P}[i]$. □

(25) Suppose u is differentiable i times on Z and $\mathrm{diff}_Z(u, i)$ is continuous on Z. Then

(i) $L \cdot u$ is differentiable i times on Z, and

(ii) $\mathrm{diff}_Z(L \cdot u, i)$ is continuous on Z.

The theorem is a consequence of (24) and (15).

(26) Let us consider real normed spaces S, T, U, a point x of S, a partial function u from S to T, a partial function v from S to U, and a partial function w from S to $T \times U$. Suppose u is differentiable in x and v is differentiable in x and $w = \langle u, v \rangle$. Then

(i) w is differentiable in x, and

(ii) $w'(x) = \langle u'(x), v'(x)\rangle$, and

(iii) for every point d_2 of S, $w'(x)(d_2) = \langle u'(x)(d_2), v'(x)(d_2)\rangle$.

PROOF: For every point t of S such that $t \in \operatorname{dom} w$ holds $w_{/t} = \langle f_{1/t}, f_{2/t}\rangle$. Consider N_1 being a neighbourhood of x_0 such that $N_1 \subseteq \operatorname{dom} f_1$ and there exists a point L of the real norm space of bounded linear operators from S into T and there exists a rest R of S, T such that for every point x of S such that $x \in N_1$ holds $f_{1/x} - f_{1/x_0} = L(x - x_0) + R_{/x-x_0}$. Consider L_1 being a point of the real norm space of bounded linear operators from S into T, R_1 being a rest of S, T such that for every point x of S such that $x \in N_1$ holds $f_{1/x} - f_{1/x_0} = L_1(x - x_0) + R_{1/x-x_0}$. Consider N_2 being a neighbourhood of x_0 such that $N_2 \subseteq \operatorname{dom} f_2$ and there exists a point L of the real norm space of bounded linear operators from S into U and there exists a rest R of S, U such that for every point x of S such that $x \in N_2$ holds $f_{2/x} - f_{2/x_0} = L(x - x_0) + R_{/x-x_0}$. Consider L_2 being a point of the real norm space of bounded linear operators from S into U, R_2 being a rest of S, U such that for every point x of S such that $x \in N_2$ holds $f_{2/x} - f_{2/x_0} = L_2(x - x_0) + R_{2/x-x_0}$. Define $\mathcal{O}(\text{object}) = \langle R_{1/\$_1}, R_{2/\$_1}\rangle$. Consider R being a function from S into $T \times U$ such that for every object d_2 such that $d_2 \in$ the carrier of S holds $R(d_2) = \mathcal{O}(d_2)$. For every real number r such that $r > 0$ there exists a real number d such that $d > 0$ and for every point z of S such that $z \neq 0_S$ and $\|z\| < d$ holds $\|z\|^{-1} \cdot \|R_{/z}\| < r$. Define $\mathcal{O}(\text{object}) = \langle L_1(\$_1), L_2(\$_1)\rangle$. For every object x such that $x \in$ the carrier of S holds $\mathcal{O}(x) \in$ the carrier of $T \times U$. Consider L being a function from S into $T \times U$ such that for every object d_2 such that $d_2 \in$ the carrier of S holds $L(d_2) = \mathcal{O}(d_2)$. For every elements x, y of S, $L(x + y) = L(x) + L(y)$. For every vector x of S and for every real number a, $L(a \cdot x) = a \cdot L(x)$. Set $K = \|L_1\| + \|L_2\|$. For every vector w of S, $\|L(w)\| \leqslant K \cdot \|w\|$. Consider N being a neighbourhood of x_0 such that $N \subseteq N_1$ and $N \subseteq N_2$. □

Let us consider real normed spaces S, T, U, a subset Z of S, a partial function u from S to T, a partial function v from S to U, and a partial function w from S to $T \times U$. Now we state the propositions:

(27) Suppose u is differentiable on Z and v is differentiable on Z and $w = \langle u, v\rangle$. Then

(i) w is differentiable on Z, and

(ii) for every point x of S such that $x \in Z$ holds $w'_{\restriction Z/x} = \langle u'_{\restriction Z/x}, v'_{\restriction Z/x}\rangle$, and

(iii) for every point x of S such that $x \in Z$ for every point d_2 of S, $(w'_{\restriction Z/x})(d_2) = \langle\!\langle (u'_{\restriction Z/x})(d_2), (v'_{\restriction Z/x})(d_2) \rangle\!\rangle$.

PROOF: For every point x of S such that $x \in Z$ holds w is differentiable in x. For every point x of S such that $x \in Z$ holds $w'_{\restriction Z/x} = \langle u'_{\restriction Z/x}, v'_{\restriction Z/x} \rangle$. For every point x of S such that $x \in Z$ for every point d_2 of S, $(w'_{\restriction Z/x})(d_2) = \langle\!\langle (u'_{\restriction Z/x})(d_2), (v'_{\restriction Z/x})(d_2) \rangle\!\rangle$. □

(28) Suppose u is differentiable on Z and $u'_{\restriction Z}$ is continuous on Z and v is differentiable on Z and $v'_{\restriction Z}$ is continuous on Z and $w = \langle u, v \rangle$. Then

(i) w is differentiable on Z, and

(ii) $w'_{\restriction Z}$ is continuous on Z.

PROOF: w is differentiable on Z and for every point x of S such that $x \in Z$ holds $w'_{\restriction Z/x} = \langle u'_{\restriction Z/x}, v'_{\restriction Z/x} \rangle$ and for every point x of S such that $x \in Z$ for every point d_2 of S, $(w'_{\restriction Z/x})(d_2) = \langle\!\langle (u'_{\restriction Z/x})(d_2), (v'_{\restriction Z/x})(d_2) \rangle\!\rangle$. Set $f = w'_{\restriction Z}$. For every point x_0 of S and for every real number r such that $x_0 \in Z$ and $0 < r$ there exists a real number s such that $0 < s$ and for every point x_1 of S such that $x_1 \in Z$ and $\|x_1 - x_0\| < s$ holds $\|f_{/x_1} - f_{/x_0}\| < r$. □

(29) Let us consider real normed spaces E, F, and a natural number i. Then $\mathrm{diff}_{\mathrm{SP}}(E^{(i+1)}, F) = \mathrm{diff}_{\mathrm{SP}}(E^{i},$ (the real norm space of bounded linear operators from E into F)).

PROOF: Define $\mathcal{P}$[natural number] $\equiv \mathrm{diff}_{\mathrm{SP}}(E^{(\$_1+1)}, F) = \mathrm{diff}_{\mathrm{SP}}(E^{\$_1},$ (the real norm space of bounded linear operators from E into F)). For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$ by [4, (10)]. For every natural number i, $\mathcal{P}[i]$. □

(30) Let us consider real normed spaces E, F, a subset Z of E, a partial function g from E to F, and a partial function f from E to the real norm space of bounded linear operators from E into F. Suppose $(g\restriction Z)'_{\restriction Z} = f\restriction Z$. Let us consider a natural number i. Then $\mathrm{diff}_Z(g, i+1) = \mathrm{diff}_Z(f, i)$.

PROOF: Define $\mathcal{P}$[natural number] $\equiv \mathrm{diff}_Z(g, \$_1 + 1) = \mathrm{diff}_Z(f, \$_1)$. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$. For every natural number i, $\mathcal{P}[i]$. □

Let us consider real normed spaces E, F, a natural number n, a subset Z of E, and a partial function g from E to F. Now we state the propositions:

(31) If $g'_{\restriction Z}$ is differentiable n times on Z and g is differentiable on Z, then g is differentiable $n+1$ times on Z.

PROOF: Set $f = g'_{\restriction Z}$. $f\restriction Z = (g\restriction Z)'_{\restriction Z}$. For every natural number i such that $i \leqslant n + 1 - 1$ holds $\mathrm{diff}_Z(g, i)$ is differentiable on Z. □

(32) Suppose $g'_{\restriction Z}$ is differentiable n times on Z and g is differentiable on Z and $\mathrm{diff}_Z(g'_{\restriction Z}, n)$ is continuous on Z. Then

(i) g is differentiable $n+1$ times on Z, and

(ii) $\mathrm{diff}_Z(g, n+1)$ is continuous on Z.

The theorem is a consequence of (31), (3), (30), and (29).

(33) Let us consider real normed spaces S, E, F, G, a Lipschitzian bilinear operator B from $E \times F$ into G, a partial function W from S to G, a partial function w from S to $E \times F$, a partial function u from S to E, a partial function v from S to F, and a point x of S. Suppose u is differentiable in x and v is differentiable in x and $x \in \mathrm{dom}\, w$ and $W = B \cdot w$ and $w = \langle u, v \rangle$. Then

(i) W is differentiable in x, and

(ii) w is differentiable in x, and

(iii) $W'(x) = B'(\langle\!\langle u_{/x}, v_{/x} \rangle\!\rangle) \cdot w'(x)$, and

(iv) $w'(x) = \langle u'(x), v'(x) \rangle$, and

(v) for every point d_1 of S, $W'(x)(d_1) =$
$B(u'(x)(d_1), v_{/x}) + B(u_{/x}, v'(x)(d_1))$.

The theorem is a consequence of (26).

(34) Let us consider real normed spaces S, E, F, G, a subset Z of S, a Lipschitzian bilinear operator B from $E \times F$ into G, a partial function W from S to G, a partial function w from S to $E \times F$, a partial function u from S to E, and a partial function v from S to F. Suppose u is differentiable on Z and v is differentiable on Z and $W = B \cdot w$ and $w = \langle u, v \rangle$. Then

(i) W is differentiable on Z, and

(ii) for every point x of S such that $x \in Z$ for every point d_1 of S, $(W'_{\restriction Z/x})(d_1) = B((u'_{\restriction Z/x})(d_1), v_{/x}) + B(u_{/x}, (v'_{\restriction Z/x})(d_1))$.

Proof: w is differentiable on Z and for every point x of S such that $x \in Z$ holds $w'_{\restriction Z/x} = \langle u'_{\restriction Z/x}, v'_{\restriction Z/x} \rangle$ and for every point x of S such that $x \in Z$ for every point d_2 of S, $(w'_{\restriction Z/x})(d_2) = \langle\!\langle (u'_{\restriction Z/x})(d_2), (v'_{\restriction Z/x})(d_2) \rangle\!\rangle$. For every point x of S such that $x \in Z$ for every point d_1 of S, $(W'_{\restriction Z/x})(d_1) = B((u'_{\restriction Z/x})(d_1), v_{/x}) + B(u_{/x}, (v'_{\restriction Z/x})(d_1))$. $\square$

Let us consider real normed spaces S, E, a subset Z of S, a partial function u from S to E, and a natural number i. Now we state the propositions:

(35) If u is differentiable $i+1$ times on Z, then $u'_{\restriction Z}$ is differentiable i times on Z.

PROOF: Define $\mathcal{P}$[natural number] $\equiv$ if u is differentiable $\$_1 + 1$ times on Z, then $u'_{\restriction Z}$ is differentiable $\$_1$ times on Z. $\mathcal{P}[0]$. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$. For every natural number i, $\mathcal{P}[i]$. □

(36) Suppose u is differentiable $i+1$ times on Z and $\mathrm{diff}_Z(u, i+1)$ is continuous on Z. Then

(i) $u'_{\restriction Z}$ is differentiable i times on Z, and

(ii) $\mathrm{diff}_Z(u'_{\restriction Z}, i)$ is continuous on Z.

The theorem is a consequence of (35), (29), (3), and (30).

(37) Let us consider real normed spaces E, F, G. Then there exists a Lipschitzian bilinear operator B from the real norm space of bounded linear operators from E into F $\times$ the real norm space of bounded linear operators from F into G into the real norm space of bounded linear operators from E into G such that for every point u of the real norm space of bounded linear operators from E into F for every point v of the real norm space of bounded linear operators from F into G, $B(u, v) = v \cdot u$.

PROOF: Set E_3 = the carrier of the real norm space of bounded linear operators from E into F. Set F_2 = the carrier of the real norm space of bounded linear operators from F into G. Set E_4 = the carrier of the real norm space of bounded linear operators from E into G. Define $\mathcal{P}$[element of E_3, element of F_2, object] $\equiv \$_3 = \$_2 \cdot \$_1$. Consider L being a function from $E_3 \times F_2$ into E_4 such that for every element x of E_3 and for every element y of F_2, $\mathcal{P}[x, y, L(x, y)]$. Set L_4 = the real norm space of bounded linear operators from E into F. Set L_5 = the real norm space of bounded linear operators from F into G. For every points x_1, x_2 of L_4 and for every point y of L_5, $L(x_1 + x_2, y) = L(x_1, y) + L(x_2, y)$. For every point x of L_4 and for every point y of L_5 and for every real number a, $L(a \cdot x, y) = a \cdot L(x, y)$. For every point x of L_4 and for every points y_1, y_2 of L_5, $L(x, y_1 + y_2) = L(x, y_1) + L(x, y_2)$. For every point x of L_4 and for every point y of L_5 and for every real number a, $L(x, a \cdot y) = a \cdot L(x, y)$. Set $K = 1$. For every vector x of L_4 and for every vector y of L_5, $\|L(x, y)\| \leqslant K \cdot \|x\| \cdot \|y\|$. □

(38) Let us consider a natural number i, real normed spaces S, E, F, G, a subset Z of S, a Lipschitzian bilinear operator B from $E \times F$ into G, a partial function u from S to E, a partial function v from S to F, a partial function w from S to $E \times F$, and a partial function W from S to G. Suppose $W = B \cdot w$ and $w = \langle u, v \rangle$ and u is differentiable i times on Z and v is differentiable i times on Z. Then W is differentiable i times on Z.

PROOF: Define $\mathcal{P}$[natural number] $\equiv$ for every real normed spaces S, E,

F, G for every subset Z of S for every Lipschitzian bilinear operator B from $E \times F$ into G for every partial function u from S to E for every partial function v from S to F for every partial function w from S to $E \times F$ for every partial function W from S to G such that $W = B \cdot w$ and $w = \langle u, v \rangle$ and u is differentiable $\$_1$ times on Z and v is differentiable $\$_1$ times on Z holds W is differentiable $\$_1$ times on Z. $\mathcal{P}[0]$. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$. For every natural number i, $\mathcal{P}[i]$. □

(39) Let us consider real normed spaces S, E, F, G, a subset Z of S, a Lipschitzian bilinear operator B from $E \times F$ into G, a partial function W from S to G, a partial function w from S to $E \times F$, a partial function u from S to E, and a partial function v from S to F. Suppose u is differentiable on Z and $u'_{\restriction Z}$ is continuous on Z and v is differentiable on Z and $v'_{\restriction Z}$ is continuous on Z and $W = B \cdot w$ and $w = \langle u, v \rangle$. Then

(i) W is differentiable on Z, and

(ii) $W'_{\restriction Z}$ is continuous on Z.

The theorem is a consequence of (28) and (22).

(40) Let us consider real normed spaces S, E, F, G, a subset Z of S, a partial function u from S to E, a partial function v from S to F, and a partial function w from S to $E \times F$. Suppose $w = \langle u, v \rangle$ and u is continuous on Z and v is continuous on Z. Then w is continuous on Z.

Proof: For every point x_0 of S and for every real number r such that $x_0 \in Z$ and $0 < r$ there exists a real number s such that $0 < s$ and for every point x_1 of S such that $x_1 \in Z$ and $\|x_1 - x_0\| < s$ holds $\|w_{/x_1} - w_{/x_0}\| < r$ by [9, (18)], (16). □

(41) Let us consider a natural number i, real normed spaces S, E, F, G, a subset Z of S, a Lipschitzian bilinear operator B from $E \times F$ into G, a partial function u from S to E, a partial function v from S to F, a partial function w from S to $E \times F$, and a partial function W from S to G. Suppose $W = B \cdot w$ and $w = \langle u, v \rangle$ and u is differentiable i times on Z and $\mathrm{diff}_Z(u, i)$ is continuous on Z and v is differentiable i times on Z and $\mathrm{diff}_Z(v, i)$ is continuous on Z. Then

(i) W is differentiable i times on Z, and

(ii) $\mathrm{diff}_Z(W, i)$ is continuous on Z.

Proof: Define $\mathcal{P}[\text{natural number}] \equiv$ for every real normed spaces S, E, F, G for every subset Z of S for every Lipschitzian bilinear operator B from $E \times F$ into G for every partial function u from S to E for every partial function v from S to F for every partial function w from S to

$E \times F$ for every partial function W from S to G such that $W = B \cdot w$ and $w = \langle u, v \rangle$ and u is differentiable $\$_1$ times on Z and $\mathrm{diff}_Z(u, \$_1)$ is continuous on Z and v is differentiable $\$_1$ times on Z and $\mathrm{diff}_Z(v, \$_1)$ is continuous on Z holds W is differentiable $\$_1$ times on Z and $\mathrm{diff}_Z(W, \$_1)$ is continuous on Z. $\mathcal{P}[0]$. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$. For every natural number i, $\mathcal{P}[i]$. □

Let us consider a natural number i, real normed spaces E, F, G, a subset Z of E, a subset T of F, a partial function u from E to F, and a partial function v from F to G. Now we state the propositions:

(42) If $u^\circ Z \subseteq T$ and u is differentiable i times on Z and v is differentiable i times on T, then $v \cdot u$ is differentiable i times on Z.

PROOF: Define $\mathcal{P}$[natural number] $\equiv$ for every real normed spaces E, F, G for every subset Z of E for every subset T of F for every partial function u from E to F for every partial function v from F to G such that $u^\circ Z \subseteq T$ and u is differentiable $\$_1$ times on Z and v is differentiable $\$_1$ times on T holds $v \cdot u$ is differentiable $\$_1$ times on Z. $\mathcal{P}[0]$. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$. For every natural number i, $\mathcal{P}[i]$. □

(43) Suppose $u^\circ Z \subseteq T$ and u is differentiable i times on Z and $\mathrm{diff}_Z(u, i)$ is continuous on Z and v is differentiable i times on T and $\mathrm{diff}_T(v, i)$ is continuous on T. Then

(i) $v \cdot u$ is differentiable i times on Z, and

(ii) $\mathrm{diff}_Z(v \cdot u, i)$ is continuous on Z.

PROOF: Define $\mathcal{P}$[natural number] $\equiv$ for every real normed spaces E, F, G for every subset Z of E for every subset T of F for every partial function u from E to F for every partial function v from F to G such that $u^\circ Z \subseteq T$ and u is differentiable $\$_1$ times on Z and $\mathrm{diff}_Z(u, \$_1)$ is continuous on Z and v is differentiable $\$_1$ times on T and $\mathrm{diff}_T(v, \$_1)$ is continuous on T holds $v \cdot u$ is differentiable $\$_1$ times on Z and $\mathrm{diff}_Z(v \cdot u, \$_1)$ is continuous on Z. $\mathcal{P}[0]$. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$. For every natural number i, $\mathcal{P}[i]$. □

3. Partial Differentiation of Vector-Valued Functions

Now we state the proposition:

(44) Let us consider real normed spaces E, F, G, a subset Z of $E \times F$, and a partial function f from $E \times F$ to G. Suppose f is differentiable on Z. Then

(i) f is partially differentiable on Z w.r.t. 1, and

(ii) f is partially differentiable on Z w.r.t. 2, and

(iii) for every point z of $E \times F$ such that $z \in Z$ holds for every point d_2 of E, $((f \restriction^1 Z)_{/z})(d_2) = f'_{\restriction Z/z}(d_2, 0_F)$ and for every point d_4 of F, $((f \restriction^2 Z)_{/z})(d_4) = f'_{\restriction Z/z}(0_E, d_4)$.

Proof: For every point z of $E \times F$ such that $z \in Z$ holds f is partially differentiable in z w.r.t. 1. For every point z of $E \times F$ such that $z \in Z$ holds f is partially differentiable in z w.r.t. 2. For every point d_2 of E, $((f \restriction^1 Z)_{/z})(d_2) = f'_{\restriction Z/z}(d_2, 0_F)$. For every point d_4 of F, $((f \restriction^2 Z)_{/z})(d_4) = f'_{\restriction Z/z}(0_E, d_4)$. □

Let us consider real normed spaces E, F. Now we state the propositions:

(45) There exists a Lipschitzian linear operator L_{10} from E into $E \times F$ such that for every point d_2 of E, $L_{10}(d_2) = \langle d_2, 0_F \rangle$.

Proof: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists a point d_2 of E such that $d_2 = \$_1$ and $\$_2 = \langle d_2, 0_F \rangle$. For every object x such that $x \in$ the carrier of E there exists an object y such that $y \in$ the carrier of $E \times F$ and $\mathcal{P}[x, y]$. Consider L_1 being a function from the carrier of E into the carrier of $E \times F$ such that for every object x such that $x \in$ the carrier of E holds $\mathcal{P}[x, L_1(x)]$. For every point d_2 of E, $L_1(d_2) = \langle d_2, 0_F \rangle$. For every elements x, y of E, $L_1(x + y) = L_1(x) + L_1(y)$. For every vector x of E and for every real number a, $L_1(a \cdot x) = a \cdot L_1(x)$. Set $K = 1$. For every vector x of E, $\|L_1(x)\| \leqslant K \cdot \|x\|$. □

(46) There exists a Lipschitzian linear operator L_{20} from F into $E \times F$ such that for every point d_4 of F, $L_{20}(d_4) = \langle 0_E, d_4 \rangle$.

Proof: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists a point d_2 of F such that $d_2 = \$_1$ and $\$_2 = \langle 0_E, d_2 \rangle$. For every object x such that $x \in$ the carrier of F there exists an object y such that $y \in$ the carrier of $E \times F$ and $\mathcal{P}[x, y]$. Consider L_1 being a function from the carrier of F into the carrier of $E \times F$ such that for every object x such that $x \in$ the carrier of F holds $\mathcal{P}[x, L_1(x)]$. For every point d_2 of F, $L_1(d_2) = \langle 0_E, d_2 \rangle$. For every elements x, y of F, $L_1(x + y) = L_1(x) + L_1(y)$. For every vector x of F and for every real number a, $L_1(a \cdot x) = a \cdot L_1(x)$. Set $K = 1$. For every vector x of F, $\|L_1(x)\| \leqslant K \cdot \|x\|$. □

(47) Let us consider real normed spaces E, F, a non empty subset Z of E, a partial function L_1 from E to F, and a point L_0 of F. Suppose Z is open and $L_1 = Z \longmapsto L_0$. Then

(i) L_1 is differentiable on Z, and

(ii) ${L_1}'_{\restriction Z}$ is continuous on Z, and

(iii) ${L_1}'_{\restriction Z} = Z \longmapsto 0_{\alpha}$, where α is the real norm space of bounded linear operators from E into F.

PROOF: For every object z such that $z \in \operatorname{dom} L_1{}'_{\restriction Z}$ holds $L_1{}'_{\restriction Z}(z) = 0_\alpha$, where α is the real norm space of bounded linear operators from E into F. □

(48) Let us consider real normed spaces E, F, a non empty subset Z of E, a partial function L_1 from E to F, and a point L_0 of F. Suppose Z is open and $L_1 = Z \longmapsto L_0$. Let us consider a natural number i. Then

(i) there exists a point P of $\operatorname{diff}_{\mathrm{SP}}(E^i, F)$ such that $\operatorname{diff}_Z(L_1, i) = Z \longmapsto P$, and

(ii) $\operatorname{diff}_Z(L_1, i)$ is differentiable on Z, and

(iii) $\operatorname{diff}_Z(L_1, i)'_{\restriction Z}$ is continuous on Z.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ there exists a point P of $\operatorname{diff}_{\mathrm{SP}}(E^{\$_1}, F)$ such that $\operatorname{diff}_Z(L_1, \$_1) = Z \longmapsto P$ and $\operatorname{diff}_Z(L_1, \$_1)$ is differentiable on Z and $\operatorname{diff}_Z(L_1, \$_1)'_{\restriction Z}$ is continuous on Z. $\mathcal{P}[0]$. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$. For every natural number i, $\mathcal{P}[i]$. □

(49) Let us consider a natural number i, real normed spaces E, F, a non empty subset Z of E, a partial function L_1 from E to F, and a point L_0 of F. Suppose Z is open and $L_1 = Z \longmapsto L_0$. Then

(i) L_1 is differentiable i times on Z, and

(ii) $\operatorname{diff}_Z(L_1, i)'_{\restriction Z}$ is continuous on Z.

The theorem is a consequence of (48).

(50) Let us consider a natural number n, a real normed space S, a subset Z of S, and a partial function f from S to S. Suppose Z is open and $f = \mathrm{id}_{\Omega_S}$. Then

(i) f is differentiable n times on Z, and

(ii) $\operatorname{diff}_Z(f, n)$ is continuous on Z.

The theorem is a consequence of (9).

Let us consider a natural number i, real normed spaces E, F, G, a non empty subset Z of $E \times F$, and a partial function f from $E \times F$ to G. Now we state the propositions:

(51) If f is differentiable $i+1$ times on Z, then $f \restriction^1 Z$ is differentiable i times on Z and $f \restriction^2 Z$ is differentiable i times on Z.

PROOF: f is differentiable on Z. f is partially differentiable on Z w.r.t. 1 and f is partially differentiable on Z w.r.t. 2 and for every point z of $E \times F$ such that $z \in Z$ holds for every point d_2 of E, $((f \restriction^1 Z)_{/z})(d_2) = f'_{\restriction Z/z}(d_2, 0_F)$ and for every point d_4 of F, $((f \restriction^2 Z)_{/z})(d_4) = f'_{\restriction Z/z}(0_E, d_4)$.

Set $P_1 = f \restriction^1 Z$. Set $P_2 = f \restriction^2 Z$. Consider L_{10} being a Lipschitzian linear operator from E into $E \times F$ such that for every point d_2 of E, $L_{10}(d_2) = \langle d_2, 0_F \rangle$. Consider L_{20} being a Lipschitzian linear operator from F into $E \times F$ such that for every point d_4 of F, $L_{20}(d_4) = \langle 0_E, d_4 \rangle$. Set $B_3 =$ the real norm space of bounded linear operators from E into $E \times F$. Set $B_2 =$ the real norm space of bounded linear operators from F into $E \times F$. Reconsider $L_1 = Z \longmapsto L_{10}$ as a partial function from $E \times F$ to B_3. L_1 is differentiable i times on Z. For every point z of $E \times F$ such that $z \in Z$ holds $(f \restriction^1 Z)_{/z} = f'_{\restriction Z/z} \cdot (L_{1/z})$. $f'_{\restriction Z}$ is differentiable i times on Z. Consider B being a Lipschitzian bilinear operator from the real norm space of bounded linear operators from E into $E \times F \times$ the real norm space of bounded linear operators from $E \times F$ into G into the real norm space of bounded linear operators from E into G such that for every point u of the real norm space of bounded linear operators from E into $E \times F$ and for every point v of the real norm space of bounded linear operators from $E \times F$ into G, $B(u, v) = v \cdot u$. Set $w_2 = \langle L_1, f'_{\restriction Z} \rangle$. Reconsider $W = B \cdot w_2$ as a partial function from $E \times F$ to the real norm space of bounded linear operators from E into G. W is differentiable i times on Z. For every object x_0 such that $x_0 \in \operatorname{dom} P_1$ holds $P_1(x_0) = W(x_0)$. Reconsider $L_2 = Z \longmapsto L_{20}$ as a partial function from $E \times F$ to B_2. L_2 is differentiable i times on Z. For every point z of $E \times F$ such that $z \in Z$ holds $(f \restriction^2 Z)_{/z} = f'_{\restriction Z/z} \cdot (L_{2/z})$. $f'_{\restriction Z}$ is differentiable i times on Z. Consider B being a Lipschitzian bilinear operator from the real norm space of bounded linear operators from F into $E \times F \times$ the real norm space of bounded linear operators from $E \times F$ into G into the real norm space of bounded linear operators from F into G such that for every point u of the real norm space of bounded linear operators from F into $E \times F$ and for every point v of the real norm space of bounded linear operators from $E \times F$ into G, $B(u, v) = v \cdot u$. Set $w_2 = \langle L_2, f'_{\restriction Z} \rangle$. Reconsider $W = B \cdot w_2$ as a partial function from $E \times F$ to the real norm space of bounded linear operators from F into G. For every object x_0 such that $x_0 \in \operatorname{dom} P_2$ holds $P_2(x_0) = W(x_0)$. □

(52) Suppose f is differentiable $i+1$ times on Z and $\operatorname{diff}_Z(f, i+1)$ is continuous on Z. Then

(i) $f \restriction^1 Z$ is differentiable i times on Z, and

(ii) $\operatorname{diff}_Z(f \restriction^1 Z, i)$ is continuous on Z, and

(iii) $f \restriction^2 Z$ is differentiable i times on Z, and

(iv) $\operatorname{diff}_Z(f \restriction^2 Z, i)$ is continuous on Z.

Proof: f is differentiable on Z. f is partially differentiable on Z w.r.t.

1 and f is partially differentiable on Z w.r.t. 2 and for every point z of $E \times F$ such that $z \in Z$ holds for every point d_2 of E, $((f \upharpoonright^1 Z)_{/z})(d_2) = f'_{\upharpoonright Z/z}(d_2, 0_F)$ and for every point d_4 of F, $((f \upharpoonright^2 Z)_{/z})(d_4) = f'_{\upharpoonright Z/z}(0_E, d_4)$. Set $P_1 = f \upharpoonright^1 Z$. Set $P_2 = f \upharpoonright^2 Z$. Consider L_{10} being a Lipschitzian linear operator from E into $E \times F$ such that for every point d_2 of E, $L_{10}(d_2) = \langle d_2, 0_F\rangle$. Consider L_{20} being a Lipschitzian linear operator from F into $E \times F$ such that for every point d_4 of F, $L_{20}(d_4) = \langle 0_E, d_4\rangle$. Set $B_3 =$ the real norm space of bounded linear operators from E into $E \times F$. Set $B_2 =$ the real norm space of bounded linear operators from F into $E \times F$. Reconsider $L_1 = Z \longmapsto L_{10}$ as a partial function from $E \times F$ to B_3. L_1 is differentiable i times on Z. L_1 is differentiable $i+1$ times on Z. For every point z of $E \times F$ such that $z \in Z$ holds $(f \upharpoonright^1 Z)_{/z} = f'_{\upharpoonright Z/z} \cdot (L_{1/z})$. $f'_{\upharpoonright Z}$ is differentiable i times on Z and $\mathrm{diff}_Z(f'_{\upharpoonright Z}, i)$ is continuous on Z. Consider B being a Lipschitzian bilinear operator from the real norm space of bounded linear operators from E into $E \times F$ $\times$ the real norm space of bounded linear operators from $E \times F$ into G into the real norm space of bounded linear operators from E into G such that for every point u of the real norm space of bounded linear operators from E into $E \times F$ and for every point v of the real norm space of bounded linear operators from $E \times F$ into G, $B(u, v) = v \cdot u$. Set $w_2 = \langle L_1, f'_{\upharpoonright Z}\rangle$. Reconsider $W = B \cdot w_2$ as a partial function from $E \times F$ to the real norm space of bounded linear operators from E into G. W is differentiable i times on Z and $\mathrm{diff}_Z(W, i)$ is continuous on Z. For every object x_0 such that $x_0 \in \mathrm{dom}\, P_1$ holds $P_1(x_0) = W(x_0)$. Reconsider $L_2 = Z \longmapsto L_{20}$ as a partial function from $E \times F$ to B_2. L_2 is differentiable i times on Z. L_2 is differentiable $i+1$ times on Z. For every point z of $E \times F$ such that $z \in Z$ holds $(f \upharpoonright^2 Z)_{/z} = f'_{\upharpoonright Z/z} \cdot (L_{2/z})$. $f'_{\upharpoonright Z}$ is differentiable i times on Z and $\mathrm{diff}_Z(f'_{\upharpoonright Z}, i)$ is continuous on Z. Consider B being a Lipschitzian bilinear operator from the real norm space of bounded linear operators from F into $E \times F$ $\times$ the real norm space of bounded linear operators from $E \times F$ into G into the real norm space of bounded linear operators from F into G such that for every point u of the real norm space of bounded linear operators from F into $E \times F$ and for every point v of the real norm space of bounded linear operators from $E \times F$ into G, $B(u, v) = v \cdot u$. Set $w_2 = \langle L_2, f'_{\upharpoonright Z}\rangle$. Reconsider $W = B \cdot w_2$ as a partial function from $E \times F$ to the real norm space of bounded linear operators from F into G. W is differentiable i times on Z and $\mathrm{diff}_Z(W, i)$ is continuous on Z. For every object x_0 such that $x_0 \in \mathrm{dom}\, P_2$ holds $P_2(x_0) = W(x_0)$. □

(53) Let us consider real normed spaces S, E, F, a partial function u from S to E, a partial function v from S to F, a partial function w from S to

$E \times F$, and a point x of S. Suppose $w = \langle u, v\rangle$ and u is differentiable in x and v is differentiable in x. Then

(i) w is differentiable in x, and

(ii) $w'(x) = \langle u'(x), v'(x)\rangle$.

Proof: Consider N_3 being a neighbourhood of x_0 such that $N_3 \subseteq \operatorname{dom} u$ and there exists a rest R_3 of S, E such that for every point x of S such that $x \in N_3$ holds $u_{/x} - u_{/x_0} = u'(x_0)(x - x_0) + R_{3/x-x_0}$. Consider R_3 being a rest of S, E such that for every point x of S such that $x \in N_3$ holds $u_{/x} - u_{/x_0} = u'(x_0)(x - x_0) + R_{3/x-x_0}$. Consider N_4 being a neighbourhood of x_0 such that $N_4 \subseteq \operatorname{dom} v$ and there exists a rest R_4 of S, F such that for every point x of S such that $x \in N_4$ holds $v_{/x} - v_{/x_0} = v'(x_0)(x - x_0) + R_{4/x-x_0}$. Consider R_4 being a rest of S, F such that for every point x of S such that $x \in N_4$ holds $v_{/x} - v_{/x_0} = v'(x_0)(x - x_0) + R_{4/x-x_0}$. Consider N being a neighbourhood of x_0 such that $N \subseteq N_3$ and $N \subseteq N_4$. Set $L = \langle u'(x_0), v'(x_0)\rangle$. For every elements x, y of S, $L(x + y) = L(x) + L(y)$. For every vector x of S and for every real number a, $L(a \cdot x) = a \cdot L(x)$. Set $K = \|u'(x_0)\| + \|v'(x_0)\|$. For every vector x of S, $\|L(x)\| \leqslant K \cdot \|x\|$. Set $R = \langle R_3, R_4\rangle$. For every point d_2 of S, $R_{/d_2} = \langle R_{3/d_2}, R_{4/d_2}\rangle$. For every real number r such that $r > 0$ there exists a real number d such that $d > 0$ and for every point z of S such that $z \neq 0_S$ and $\|z\| < d$ holds $\|z\|^{-1} \cdot \|R_{/z}\| < r$. For every point x of S such that $x \in N$ holds $w_{/x} - w_{/x_0} = L(x - x_0) + R_{/x-x_0}$. □

(54) Let us consider real normed spaces S, E, F, a partial function u from S to E, a partial function v from S to F, a partial function w from S to $E \times F$, and a subset Z of S. Suppose $w = \langle u, v\rangle$ and u is differentiable on Z and v is differentiable on Z. Then

(i) w is differentiable on Z, and

(ii) for every point x of S such that $x \in Z$ holds $w'_{\restriction Z/x} = \langle u'_{\restriction Z/x}, v'_{\restriction Z/x}\rangle$.

Proof: For every point x of S such that $x \in Z$ holds w is differentiable in x and $w'(x) = \langle u'(x), v'(x)\rangle$. For every point x of S such that $x \in Z$ holds $w'_{\restriction Z/x} = \langle u'_{\restriction Z/x}, v'_{\restriction Z/x}\rangle$. □

Let S, E, F be real normed spaces. The functor $\mathrm{CTP}(S, E, F)$ yielding a Lipschitzian linear operator from (the real norm space of bounded linear operators from S into E) $\times$ (the real norm space of bounded linear operators from S into F) into the real norm space of bounded linear operators from S into $E \times F$ is defined by

(Def. 3) for every Lipschitzian linear operator f from S into E and for every Lipschitzian linear operator g from S into F, $it(f, g) = \langle f, g\rangle$.

Now we state the proposition:

(55) Let us consider real normed spaces S, E, F, and a natural number i. Then $\mathrm{CTP}(S, \mathrm{diff}_{\mathrm{SP}}(S^i, E), \mathrm{diff}_{\mathrm{SP}}(S^i, F))$ is a Lipschitzian linear operator from $\mathrm{diff}_{\mathrm{SP}}(S^{(i+1)}, E) \times \mathrm{diff}_{\mathrm{SP}}(S^{(i+1)}, F)$ into the real norm space of bounded linear operators from S into $\mathrm{diff}_{\mathrm{SP}}(S^i, E) \times \mathrm{diff}_{\mathrm{SP}}(S^i, F)$.

Let us consider real normed spaces S, E, F, a partial function u from S to E, a partial function v from S to F, a partial function w from S to $E \times F$, and a subset Z of S. Now we state the propositions:

(56) Suppose $w = \langle u, v \rangle$ and u is differentiable on Z and v is differentiable on Z. Then

(i) w is differentiable on Z, and

(ii) $w'_{\restriction Z} = (\mathrm{CTP}(S, \mathrm{diff}_{\mathrm{SP}}(S^0, E), \mathrm{diff}_{\mathrm{SP}}(S^0, F))) \cdot \langle u'_{\restriction Z}, v'_{\restriction Z} \rangle$.

PROOF: w is differentiable on Z and for every point x of S such that $x \in Z$ holds $w'_{\restriction Z/x} = \langle u'_{\restriction Z/x}, v'_{\restriction Z/x} \rangle$. $\mathrm{CTP}(S, \mathrm{diff}_{\mathrm{SP}}(S^0, E), \mathrm{diff}_{\mathrm{SP}}(S^0, F))$ is a Lipschitzian linear operator from $\mathrm{diff}_{\mathrm{SP}}(S^{(0+1)}, E) \times \mathrm{diff}_{\mathrm{SP}}(S^{(0+1)}, F)$ into the real norm space of bounded linear operators from S into $\mathrm{diff}_{\mathrm{SP}}(S^0, E)$ $\times\, \mathrm{diff}_{\mathrm{SP}}(S^0, F)$. For every object x_0 such that $x_0 \in \mathrm{dom}\, w'_{\restriction Z}$ holds $w'_{\restriction Z}(x_0) = ((\mathrm{CTP}(S, \mathrm{diff}_{\mathrm{SP}}(S^0, E), \mathrm{diff}_{\mathrm{SP}}(S^0, F))) \cdot \langle u'_{\restriction Z}, v'_{\restriction Z} \rangle)(x_0)$. □

(57) Suppose $w = \langle u, v \rangle$ and u is differentiable on Z and v is differentiable on Z. Then

(i) $\mathrm{diff}_Z(w, 0)$ is differentiable on Z, and

(ii) there exists a Lipschitzian linear operator T from $\mathrm{diff}_{\mathrm{SP}}(S^1, E) \times \mathrm{diff}_{\mathrm{SP}}(S^1, F)$ into $\mathrm{diff}_{\mathrm{SP}}(S^1, (E \times F))$ such that $T = \mathrm{CTP}(S, \mathrm{diff}_{\mathrm{SP}}(S^0, E), \mathrm{diff}_{\mathrm{SP}}(S^0, F))$ and $\mathrm{diff}_Z(w, 1) = T \cdot \langle \mathrm{diff}_Z(u, 1), \mathrm{diff}_Z(v, 1) \rangle$.

The theorem is a consequence of (56), (3), (55), and (2).

(58) Suppose $w = \langle u, v \rangle$ and u is differentiable 2 times on Z and v is differentiable 2 times on Z. Then

(i) w is differentiable 2 times on Z, and

(ii) there exists a Lipschitzian linear operator L_1 from $\mathrm{diff}_{\mathrm{SP}}(S^1, E) \times \mathrm{diff}_{\mathrm{SP}}(S^1, F)$ into $\mathrm{diff}_{\mathrm{SP}}(S^1, (E \times F))$ and there exists a Lipschitzian linear operator L_2 from $\mathrm{diff}_{\mathrm{SP}}(S^2, E) \times \mathrm{diff}_{\mathrm{SP}}(S^2, F)$ into $\mathrm{diff}_{\mathrm{SP}}(S^1, (\mathrm{diff}_{\mathrm{SP}}(S^1, E) \times \mathrm{diff}_{\mathrm{SP}}(S^1, F)))$ and there exists a Lipschitzian linear operator T from $\mathrm{diff}_{\mathrm{SP}}(S^2, E) \times \mathrm{diff}_{\mathrm{SP}}(S^2, F)$ into $\mathrm{diff}_{\mathrm{SP}}(S^2, (E \times F))$ such that $L_1 = \mathrm{CTP}(S, \mathrm{diff}_{\mathrm{SP}}(S^0, E), \mathrm{diff}_{\mathrm{SP}}(S^0, F))$ and $L_2 = \mathrm{CTP}(S, \mathrm{diff}_{\mathrm{SP}}(S^1, E), \mathrm{diff}_{\mathrm{SP}}(S^1, F))$ and $T = (\mathrm{LTRN}(1, L_1, S)) \cdot L_2$ and $\mathrm{diff}_Z(w, 2) = T \cdot \langle \mathrm{diff}_Z(u, 2), \mathrm{diff}_Z(v, 2) \rangle$.

The theorem is a consequence of (2), (57), (3), (55), and (24).

(59) Let us consider a natural number i, real normed spaces S, E, F, a partial function u from S to E, a partial function v from S to F, a partial function w from S to $E \times F$, and a subset Z of S. Suppose $w = \langle u, v \rangle$ and u is differentiable $i+1$ times on Z and v is differentiable $i+1$ times on Z. Then

(i) w is differentiable $i+1$ times on Z, and

(ii) there exists a Lipschitzian linear operator T from $\mathrm{diff}_{\mathrm{SP}}(S^{(i+1)}, E) \times \mathrm{diff}_{\mathrm{SP}}(S^{(i+1)}, F)$ into $\mathrm{diff}_{\mathrm{SP}}(S^{(i+1)}, (E \times F))$ such that $\mathrm{diff}_Z(w, i+1) = T \cdot \langle \mathrm{diff}_Z(u, i+1), \mathrm{diff}_Z(v, i+1) \rangle$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every real normed spaces S, E, F for every partial function u from S to E for every partial function v from S to F for every partial function w from S to $E \times F$ for every subset Z of S such that $w = \langle u, v \rangle$ and u is differentiable $\$_1 + 1$ times on Z and v is differentiable $\$_1 + 1$ times on Z holds w is differentiable $\$_1 + 1$ times on Z and there exists a Lipschitzian linear operator T from $\mathrm{diff}_{\mathrm{SP}}(S^{(\$_1+1)}, E) \times \mathrm{diff}_{\mathrm{SP}}(S^{(\$_1+1)}, F)$ into $\mathrm{diff}_{\mathrm{SP}}(S^{(\$_1+1)}, (E \times F))$ such that $\mathrm{diff}_Z(w, \$_1 + 1) = T \cdot \langle \mathrm{diff}_Z(u, \$_1 + 1), \mathrm{diff}_Z(v, \$_1 + 1) \rangle$. $\mathcal{P}[0]$. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$. For every natural number i, $\mathcal{P}[i]$. □

(60) Let us consider real normed spaces S, E, F, a partial function u from S to E, a partial function v from S to F, a partial function w from S to $E \times F$, a subset Z of S, and a natural number i. Suppose $w = \langle u, v \rangle$ and u is differentiable $i+1$ times on Z and $\mathrm{diff}_Z(u, i+1)$ is continuous on Z and v is differentiable $i+1$ times on Z and $\mathrm{diff}_Z(v, i+1)$ is continuous on Z. Then

(i) w is differentiable $i+1$ times on Z, and

(ii) $\mathrm{diff}_Z(w, i+1)$ is continuous on Z.

PROOF: Consider T being a Lipschitzian linear operator from $\mathrm{diff}_{\mathrm{SP}}(S^{(i+1)}, E) \times \mathrm{diff}_{\mathrm{SP}}(S^{(i+1)}, F)$ into $\mathrm{diff}_{\mathrm{SP}}(S^{(i+1)}, (E \times F))$ such that $\mathrm{diff}_Z(w, i+1) = T \cdot \langle \mathrm{diff}_Z(u, i+1), \mathrm{diff}_Z(v, i+1) \rangle$. Set $u_1 = \mathrm{diff}_Z(u, i+1)$. Set $v_1 = \mathrm{diff}_Z(v, i+1)$. Set $G = \langle u_1, v_1 \rangle$. For every point x_0 of S and for every real number r such that $x_0 \in Z$ and $0 < r$ there exists a real number s such that $0 < s$ and for every point x_1 of S such that $x_1 \in Z$ and $\|x_1 - x_0\| < s$ holds $\|G_{/x_1} - G_{/x_0}\| < r$. T is continuous on $\Omega_{\mathrm{diff}_{\mathrm{SP}}(S^{(i+1)}, E) \times \mathrm{diff}_{\mathrm{SP}}(S^{(i+1)}, F)}$. □

(61) Let us consider real normed spaces X, Y, a subset V of $X \times Y$, a subset D of X, and a subset E of Y. Suppose D is open and E is open and $V = D \times E$. Then V is open.

Proof: For every point x of X and for every point y of Y such that $\langle x, y\rangle \in V$ there exist real numbers r_1, r_2 such that $0 < r_1$ and $0 < r_2$ and $\mathrm{Ball}(x, r_1) \times \mathrm{Ball}(y, r_2) \subseteq V$. □

4. Higher-Order Differentiability of Inverse Function Theorem

Now we state the propositions:

(62) Let us consider real normed spaces E, F, G, a point x of the real norm space of bounded linear operators from E into F, and a point L of the real norm space of bounded linear operators from the real norm space of bounded linear operators from F into E into the real norm space of bounded linear operators from E into E. Suppose x is invertible and for every point y of the real norm space of bounded linear operators from F into E, $L(y) = y \cdot x$. Then L is invertible.

Proof: Set $F_4 =$ the real norm space of bounded linear operators from F into E. Set $E_2 =$ the real norm space of bounded linear operators from E into E. Reconsider $L_1 = L$ as a Lipschitzian linear operator from F_4 into E_2. Reconsider $d_2 = x^{-1}$ as a point of the real norm space of bounded linear operators from F into E. For every objects x_1, x_2 such that x_1, $x_2 \in \Omega_{F_4}$ and $L_1(x_1) = L_1(x_2)$ holds $x_1 = x_2$. For every object y such that $y \in \Omega_{E_2}$ there exists an object z such that $z \in \Omega_{F_4}$ and $y = L_1(z)$. Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists a point y of E_2 such that $y = \$_1$ and $\$_2 = y \cdot d_2$. For every object y such that $y \in$ the carrier of E_2 there exists an object z such that $z \in$ the carrier of F_4 and $\mathcal{P}[y, z]$. Consider R being a function from the carrier of E_2 into the carrier of F_4 such that for every object y such that $y \in$ the carrier of E_2 holds $\mathcal{P}[y, R(y)]$. For every point y of E_2, $R(y) = y \cdot d_2$. For every element y of F_4, $(R \cdot L_1)(y) = y$. Set $K = \|d_2\|$. For every vector y of E_2, $\|R(y)\| \leqslant K \cdot \|y\|$. □

(63) Let us consider a non trivial real Banach space F. Then the real norm space of bounded linear operators from F into F is a non trivial real Banach space.

(64) Let us consider a real Banach space E, non trivial real Banach spaces F, G, a non empty subset Z of $E \times F$, a point c of G, a subset A of E, and a subset B of F. Suppose Z is open and A is open and B is open and $A \times B \subseteq Z$. Let us consider a natural number i, a partial function f from $E \times F$ to G, and a partial function g from E to F. Suppose $\mathrm{dom}\, f = Z$ and f is differentiable $i+1$ times on Z and $\mathrm{diff}_Z(f, i+1)$ is continuous on Z and $\mathrm{dom}\, g = A$ and $\mathrm{rng}\, g \subseteq B$ and g is continuous on A and for every point x of E such that $x \in A$ holds $f(x, g(x)) = c$ and for every point x

of E and for every point z of $E \times F$ such that $x \in A$ and $z = \langle x, g(x) \rangle$ holds partdiff(f, z) w.r.t. 2 is invertible. Then

(i) g is differentiable $i + 1$ times on A, and

(ii) $\mathrm{diff}_A(g, i + 1)$ is continuous on A, and

(iii) for every point x of E and for every point z of $E \times F$ such that $x \in A$ and $z = \langle x, g(x) \rangle$ holds $g'(x) = -$(Inv partdiff(f, z) w.r.t. 2) $\cdot$ (partdiff (f, z) w.r.t. 1).

PROOF: Define $\mathcal{P}$[natural number] $\equiv$ for every real Banach space E for every non trivial real Banach spaces F, G for every non empty subset Z of $E \times F$ for every point c of G for every subset A of E for every subset B of F such that Z is open and A is open and B is open and $A \times B \subseteq Z$ for every partial function f from $E \times F$ to G for every partial function g from E to F such that $\mathrm{dom}\, f = Z$ and f is differentiable $\$_1 + 1$ times on Z and $\mathrm{diff}_Z(f, \$_1 + 1)$ is continuous on Z and $\mathrm{dom}\, g = A$ and $\mathrm{rng}\, g \subseteq B$ and g is continuous on A and for every point x of E such that $x \in A$ holds $f(x, g(x)) = c$ and for every point x of E and for every point z of $E \times F$ such that $x \in A$ and $z = \langle x, g(x) \rangle$ holds partdiff(f, z) w.r.t. 2 is invertible holds g is differentiable $\$_1 + 1$ times on A and $\mathrm{diff}_A(g, \$_1 + 1)$ is continuous on A and for every point x of E and for every point z of $E \times F$ such that $x \in A$ and $z = \langle x, g(x) \rangle$ holds $g'(x) = -$(Inv partdiff(f, z) w.r.t. 2) $\cdot$ (partdiff(f, z) w.r.t. 1). $\mathcal{P}[0]$. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$. For every natural number i, $\mathcal{P}[i]$. □

(65) Let us consider non trivial real Banach spaces F, G. Then there exists a partial function I from the real norm space of bounded linear operators from F into G to the real norm space of bounded linear operators from G into F such that

(i) $\mathrm{dom}\, I = \mathrm{InvertOpers}(F, G)$, and

(ii) $\mathrm{rng}\, I = \mathrm{InvertOpers}(G, F)$, and

(iii) I is one-to-one and continuous on $\mathrm{InvertOpers}(F, G)$, and

(iv) there exists a partial function J from the real norm space of bounded linear operators from G into F to the real norm space of bounded linear operators from F into G such that $J = I^{-1}$ and J is one-to-one and $\mathrm{dom}\, J = \mathrm{InvertOpers}(G, F)$ and $\mathrm{rng}\, J = \mathrm{InvertOpers}(F, G)$ and J is continuous on $\mathrm{InvertOpers}(G, F)$, and

(v) for every point u of the real norm space of bounded linear operators from F into G such that $u \in \mathrm{InvertOpers}(F, G)$ holds $I(u) = \mathrm{Inv}\, u$, and

(vi) for every natural number n, I is differentiable $n+1$ times on InvertOpers(F, G) and $\mathrm{diff}_{\mathrm{InvertOpers}(F,G)}(I, n+1)$ is continuous on InvertOpers(F, G).

PROOF: Set E_1 = the real norm space of bounded linear operators from F into G. Set F_1 = the real norm space of bounded linear operators from G into F. Set G_1 = the real norm space of bounded linear operators from F into F. G_1 is a non trivial real Banach space. Set A_1 = InvertOpers(F, G). Set B_1 = InvertOpers(G, F). Consider g_1 being a partial function from E_1 to F_1 such that $\mathrm{dom}\, g_1 = A_1$ and $\mathrm{rng}\, g_1 = B_1$ and g_1 is one-to-one and continuous on A_1 and there exists a partial function J from F_1 to E_1 such that $J = {g_1}^{-1}$ and J is one-to-one and $\mathrm{dom}\, J = B_1$ and $\mathrm{rng}\, J = A_1$ and J is continuous on B_1 and for every point u of E_1 such that $u \in A_1$ holds $g_1(u) = \mathrm{Inv}\, u$. Set $Z_1 = \Omega_{E_1 \times F_1}$. Reconsider $a = \mathrm{id}_{\Omega_F}$ as a Lipschitzian linear operator from F into F. Consider f_0 being a Lipschitzian bilinear operator from $E_1 \times F_1$ into G_1 such that for every point u of E_1 and for every point v of F_1, $f_0(u, v) = v \cdot u$. Reconsider $f_1 = f_0 \restriction Z_1$ as a partial function from $E_1 \times F_1$ to G_1. For every point x of E_1 such that $x \in A_1$ holds $f_1(x, g_1(x)) = a$ by [6, (22)]. For every point x of E_1 and for every point z of $E_1 \times F_1$ such that $x \in A_1$ and $z = \langle x, g_1(x) \rangle$ for every point y of F_1, (partdiff(f_1, z) w.r.t. 2)$(y) = y \cdot x$ by [8, (4)]. For every point x of E_1 and for every point z of $E_1 \times F_1$ such that $x \in A_1$ and $z = \langle x, g_1(x) \rangle$ holds partdiff(f_1, z) w.r.t. 2 is invertible. g_1 is differentiable $i + 1$ times on A_1 and $\mathrm{diff}_{A_1}(g_1, i + 1)$ is continuous on A_1. □

(66) Let us consider non trivial real Banach spaces E, F, a subset Z of E, a partial function f from E to F, a point a of E, a point b of F, and a natural number n. Suppose Z is open and $\mathrm{dom}\, f = Z$ and f is differentiable $n + 1$ times on Z and $\mathrm{diff}_Z(f, n + 1)$ is continuous on Z and $a \in Z$ and $f(a) = b$ and $f'(a)$ is invertible. Then there exists a subset A of E and there exists a subset B of F and there exists a partial function g from F to E such that A is open and B is open and $A \subseteq \mathrm{dom}\, f$ and $a \in A$ and $b \in B$ and $f^{\circ}A = B$ and $\mathrm{dom}\, g = B$ and $\mathrm{rng}\, g = A$ and $\mathrm{dom}(f \restriction A) = A$ and $\mathrm{rng}(f \restriction A) = B$ and $f \restriction A$ is one-to-one and g is one-to-one and $g = (f \restriction A)^{-1}$ and $f \restriction A = g^{-1}$ and $g(b) = a$ and for every point y of F such that $y \in B$ holds $f'(g_{/y})$ is invertible and for every point y of F such that $y \in B$ holds $g'(y) = \mathrm{Inv}\, f'(g_{/y})$ and f is differentiable $n + 1$ times on A and $\mathrm{diff}_A(f, n+1)$ is continuous on A and g is differentiable $n+1$ times on B and $\mathrm{diff}_B(g, n + 1)$ is continuous on B.

PROOF: Define $\mathcal{P}$[natural number] $\equiv$ if Z is open and $\mathrm{dom}\, f = Z$ and f is differentiable $\$_1 + 1$ times on Z and $\mathrm{diff}_Z(f, \$_1 + 1)$ is continuous on Z

and $a \in Z$ and $f(a) = b$ and $f'(a)$ is invertible, then there exists a subset A of E and there exists a subset B of F and there exists a partial function g from F to E such that A is open and B is open and $A \subseteq \operatorname{dom} f$ and $a \in A$ and $b \in B$ and $f^{\circ}A = B$ and $\operatorname{dom} g = B$ and $\operatorname{rng} g = A$ and $\operatorname{dom}(f \restriction A) = A$ and $\operatorname{rng}(f \restriction A) = B$ and $f \restriction A$ is one-to-one and g is one-to-one and $g = (f \restriction A)^{-1}$ and $f \restriction A = g^{-1}$ and $g(b) = a$ and for every point y of F such that $y \in B$ holds $f'(g_{/y})$ is invertible and for every point y of F such that $y \in B$ holds $g'(y) = \operatorname{Inv} f'(g_{/y})$ and f is differentiable $\$_1 + 1$ times on A and $\operatorname{diff}_A(f, \$_1 + 1)$ is continuous on A and g is differentiable $\$_1 + 1$ times on B and $\operatorname{diff}_B(g, \$_1 + 1)$ is continuous on B. $\mathcal{P}[0]$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every natural number n, $\mathcal{P}[n]$. □

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