FORMALIZED MATHEMATICS
Vol. **32**, No. **1**, pp. 223–233, **2024**
DOI: 10.2478/forma-2024-0019
e-ISSN: 1898–9934

sciendo
sciendo.com/journal/forma

Some Standard Examples of Vector Spaces

Christoph Schwarzweller
Institute of Informatics
University of Gdańsk
Poland

Agnieszka Rowińska-Schwarzweller
Institute of Informatics
University of Gdańsk
Poland

Summary. In this article, using the Mizar system, we introduce some standard examples of vector spaces, e.g., the vector space of linear transformations between vector spaces. We formulate some conditions for the isomorphism of finite-dimensional vector spaces and prove that linear transformations are uniquely determined by their values with respect to the basis.

MSC: 15A03 15A04 68V20

Keywords: vector space; linear transformation; dimension of a vector space

MML identifier: `VECTSP13`, version: `8.1.14 5.87.1483`

Introduction

In this article we continue the development of the hierarchy of algebraic structures [5] in the repository of automatically verified repository of mathematical texts. Using the Mizar system [2], [3], we introduce three standard [7] examples of vector spaces: the vector space of n-tuples over a commutative ring R (in Section 3), the vector space of of maps from a set X into a ring R (Section 4), and the vector space of linear transformations between vector spaces (Section 5). Additionally, we prove that two finite-dimensional vector spaces are isomorphic if and only if they are of the same dimension, and hence each vector space of dimension n is isomorphic to a vector space of n-tuples [4], [6]. In the course of proving the above we also showed, that linear transformations are uniquely determined by their values with respect to the basis. Constructed examples and proven facts will be useful to continue the formalization of `FIELD` series of Mizar articles [8] or more advanced structures [9].

1. Preliminaries

Now we state the proposition:

(1) Let us consider non empty, finite sets X, Y. Suppose $\overline{\overline{Y}} = \overline{\overline{X}}$. Then there exists a function f from X into Y such that f is bijective.

Let L be a non empty additive loop structure, n be a natural number, and u, v be n-element finite sequences of elements of the carrier of L. One can verify that $u+v$ is n-element.

Let M be a non empty multiplicative magma, u be an n-element finite sequence of elements of the carrier of M, and a be an element of M. Let us note that $a \cdot u$ is n-element. Now we state the propositions:

(2) Let us consider a non empty, Abelian additive loop structure R, a natural number n, and n-tuples u, v of the carrier of R. Then $u+v=v+u$.

(3) Let us consider a non empty, add-associative additive loop structure R, a natural number n, and n-tuples u, v, w of the carrier of R. Then $(u+v)+w=u+(v+w)$.

Let F be a field. Let us note that every trivial vector space over F is finite dimensional. Let V be a non trivial vector space over F. Note that there exists a subset of V which is non empty, finite, and linearly independent. Let V be a non trivial, finite dimensional vector space over F. One can check that every basis of V is non empty.

2. On Linear Transformations

Let F be a field, U, V be vector spaces over F, B be a non empty subset of U, and f be a function from B into V. Let us note that the functor $\operatorname{rng} f$ yields a subset of V. Now we state the propositions:

(4) Let us consider a field F, a vector space U over F, a linearly independent subset B of U, and an element w of U. Suppose $w \in B$. Let us consider a linear combination l of B. Suppose $\sum l = w$. Then

(i) the support of $l = \{w\}$, and

(ii) $l(w) = 1_F$.

Proof: Define $\mathcal{Q}[\text{object}, \text{object}] \equiv \$_1 = w$ and $\$_2 = 1_F$ or $\$_1 \neq w$ and $\$_2 = 0_F$. For every object x such that $x \in$ the carrier of U there exists an object y such that $y \in$ the carrier of F and $\mathcal{Q}[x,y]$. Consider l_1 being a function from the carrier of U into the carrier of F such that for every object x such that $x \in$ the carrier of U holds $\mathcal{Q}[x, l_1(x)]$. For every element v of U such that $v \notin \{w\}$ holds $l_1(v) = 0_F$. The support of $l_1 = \{w\}$. $\square$

(5) Let us consider a field F, vector spaces U, V over F, a subset B of U, a subset A of V, and a linear transformation T from U to V. Suppose $T^\circ B \subseteq A$. Let us consider a linear combination l of B. Then $T(\sum l) \in \mathrm{Lin}(A)$.

Proof: Define $\mathcal{P}[\text{natural number}] \equiv$ for every linear combination l of B such that $\overline{\overline{\text{the support of } l}} = \$_1$ holds $T(\sum l) \in \mathrm{Lin}(A)$. $\mathcal{P}[0]$. For every natural number k, $\mathcal{P}[k]$. Consider n being a natural number such that $\overline{\overline{\alpha}} = n$, where α is the support of l. □

(6) Let us consider a field F, vector spaces U, V over F, a basis B of U, and linear transformations T_1, T_2 from U to V. If $T_1 \restriction B = T_2 \restriction B$, then $T_1 = T_2$.

Proof: Define $\mathcal{P}[\text{natural number}] \equiv$ for every linear combination l of B such that $\overline{\overline{\text{the support of } l}} = \$_1$ holds $T_1(\sum l) = T_2(\sum l)$. $\mathcal{P}[0]$. For every natural number k, $\mathcal{P}[k]$. □

Let F be a field, U, V be vector spaces over F, B be a non empty, finite subset of U, l be a linear combination of B, f be a function from B into V, and v be an element of V. The functor $\mathrm{Expand}(f, l, v)$ yielding a finite sequence of elements of the carrier of F is defined by the term

(Def. 1) $l \cdot (\mathrm{CFS}(f^{-1}(\{v\})))$.

The functor $f \cdot l$ yielding a linear combination of $\mathrm{rng}\, f$ is defined by

(Def. 2) for every element v of V, $it(v) = \sum \mathrm{Expand}(f, l, v)$.

Let l be a linear combination of $\mathrm{rng}\, f$. The functor $l \cdot f$ yielding a linear combination of B is defined by

(Def. 3) for every element u of U such that $u \in B$ holds $it(u) = l(f(u))$.

Now we state the propositions:

(7) Let us consider a field F, vector spaces U, V over F, a non empty, finite subset B of U, and a function f from B into V. Suppose f is one-to-one. Let us consider a linear combination l of B, and an element v of V. If $v \in \mathrm{rng}\, f$, then $(f \cdot l)(v) = l((f^{-1})(v))$.

Proof: Consider x being an object such that $x \in \mathrm{dom}\, f$ and $f(x) = v$. Set $G = l \cdot (\mathrm{CFS}(f^{-1}(\{v\})))$. $\mathrm{dom}\, G = \mathrm{Seg}\, 1$. □

(8) Let us consider a field F, vector spaces U, V over F, a non empty, finite subset B of U, and a function f from B into V. Suppose f is one-to-one. Let us consider a linear combination l of B. Then $(f \cdot l) \cdot f = l$. The theorem is a consequence of (7).

(9) Let us consider a field F, vector spaces U, V over F, a non empty, finite subset B of U, and a function f from B into V. Suppose f is one-to-one. Let us consider a linear combination l of $\mathrm{rng}\, f$. Then $f \cdot (l \cdot f) = l$. The theorem is a consequence of (7).

(10) Let us consider a field F, vector spaces U, V over F, a non empty, finite subset B of U, a function f from B into V, and a linear combination l of B. Then the support of $f \cdot l \subseteq f^{\circ}$(the support of l).

(11) Let us consider a field F, vector spaces U, V over F, a non empty, finite subset B of U, an element b of B, a function f from B into V, and a linear combination l of B. Suppose the support of $l = \{b\}$. Then

  (i) the support of $f \cdot l = \{f(b)\}$, and

  (ii) $\sum(f \cdot l) = l(b) \cdot f(b)$.

The theorem is a consequence of (10).

(12) Let us consider a field F, vector spaces U, V over F, a non empty, finite subset B of U, a function f from B into V, and linear combinations l_1, l_2, l_3 of B. If $l_3 = l_1 + l_2$, then $f \cdot l_3 = f \cdot l_1 + f \cdot l_2$.

(13) Let us consider a field F, vector spaces U, V over F, a non empty, finite subset B of U, a function f from B into V, linear combinations l_1, l_2 of B, and an element a of F. If $l_2 = a \cdot l_1$, then $f \cdot l_2 = a \cdot (f \cdot l_1)$.

(14) Let us consider a field F, vector spaces U, V over F, a non empty, finite subset B of U, a function f from B into V, and linear combinations l_1, l_2 of B. If $l_2 = -l_1$, then $f \cdot l_2 = -f \cdot l_1$.

(15) Let us consider a field F, vector spaces U, V over F, a non empty, finite subset B of U, a function f from B into V, and linear combinations l_1, l_2, l_3 of B. If $l_3 = l_1 - l_2$, then $f \cdot l_3 = f \cdot l_1 - f \cdot l_2$. The theorem is a consequence of (13) and (12).

(16) Let us consider a field F, vector spaces U, V over F, and a non empty, finite subset B of U. Suppose B is linearly independent. Let us consider an element w of U. Suppose $w \in B$. Let us consider a linear combination l of B. If $\sum l = w$, then for every function f from B into V, $\sum(f \cdot l) = f(w)$. The theorem is a consequence of (4) and (11).

Let F be a field, U be a finite dimensional vector space over F, V be a vector space over F, B be a basis of U, and f be a function from B into V. The functor canLinTrans(f) yielding a linear transformation from U to V is defined by

(Def. 4) $it \restriction B = f$.

Now we state the propositions:

(17) Let us consider a field F, a non trivial, finite dimensional vector space U over F, a vector space V over F, a basis B of U, a function f from B into V, and a linear combination l of B. Then $(\text{canLinTrans}(f))(\sum l) = \sum(f \cdot l)$.
PROOF: Set $B_1 = B$. Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ for every linear combination l of B_1 such that $\$_1 = \sum l$ holds $\$_2 = \sum(f \cdot l)$. Consider T being

a function from the carrier of U into the carrier of V such that for every object u such that $u \in$ the carrier of U holds $\mathcal{P}[u, T(u)]$. $T = \mathrm{canLinTrans}(f)$. $\square$

(18) Let us consider a field F, finite dimensional vector spaces U, V over F, a basis B of U, a function f from B into V, and a linear transformation T from U to V. Then $T = \mathrm{canLinTrans}(f)$ if and only if for every element u of U such that $u \in B$ holds $T(u) = f(u)$.

(19) Let us consider a field F, finite dimensional vector spaces U, V over F, a basis B of U, and a function f from B into V. Then $(\mathrm{canLinTrans}(f))^{\circ}B \subseteq \mathrm{rng}\, f$.

(20) Let us consider a field F, a non trivial, finite dimensional vector space U over F, a finite dimensional vector space V over F, a basis B of U, a function f from B into V, and a linear combination l_2 of $\mathrm{rng}\, f$. Then there exists a linear combination l_1 of B such that $(\mathrm{canLinTrans}(f))(\sum l_1) = \sum l_2$.
Proof: Define $\mathcal{P}[\text{natural number}] \equiv$ for every linear combination l_2 of $\mathrm{rng}\, f$ such that $\overline{\overline{\text{the support of } l_2}} = \$_1$ there exists a linear combination l_1 of B such that $(\mathrm{canLinTrans}(f))(\sum l_1) = \sum l_2$. $\mathcal{P}[0]$ by [1, (9)]. $\mathcal{P}[1]$. For every natural number k, $\mathcal{P}[k]$. Consider n being a natural number such that $\overline{\overline{\alpha}} = n$, where α is the support of l_2. $\square$

Let us consider a field F, a non trivial, finite dimensional vector space U over F, a finite dimensional vector space V over F, a basis B of U, and a function f from B into V. Now we state the propositions:

(21) $\mathrm{im}\, \mathrm{canLinTrans}(f) = \mathrm{Lin}(\mathrm{rng}\, f)$. The theorem is a consequence of (19), (5), and (20).

(22) $\mathrm{canLinTrans}(f)$ is one-to-one if and only if $\mathrm{rng}\, f$ is linearly independent and f is one-to-one.

3. The Vector Space R^n

Let R be a ring and n be a natural number. The functor $\mathrm{vectorAdd}(n, R)$ yielding a binary operation on (the carrier of $R)^n$ is defined by

(Def. 5) for every n-tuples u, v of the carrier of R, $it(u, v) = u + v$.

The functor $\mathrm{vectorMult}(n, R)$ yielding a function from (the carrier of $R) \times$ (the carrier of $R)^n$ into (the carrier of $R)^n$ is defined by

(Def. 6) for every element a of R and for every n-tuple u of the carrier of R, $it(a, u) = a \cdot u$.

The functor R^n_{VS} yielding a strict vector space structure over R is defined by

(Def. 7) the carrier of it = (the carrier of $R)^n$ and the addition of it = vectorAdd (n, R) and the zero of $it = n \mapsto 0_R$ and the left multiplication of it = vectorMult(n, R).

One can verify that R^n_{VS} is non empty and R^n_{VS} is Abelian, add-associative, right zeroed, and right complementable and R^n_{VS} is scalar distributive, scalar associative, and vector distributive. Let R be a commutative ring. Observe that R^n_{VS} is scalar unital.

Let R be a ring and n, i be natural numbers. The functor $i^{\,\mathrm{th}}$unitVector(n, R) yielding an element of (the carrier of $R)^n$ is defined by the term

(Def. 8) Replace$(n \mapsto 0_R, i, 1_R)$.

Now we state the propositions:

(23) Let us consider a ring R, and natural numbers n, i. Suppose $1 \leqslant i \leqslant n$. Then

(i) $(i^{\,\mathrm{th}}$unitVector$(n, R))(i) = 1_R$, and

(ii) for every natural number j such that $1 \leqslant j \leqslant n$ and $j \neq i$ holds $(i^{\,\mathrm{th}}$unitVector$(n, R))(j) = 0_R$.

(24) Let us consider a non degenerated ring R, and natural numbers n, i, j. Suppose $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant n$. Then $i^{\,\mathrm{th}}$unitVector$(n, R) = j^{\,\mathrm{th}}$unitVector$(n, R)$ if and only if $i = j$. The theorem is a consequence of (23).

Let R be a ring and n be a natural number. The functor Base(R, n) yielding a subset of R^n_{VS} is defined by the term

(Def. 9) $\{i^{\,\mathrm{th}}$unitVector(n, R), where i is a natural number : $1 \leqslant i \leqslant n\}$.

One can check that Base(R, n) is finite. Now we state the propositions:

(25) Let us consider a non degenerated ring R, and a natural number n. Then $\overline{\overline{\mathrm{Base}(R, n)}} = n$.

PROOF: Define $\mathcal{P}$[object, object] $\equiv$ there exists a natural number x such that $\$_1 = x$ and $\$_2 = x^{\,\mathrm{th}}$unitVector$(n, R)$. Consider f being a function such that dom f = Seg n and for every object x such that $x \in$ Seg n holds $\mathcal{P}[x, f(x)]$. $\square$

(26) Let us consider a non degenerated, commutative ring R, a natural number n, a linear combination l of Base(R, n), an n-tuple v of the carrier of R, and a natural number i. If $v = \sum l$ and $1 \leqslant i \leqslant n$, then $v(i) = l(i^{\,\mathrm{th}}$unitVector$(n, R))$. The theorem is a consequence of (24) and (23).

Let R be a non degenerated, commutative ring and n be a natural number. Note that Base(R, n) is linearly independent. Now we state the proposition:

(27) Let us consider a non degenerated, commutative ring R, and a natural number n. Then $\mathrm{Lin}(\mathrm{Base}(R,n)) = R^n_{\mathrm{VS}}$.

Let R be a non degenerated, commutative ring and n be a natural number. One can check that $\mathrm{Base}(R,n)$ is base. Let F be a field. One can check that F^n_{VS} is finite dimensional. Now we state the proposition:

(28) Let us consider a field F, and a natural number n. Then $\dim(F^n_{\mathrm{VS}}) = n$. The theorem is a consequence of (25) and (27).

Let R be a ring and U, V be vector spaces over R. We say that U and V are isomorphic if and only if

(Def. 10) there exists a linear transformation T from U to V such that T is bijective.

Now we state the propositions:

(29) Let us consider a field F, and finite dimensional vector spaces U, V over F. Then U and V are isomorphic if and only if $\dim(U) = \dim(V)$. The theorem is a consequence of (1), (22), and (21).

(30) Let us consider a field F, and a finite dimensional vector space U over F. Then U and $F^{\dim(U)}_{\mathrm{VS}}$ are isomorphic. The theorem is a consequence of (28) and (29).

(31) Let us consider a finite ring R, and a natural number n. Then $\overline{\overline{\alpha}} = \overline{\overline{\beta}}^n$, where α is the carrier of R^n_{VS} and β is the carrier of R.

Let R be a finite ring and n be a natural number. One can check that R^n_{VS} is finite.

4. The Vector Space of Maps into a Ring R

Let X be a non empty set, L be a non empty additive loop structure, and f, g be functions from X into L. The functor $f' +' g$ yielding a function from X into L is defined by

(Def. 11) for every element x of X, $it(x) = f(x) + g(x)$.

Let L be a non empty, Abelian additive loop structure. Observe that the functor $f' +' g$ is commutative.

Let L be a non empty additive loop structure and f be a function from X into L. The functor $-f$ yielding a function from X into L is defined by

(Def. 12) for every element x of X, $it(x) = -f(x)$.

Let L be a non empty multiplicative loop structure and a be an element of L. The functor $a \star f$ yielding a function from X into L is defined by

(Def. 13) for every element x of X, $it(x) = a \cdot f(x)$.

Let L be a left unital, non empty multiplicative loop structure. One can verify that $1_L \star f$ reduces to f. Now we state the propositions:

(32) Let us consider a non empty set X, a non empty, add-associative additive loop structure L, and functions f, g, h from X into L. Then $f' +' g' +' h = f' +' g' +' h$.

(33) Let us consider a non empty set X, a non empty, add-associative, right zeroed, right complementable additive loop structure L, and a function f from X into L. Then $f' +' - f = X \longmapsto 0_L$.

(34) Let us consider a non empty set X, a left distributive, non empty double loop structure L, elements a, b of L, and a function f from X into L. Then $(a + b) \star f = a \star f' +' b \star f$.

(35) Let us consider a non empty set X, an associative, non empty multiplicative loop structure L, elements a, b of L, and a function f from X into L. Then $a \cdot b \star f = a \star (b \star f)$.

(36) Let us consider a non empty set X, a right distributive, non empty double loop structure L, an element a of L, and functions f, g from X into L. Then $a \star (f' +' g) = a \star f' +' a \star g$.

Let X be a non empty set and L be a non empty additive loop structure. The functor mapAdd(X, L) yielding a binary operation on (the carrier of $L)^X$ is defined by

(Def. 14) for every functions f, g from X into L, $it(f, g) = f' +' g$.

Let L be a non empty multiplicative loop structure. The functor mapMult(X, L) yielding a function from (the carrier of L) $\times$ (the carrier of $L)^X$ into (the carrier of $L)^X$ is defined by

(Def. 15) for every function f from X into L and for every element a of L, $it(a, f) = a \star f$.

Let L be a non empty double loop structure. The functor Maps(X, L) yielding a strict vector space structure over L is defined by

(Def. 16) the carrier of it = (the carrier of $L)^X$ and the addition of it = mapAdd (X, L) and the zero of $it = X \longmapsto 0_L$ and the left multiplication of it = mapMult(X, L).

Let X, L be non empty double loop structures. The functor Maps(X, L) is defined by the term

(Def. 17) Maps((the carrier of X), L).

Let X be a non empty set and L be a non empty double loop structure. Let us note that Maps(X, L) is non empty. Let L be a non empty, Abelian double loop structure. Let us note that Maps(X, L) is Abelian. Let L be a non empty, add-associative double loop structure. Observe that $\mathrm{Maps}(X, L)$ is add-associative. Let L be a non empty, right zeroed double loop structure. Observe that $\mathrm{Maps}(X, L)$ is right zeroed.

Let L be a non empty, add-associative, right zeroed, right complementable double loop structure. Note that $\mathrm{Maps}(X, L)$ is right complementable. Let L be a left distributive, non empty double loop structure. One can check that $\mathrm{Maps}(X, L)$ is scalar distributive. Let L be an associative, non empty double loop structure. One can check that $\mathrm{Maps}(X, L)$ is scalar associative. Let L be a right distributive, non empty double loop structure. One can check that $\mathrm{Maps}(X, L)$ is vector distributive. Let L be a left unital, non empty double loop structure. One can verify that $\mathrm{Maps}(X, L)$ is scalar unital.

5. The Vector Space of Linear Transformations

Let X be a non empty set, R be a non empty 1-sorted structure, L be a non empty vector space structure over R, f be a function from X into L, and a be an element of R. The functor $a \star f$ yielding a function from X into L is defined by

(Def. 18) for every element x of X, $it(x) = a \cdot f(x)$.

Let R be a non empty ring and L be a scalar unital, non empty vector space structure over R. One can check that $1_R \star f$ reduces to f. Now we state the propositions:

(37) Let us consider a non empty set X, a ring R, a scalar distributive, non empty vector space structure L over R, elements a, b of R, and a function f from X into L. Then $(a + b) \star f = a \star f' +' b \star f$.

(38) Let us consider a non empty set X, a ring R, a scalar associative, non empty vector space structure L over R, elements a, b of R, and a function f from X into L. Then $a \cdot b \star f = a \star (b \star f)$.

(39) Let us consider a non empty set X, a ring R, a vector distributive, non empty vector space structure L over R, elements a, b of R, and functions f, g from X into L. Then $a \star (f' +' g) = a \star f' +' a \star g$.

Let X be a non empty set, R be a non empty additive loop structure, and L be a non empty vector space structure over R. The functor $\mathrm{mapAdd}(X, L)$ yielding a binary operation on (the carrier of $L)^X$ is defined by

(Def. 19) for every functions f, g from X into L, $it(f, g) = f' +' g$.

Let R be a non empty multiplicative loop structure. The functor $\mathrm{mapMult}(X,$ $L)$ yielding a function from (the carrier of R) $\times$ (the carrier of $L)^X$ into (the carrier of $L)^X$ is defined by

(Def. 20) for every function f from X into L and for every element a of R, $it(a, f) = a \star f$.

Let R be a non empty double loop structure. The functor $\mathrm{Maps}(X, L)$ yielding a strict vector space structure over R is defined by

(Def. 21) the carrier of $it =$ (the carrier of $L)^X$ and the addition of $it = \mathrm{mapAdd}(X, L)$ and the zero of $it = X \longmapsto 0_L$ and the left multiplication of $it = \mathrm{mapMult}(X, L)$.

Let L_1, L_2 be non empty vector space structures over R. The functor $\mathrm{Maps}(L_1, L_2)$ is defined by the term

(Def. 22) $\mathrm{Maps}(($the carrier of $L_1), L_2)$.

Let X be a non empty set and L be a non empty vector space structure over R. One can verify that $\mathrm{Maps}(X, L)$ is non empty.

Let R be a ring and L be a non empty, Abelian vector space structure over R. Observe that $\mathrm{Maps}(X, L)$ is Abelian.

Let L be a non empty, add-associative vector space structure over R. Note that $\mathrm{Maps}(X, L)$ is add-associative.

Let L be a non empty, right zeroed vector space structure over R. Let us observe that $\mathrm{Maps}(X, L)$ is right zeroed.

Let L be an add-associative, right zeroed, right complementable, non empty vector space structure over R. One can verify that $\mathrm{Maps}(X, L)$ is right complementable.

Let L be a scalar distributive, non empty vector space structure over R. Note that $\mathrm{Maps}(X, L)$ is scalar distributive.

Let L be a scalar associative, non empty vector space structure over R. Let us observe that $\mathrm{Maps}(X, L)$ is scalar associative.

Let L be a vector distributive, non empty vector space structure over R. Note that $\mathrm{Maps}(X, L)$ is vector distributive.

Let L be a scalar unital, non empty vector space structure over R. Let us observe that $\mathrm{Maps}(X, L)$ is scalar unital.

Let U, V be vector spaces over R. One can check that the functor $\mathrm{Maps}(U, V)$ yields a vector space over R. Now we state the propositions:

(40) Let us consider a commutative ring R, vector spaces U, V over R, and linear transformations f, g from U to V. Then $f\ {}'{+}'\ g$ is a linear transformation from U to V.

(41) Let us consider a commutative ring R, vector spaces U, V over R, a linear transformation f from U to V, and an element a of R. Then $a \star f$ is a linear transformation from U to V.

Let F be a field and U, V be vector spaces over F. The functor $\mathrm{LinTrans}(U, V)$ yielding a strict subspace of $\mathrm{Maps}(U, V)$ is defined by

(Def. 23) the carrier of *it* = the set of all f where f is a linear transformation from U to V.

Let V be a vector space over F. The functor $\mathrm{End}(V)$ yielding a subspace of $\mathrm{Maps}(V,V)$ is defined by the term

(Def. 24) $\mathrm{LinTrans}(V,V)$.

References

[1] Jesse Alama. The rank+nullity theorem. *Formalized Mathematics*, 15(**3**):137–142, 2007. doi:10.2478/v10037-007-0015-6.

[2] Grzegorz Bancerek, Czesław Byliński, Adam Grabowski, Artur Korniłowicz, Roman Matuszewski, Adam Naumowicz, Karol Pąk, and Josef Urban. Mizar: State-of-the-art and beyond. In Manfred Kerber, Jacques Carette, Cezary Kaliszyk, Florian Rabe, and Volker Sorge, editors, *Intelligent Computer Mathematics*, volume 9150 of *Lecture Notes in Computer Science*, pages 261–279. Springer International Publishing, 2015. ISBN 978-3-319-20614-1. doi:10.1007/978-3-319-20615-8_17.

[3] Grzegorz Bancerek, Czesław Byliński, Adam Grabowski, Artur Korniłowicz, Roman Matuszewski, Adam Naumowicz, and Karol Pąk. The role of the Mizar Mathematical Library for interactive proof development in Mizar. *Journal of Automated Reasoning*, 61(1):9–32, 2018. doi:10.1007/s10817-017-9440-6.

[4] Norbert Christmann. *Einführung in die (lineare) Algebra.* Lecture Notes, University of Kaiserslautern, Germany, 1989.

[5] Adam Grabowski, Artur Korniłowicz, and Christoph Schwarzweller. On algebraic hierarchies in mathematical repository of Mizar. In M. Ganzha, L. Maciaszek, and M. Paprzycki, editors, *Proceedings of the 2016 Federated Conference on Computer Science and Information Systems (FedCSIS)*, volume 8 of *Annals of Computer Science and Information Systems*, pages 363–371, 2016. doi:10.15439/2016F520.

[6] I. Martin Isaacs. *Algebra, A Graduate Course.* American Mathematical Society, 2009.

[7] Serge Lang. *Algebra (Revised Third Edition).* Springer Verlag, 2002.

[8] Christoph Schwarzweller. Normal extensions. *Formalized Mathematics*, 31(1):121–130, 2023. doi:10.2478/forma-2023-0011.

[9] Yasushige Watase. Ring of endomorphisms and modules over a ring. *Formalized Mathematics*, 30(**3**):211–221, 2022. doi:10.2478/forma-2022-0016.

Accepted December 24, 2024