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Fundamentals of Finitary Proofs[1]

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Summary. An abstract, generic textbook notion of a finitary proof and some of its basic properties are presented, using the Mizar system. A general form of Lindenbaum's lemma is included.

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Introduction

An abstract, generic textbook notion of a finitary proof and some of its basic properties are presented, using the Mizar system [1], [2]. The approach is analogous to that of many textbooks, such as [11] or [3]. A general form of Lindenbaum's lemma is included.

The outline of the paper is as follows: first three sections define formulas and rules, proof steps and derivability. Section 4 describes formally the behaviour of supersets of formulas and rules. Section 5 contains the key definition in this article: the structure defining proof systems (prefixed by `1-sorted`); one can notice that due to the set-theoretic approach claimed in the Mizar Mathematical Library, a binary relation can denote either a single, or more rules (hence a type is just a rule, but the selector in the structure has the name "rules"). Closing sections contain Lindenbaum's and Teichmüller-Tukey lemmas.

Part of the contents were taken from [9], which was written to formalize some ideas from [4], [5], and [8]. This general approach could allow either to

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rewrite previous articles in this uniform language in the process of revisions [7], or to develop other logics, such as Suszko's logics ([12], [13], [14]), intuitionistic logic [10], or even fuzzy logics [6].

1. Preliminaries: Formulas and Rules

From now on i, j, k, l, m, n denote natural numbers, a, b, c, t, u denote objects, X, Y, Z denote sets, D, D_1, D_2, H denote non empty sets, and p, q, r, s denote finite sequences.

Let R be a binary relation. We say that R is finitary if and only if

(Def. 1) for every a such that $a \in \operatorname{dom} R$ holds a is a finite set.

Let us observe that every binary relation which is empty is also finitary and there exists a binary relation which is finitary.

We introduce the notation formula as a synonym of object.

A rule is a finitary binary relation.

A formula-finset is a finite set.

A formula-sequence is a finite sequence. Let H be a set.

A rule of H is a rule defined by

(Def. 2) $\operatorname{dom} it \subseteq \operatorname{Fin} H$ and $\operatorname{rng} it \subseteq H$.

Let H be a non empty set.

A formula of H is a formula defined by

(Def. 3) $it \in H$.

Let H be a set.

A formula-finset of H is a formula-finset defined by

(Def. 4) $it \subseteq H$.

A formula-sequence of H is a formula-sequence defined by

(Def. 5) it is a finite sequence of elements of H.

In the sequel R, R_1, R_2 denote rules, A, A_1, A_2 denote non empty sets, B, B_1, B_2 denote sets, P, P_1, P_2 denote formula-sequences, and S, S_1, S_2 denote formula-finsets.

Let us consider P. Observe that the functor $\operatorname{rng} P$ yields a formula-finset. Let us consider H. Let B_1, B_2 be subsets of H. Note that the functor $B_1 \cup B_2$ yields a subset of H. Let us consider S_1 and S_2. One can check that the functor $S_1 \cup S_2$ yields a formula-finset. Let us consider H. Let S_1, S_2 be formula-finsets of H. Let us note that the functor $S_1 \cup S_2$ yields a formula-finset of H. Let us consider R_1 and R_2. Note that the functor $R_1 \cup R_2$ yields a rule. Let us consider H. Let R_1, R_2 be rules of H. Let us observe that the functor $R_1 \cup R_2$ yields a rule of H.

2. Proof Steps

Let us consider B, R, P, and n. We say that (P, n) is a correct step w.r.t. B, R if and only if

(Def. 6) $P(n) \in B$ or there exists a formula-finset Q such that $\langle Q, P(n)\rangle \in R$ and for every t such that $t \in Q$ there exists k such that $k \in \operatorname{dom} P$ and $k < n$ and $P(k) = t$.

We say that P is (B, R)-correct if and only if

(Def. 7) for every k such that $k \in \operatorname{dom} P$ holds (P, k) is a correct step w.r.t. B, R.

Let us observe that every formula-sequence which is non (B, R)-correct is also non empty.

Let us consider H. Let us observe that there exists a formula-sequence of H which is (B, R)-correct and there exists a formula-sequence which is (B, R)-correct. Now we state the proposition:

(1) Let us consider an element a of A. Then $\langle a \rangle$ is (A, R)-correct.

Let us consider A and R. Let us observe that there exists a formula-sequence which is non empty and (A, R)-correct.

3. Derivability

Let us consider B, R, and S. We say that S is (B, R)-derivable if and only if

(Def. 8) there exists P such that $S = \operatorname{rng} P$ and P is (B, R)-correct.

Now we state the propositions:

(2) If P is (B, R)-correct and $P = P_1 \frown P_2$, then P_1 is (B, R)-correct.

(3) If P_1 is (B, R)-correct and P_2 is (B, R)-correct, then $P_1 \frown P_2$ is (B, R)-correct.

(4) If S_1 is (B, R)-derivable and S_2 is (B, R)-derivable, then $S_1 \cup S_2$ is (B, R)-derivable. The theorem is a consequence of (3).

(5) If $B \subseteq B_1$ and $R \subseteq R_1$ and P is (B, R)-correct, then P is (B_1, R_1)-correct.

Let us consider B and a. We say that a is B-axiomatic if and only if

(Def. 9) $a \in B$.

Let us consider R. We say that $B, R \vdash a$ if and only if

(Def. 10) there exists P such that $a \in \operatorname{rng} P$ and P is (B, R)-correct.

We say that a is (B, R)-provable if and only if

(Def. 11) $B, R \vdash a$.

4. Extensions

Let us consider B and B_1. We say that B_1 is B-extending if and only if

(Def. 12) $B \subseteq B_1$.

Let us consider R and R_1. We say that R_1 is R-extending if and only if

(Def. 13) $R \subseteq R_1$.

Let us consider B. Observe that there exists a set which is B-extending. Let us consider R. Let us observe that there exists a rule which is R-extending. Let us consider B.

An extension of B is a B-extending set. Let us consider H. Let B be a subset of H. Note that there exists a subset of H which is B-extending.

An extension of B is a B-extending subset of H. Let us consider R.

An extension of R is an R-extending rule. Let us consider H. Let B be a subset of H and t be a formula of H. The functor $B + t$ yielding an extension of B is defined by the term

(Def. 14) $B \cup \{t\}$.

Now we state the proposition:

(6) a is $(B \cup \{t\})$-axiomatic if and only if a is B-axiomatic or $a = t$.

From now on C denotes an extension of B and E denotes an extension of R.

Let us consider B and C. Let us note that every set which is C-extending is also B-extending and every object which is non C-axiomatic is also non B-axiomatic.

Let us consider R and E. Let us note that every rule which is E-extending is also R-extending.

Let us consider B and R_1. We say that R_1 is (B, R)-derivable if and only if

(Def. 15) for every S and t such that $\langle S, t\rangle \in R_1$ holds $B \cup S, R \vdash t$.

Now we state the propositions:

(7) $B, R \vdash t$ if and only if there exists S such that $t \in S$ and S is (B, R)-derivable.

(8) If $a \in B$, then $B, R \vdash a$. The theorem is a consequence of (1).

Let us consider B and R. One can verify that every object which is non (B, R)-provable is also non B-axiomatic. Now we state the propositions:

(9) If for every a such that $a \in S$ holds $B, R \vdash a$, then there exists S_1 such that $S \subseteq S_1$ and S_1 is (B, R)-derivable.

PROOF: Define $\mathcal{X}[\text{set}] \equiv$ there exists S_1 such that $\$_1 \subseteq S_1$ and S_1 is (B, R)-derivable. $\mathcal{X}[\emptyset]$. For every sets x, B_1 such that $x \in S$ and $B_1 \subseteq S$ and $\mathcal{X}[B_1]$ holds $\mathcal{X}[B_1 \cup \{x\}]$. $\mathcal{X}[S]$. $\square$

(10) If S is (B, R)-derivable and $B \cap S \subseteq B_1$, then S is (B_1, R)-derivable.
PROOF: Consider P such that $S = \operatorname{rng} P$ and P is (B, R)-correct. P is (B_1, R)-correct. $\square$

(11) If for every a such that $a \in S$ holds $B, R \vdash a$ and $\langle S, t\rangle \in R$, then $B, R \vdash t$. The theorem is a consequence of (9).

(12) If $B, R \vdash a$, then $a \in B$ or there exists S such that $\langle S, a\rangle \in R$ and for every b such that $b \in S$ holds $B, R \vdash b$.

(13) If S_1 is (B, R)-derivable and S_2 is (S_1, R)-derivable, then $S_1 \cup S_2$ is (B, R)-derivable.
PROOF: Consider P_1, P_2 such that P_1 is (B, R)-correct and $S_1 = \operatorname{rng} P_1$ and P_2 is (S_1, R)-correct and $S_2 = \operatorname{rng} P_2$. Set $P = P_1 \frown P_2$. For every k such that $k \in \operatorname{dom} P_1$ holds (P, k) is a correct step w.r.t. B, R. P is (B, R)-correct. $\square$

(14) If $B_1, R \vdash a$ and for every b such that $b \in B_1$ holds $B, R \vdash b$, then $B, R \vdash a$. The theorem is a consequence of (7), (9), (10), and (13).

(15) If $B, R \vdash a$, then $C, E \vdash a$. The theorem is a consequence of (5).

Let us consider B, R, and a. Note that a is (B, R)-provable if and only if the condition (Def. 16) is satisfied.

(Def. 16) for every C and E, $C, E \vdash a$.

Let us consider C. Note that every object which is non (C, R)-provable is also non (B, R)-provable. Let us consider E. Observe that every object which is non (C, E)-provable is also non (B, R)-provable. Now we state the propositions:

(16) $R_1 \cup R_2$ is (B, R)-derivable if and only if R_1 is (B, R)-derivable and R_2 is (B, R)-derivable.

(17) Let us consider a subset B of H, a rule R of H, and a. If $B, R \vdash a$, then $a \in H$.

5. Proof Systems

We consider proof systems which extend 1-sorted structures and are systems

$$\langle \text{a carrier}, \text{axioms}, \text{rules} \rangle$$

where the carrier is a set, the axioms constitute a subset of the carrier, the rules constitute a rule of the carrier.

Let P be a proof system. A formula-finset of P is a formula-finset of the carrier of P. Let a be an object. We say that $P \vdash a$ if and only if

(Def. 17) the axioms of P, the rules of $P \vdash a$.

Note that there exists a proof system which is non empty.

From now on P denotes a non empty proof system, B, B_1, B_2 denote subsets of P, and F denotes a finite subset of P. Now we state the proposition:

(18) If $P \vdash a$, then a is an element of P.

Let us consider P and B. We say that $P \vdash B$ if and only if

(Def. 18) for every a such that $a \in B$ holds $P \vdash a$.

Let us consider B_1 and B_2. One can check that the functor $B_1 \cup B_2$ yields a subset of P.

6. Consistency

Let us consider P. We say that P is consistent if and only if

(Def. 19) there exists a such that $a \in P$ and $P \nvdash a$.

Let us consider B. The functor $P \cup B$ yielding a non empty proof system is defined by the term

(Def. 20) $\langle$the carrier of P, (the axioms of $P) \cup B$, the rules of $P\rangle$.

Let us note that there exists a non empty proof system which is consistent and strict. Let P be a strict proof system and E be an empty subset of P. Let us observe that $P \cup E$ reduces to P. Let us consider P. We introduce the notation P is inconsistent as an antonym for P is consistent.

Let us consider B. We say that B is consistent if and only if

(Def. 21) $P \cup B$ is consistent.

Let P be a consistent, non empty proof system. Note that there exists a subset of P which is consistent.

Let us consider P and B. We introduce the notation B is inconsistent as an antonym for B is consistent.

One can check that there exists a subset of P which is inconsistent.

We say that P is paraconsistent if and only if

(Def. 22) every finite subset of P is consistent.

One can verify that every non empty proof system which is paraconsistent is also consistent and there exists a non empty proof system which is consistent and non paraconsistent.

7. Contradictions and Lindenbaum's Lemma

Let us consider P, B, and B_1. We say that B_1 is B-omitting if and only if

(Def. 23) there exists a such that $a \in B$ and $P \cup B_1 \nvdash a$.

Now we state the proposition:

(19) If B is inconsistent, then B_1 is consistent iff B_1 is B-omitting. The theorem is a consequence of (8) and (14).

Let us consider P. Let B be an inconsistent subset of P. One can verify that every subset of P which is B-omitting is also consistent and every subset of P which is non B-omitting is also inconsistent.

Let us consider B. One can verify that there exists a subset of P which is non B-omitting. Now we state the proposition:

(20) If B_1 is B-omitting and $B_2 \subseteq B_1$, then B_2 is B-omitting. The theorem is a consequence of (15).

Let us consider P and B. The functor $\mathrm{Omit}(P, B)$ yielding a family of subsets of P is defined by the term

(Def. 24) $\{B_1$, where B_1 is a subset of P : B_1 is B-omitting$\}$.

One can verify that the functor $\mathrm{Omit}(P, B)$ is defined by

(Def. 25) for every B_1, $B_1 \in it$ iff B_1 is B-omitting.

Let us consider B_1. We say that B_1 is B-maximally-omitting if and only if

(Def. 26) B_1 is B-omitting and for every B_2 such that $B_1 \subset B_2$ holds B_2 is not B-omitting.

Observe that every subset of P which is B-maximally-omitting is also B-omitting.

Let us consider X. We say that X is finite-character if and only if

(Def. 27) for every a, $a \in X$ iff there exists a set B such that $B = a$ and for every finite subset S of B, $S \in X$.

Let us observe that X is finite-character if and only if the condition (Def. 28) is satisfied.

(Def. 28) for every Y, $Y \in X$ iff for every finite subset S of Y, $S \in X$.

Let F be a family of subsets of X. Observe that F is finite-character if and only if the condition (Def. 29) is satisfied.

(Def. 29) for every subset B of X, $B \in F$ iff for every finite subset S of B, $S \in F$.

One can check that there exists a family of subsets of X which is non empty and finite-character and every set which is empty is also finite-character and there exists a set which is non empty and finite-character.

8. Teichmüller-Tukey Lemma

Now we state the proposition:

(21) Let us consider a non empty, finite-character set X. Then there exists an element Y of X such that for every element Z of X, $Y \not\subset Z$.
Proof: For every set C such that $C \subseteq X$ and C is $\subseteq$-linear there exists Y such that $Y \in X$ and for every Z such that $Z \in C$ holds $Z \subseteq Y$. Consider Y such that $Y \in X$ and for every Z such that $Z \in X$ and $Z \neq Y$ holds $Y \not\subseteq Z$. □

Let us consider P and F. One can check that $\mathrm{Omit}(P, F)$ is finite-character.

Now we state the proposition:

(22) If B is F-omitting, then there exists B_1 such that $B \subseteq B_1$ and B_1 is F-maximally-omitting. The theorem is a consequence of (21).

Let us consider P and B. We say that B is maximally-consistent if and only if

(Def. 30) B is consistent and for every B_1 such that $B \subset B_1$ holds B_1 is inconsistent.

Now we state the proposition:

(23) If P is consistent and non paraconsistent and B is consistent, then there exists B_1 such that $B \subseteq B_1$ and B_1 is maximally-consistent. The theorem is a consequence of (22).

The scheme *UnOpCongr* deals with a non empty set $\mathcal{X}$ and a unary functor $\mathcal{F}$ yielding an element of $\mathcal{X}$ and an equivalence relation $\mathcal{E}$ of $\mathcal{X}$ and states that

(Sch. 1) There exists a unary operation f on $\mathrm{Classes}\,\mathcal{E}$ such that for every element x of $\mathcal{X}$, $f([x]_{\mathcal{E}}) = [\mathcal{F}(x)]_{\mathcal{E}}$

provided

- for every elements x, y of $\mathcal{X}$ such that $\langle x, y\rangle \in \mathcal{E}$ holds $\langle \mathcal{F}(x), \mathcal{F}(y)\rangle \in \mathcal{E}$.

The scheme *BinOpCongr* deals with a non empty set $\mathcal{X}$ and a binary functor $\mathcal{F}$ yielding an element of $\mathcal{X}$ and an equivalence relation $\mathcal{E}$ of $\mathcal{X}$ and states that

(Sch. 2) There exists a binary operation f on $\mathrm{Classes}\,\mathcal{E}$ such that for every elements x, y of $\mathcal{X}$, $f([x]_{\mathcal{E}}, [y]_{\mathcal{E}}) = [\mathcal{F}(x, y)]_{\mathcal{E}}$

provided

- for every elements x_1, x_2, y_1, y_2 of $\mathcal{X}$ such that $\langle x_1, x_2\rangle$, $\langle y_1, y_2\rangle \in \mathcal{E}$ holds $\langle \mathcal{F}(x_1, y_1), \mathcal{F}(x_2, y_2)\rangle \in \mathcal{E}$.

The scheme *ProofInduction* deals with a set $\mathcal{B}$ and a rule $\mathcal{R}$ and a unary predicate $\mathcal{S}$ and states that

(Sch. 3) For every a such that $\mathcal{B}, \mathcal{R} \vdash a$ holds $\mathcal{S}[a]$

provided

- for every a such that $a \in \mathcal{B}$ holds $\mathcal{S}[a]$ and
- for every X and a such that $\langle X, a \rangle \in \mathcal{R}$ and for every b such that $b \in X$ holds $\mathcal{S}[b]$ holds $\mathcal{S}[a]$.

Let us consider R and X. We say that X is R-closed if and only if

(Def. 31) for every Y and a such that $\langle Y, a \rangle \in R$ and $Y \subseteq X$ holds $a \in X$.

9. Theorems

Let us consider D and R. A theorem of D, R is an object defined by

(Def. 32) $D, R \vdash \mathit{it}$.

Note that the type possesses the sethood property. Let us consider X. The functor Theorems(X, R) yielding a set is defined by the term

(Def. 33) $\{t$, where t is an element of $X \cup \operatorname{rng} R : X, R \vdash t\}$.

Note that the functor Theorems(X, R) is defined by

(Def. 34) for every a, $a \in \mathit{it}$ iff $X, R \vdash a$.

Note that Theorems(X, R) is X-extending and R-closed and there exists a set which is X-extending and R-closed. Now we state the proposition:

(24) $X, R \vdash a$ if and only if for every R-closed, X-extending set Y, $a \in Y$.

Let us consider P. The functor Theorems(P) yielding a subset of P is defined by the term

(Def. 35) Theorems((the axioms of P), (the rules of P)).

Let us consider X. We say that X is P-closed if and only if

(Def. 36) X is (the rules of P)-closed and (the axioms of P)-extending.

Let us note that Theorems(P) is P-closed and every set which is P-closed is also (the rules of P)-closed and (the axioms of P)-extending and every set which is (the rules of P)-closed and (the axioms of P)-extending is also P-closed and there exists a subset of P which is P-closed and there exists a set which is P-closed. Now we state the proposition:

(25) $P \vdash a$ if and only if for every P-closed set X, $a \in X$. The theorem is a consequence of (24).

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