


Classical Isoperimetric Theorem¹

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Summary. In this article we present the Mizar proof of the isoperimetric theorem (one of the theorems listed among Wiedijk’s Top 100 mathematical theorems), inspired by Peter D. Lax’s paper “A Short Path to the Shortest Path”. Using relatively simple formal apparatus of continuous and differentiable functions, we show that among all curves of fixed length connecting two points on the x-axis, a semicircle is the curve which maximizes the area between the curve and the x-axis.

MSC: 26B15 52A40 68V20

Keywords: isoperimetric theorem; calculus of variations; parametric curve

MML identifier: PDLAX, version: 8.1.14 5.86.1479

INTRODUCTION

In this article we formalize in Mizar [1], [2] the isoperimetric theorem, inspired by Peter D. Lax’s “A Short Path to the Shortest Path” [12]. Notably, Lax’s proof is remarkably concise, spanning just one page, demonstrating the elegance of his approach.

Our formalization begins by establishing fundamental properties of continuous and differentiable functions (although most of useful properties are already present in the Mizar Mathematical Library, for more advanced recent results in this area see [4], [5], [6]), including theorems on integrals and differentiation rules. Building upon these, it progresses to the proof of the isoperimetric theorem, addressing the following question: Among all curves of fixed length connecting

¹This work was supported by JSPS KAKENHI Grant Number 24K14897.

two points on the x-axis, which curve maximizes the area between the curve and the x-axis? The formalization proves that for parametric curves $(x(t), y(t))$ with fixed length and endpoints on the x-axis, the integral $\int_0^\pi y(t)x'(t) dt$ is maximized when the curve is a semicircle [14]. The historical background on the isoperimetric theorem is detailed in [3] and [11].

This work represents the solution of the problem #43 of Freek Wiedijk's "Formalizing 100 Theorems" project [16], underscoring the significance of this effort in the context of formalization of mathematics (compare the formal development in HOL Light [9]: [10] and [15]). The survey of some implications of the theorem is contained in [13].

1. FOUNDATIONS OF CONTINUITY AND INTEGRATION

From now on a, b, r denote real numbers, A denotes a non empty set, X, x denote sets, f, g, F, G denote partial functions from \mathbb{R} to \mathbb{R} , and n denotes an element of \mathbb{N} . Now we state the propositions:

- (1) Let us consider real numbers a, b, C , and a partial function u from \mathbb{R} to \mathbb{R} . Suppose $a < b$ and $[a, b] \subseteq \text{dom } u$ and u is continuous and for every real number t such that $t \in]a, b[$ holds $u(t) = C$. Let us consider a real number t . If $t \in [a, b]$, then $u(t) = C$.

PROOF: Define $\mathcal{M}(\text{natural number}) = \frac{b-a}{2} (\in \mathbb{R})$. Consider S_4 being a function from \mathbb{N} into \mathbb{R} such that for every element x of \mathbb{N} , $S_4(x) = \mathcal{M}(x)$.

For every natural number n , $S_4(n) = \frac{b-a}{n+1}$. Consider S_2 being a constant function from \mathbb{N} into \mathbb{R} such that for every natural number x , $S_2(x) = a$. Set $S_0 = S_2 + S_4$. $\text{rng } S_0 \subseteq]a, b[$. For every natural number n , $(u_* S_0)(n) = C$. For every objects x, y such that $x, y \in \text{dom}(u_* S_0)$ holds $(u_* S_0)(x) = (u_* S_0)(y)$. Consider S_3 being a constant function from \mathbb{N} into \mathbb{R} such that for every natural number x , $S_3(x) = b$. Set $S_1 = S_3 - S_4$. $\text{rng } S_1 \subseteq]a, b[$. For every natural number n , $(u_* S_1)(n) = C$. For every objects x, y such that $x, y \in \text{dom}(u_* S_1)$ holds $(u_* S_1)(x) = (u_* S_1)(y)$. For every real number t such that $t \in [a, b]$ holds $u(t) = C$. \square

- (2) Let us consider real numbers a, b, c, d , and a partial function f from \mathbb{R} to \mathbb{R} . Suppose $a \leq b$ and $c \leq d$ and $[a, b] \subseteq \text{dom } f$ and $c, d \in [a, b]$ and $f \upharpoonright [a, b]$ is continuous and for every real number t such that $t \in [c, d]$ holds

$$0 \leq f(t). \text{ Then } 0 \leq \int_c^d f(x) dx.$$

PROOF: For every object x such that $x \in \text{dom}(f \upharpoonright [c, d])$ holds $(f \upharpoonright [c, d])(x) = (|f| \upharpoonright [c, d])(x)$. \square

- (3) Let us consider real numbers a, b, c, d , and partial functions f, g from \mathbb{R} to \mathbb{R} . Suppose $a \leq b$ and $c \leq d$ and $[a, b] \subseteq \text{dom } f$ and $[a, b] \subseteq \text{dom } g$ and $c, d \in [a, b]$ and $f|_{[a, b]}$ is continuous and $g|_{[a, b]}$ is continuous and for every real number t such that $t \in [c, d]$ holds $f(t) \leq g(t)$. Then $\int_c^d f(x)dx \leq \int_c^d g(x)dx$. The theorem is a consequence of (2).

- (4) Let us consider real numbers a, b, c, d, e , and a partial function f from \mathbb{R} to \mathbb{R} . Suppose $a \leq b$ and $c \leq d$ and $c, d \in [a, b]$ and $[a, b] \subseteq \text{dom } f$ and $f|_{[a, b]}$ is continuous and for every real number t such that $t \in [c, d]$ holds $e \leq f(t)$. Then $e \cdot (d - c) \leq \int_c^d f(x)dx$.

PROOF: Set $g = \mathbb{R} \mapsto e$. For every real number t such that $t \in [c, d]$ holds $g(t) \leq f(t)$. $\int_c^d g(x)dx \leq \int_c^d f(x)dx$. \square

- (5) Let us consider real numbers a, b, c, d, e , and a partial function f from \mathbb{R} to \mathbb{R} . Suppose $0 < e$ and $a \leq b$ and $c < d$ and $c, d \in [a, b]$ and $[a, b] \subseteq \text{dom } f$ and $f|_{[a, b]}$ is continuous and for every real number t such that $t \in [a, b]$ holds $0 \leq f(t)$ and for every real number t such that $t \in [c, d]$ holds $e \leq f(t)$. Then $0 < e \cdot (d - c) \leq \int_a^b f(x)dx$. The theorem is a consequence of (2) and (4).

- (6) Let us consider real numbers a, b , and a partial function f from \mathbb{R} to \mathbb{R} . Suppose $a \leq b$ and $[a, b] \subseteq \text{dom } f$ and $f|_{[a, b]}$ is continuous and for every real number t such that $t \in [a, b]$ holds $0 \leq f(t)$ and there exists a real number t_0 such that $t_0 \in]a, b[$ and $0 < f(t_0)$. Then there exist real numbers d, c, e such that

- (i) $0 < e$, and
- (ii) $c < d$, and
- (iii) $c, d \in [a, b]$, and
- (iv) $0 < e \cdot (d - c) \leq \int_a^b f(x)dx$.

PROOF: Consider t_0 being a real number such that $t_0 \in]a, b[$ and $0 < f(t_0)$. Set $e = \frac{f(t_0)}{2}$. Consider s_0 being a real number such that $0 < s_0$

and for every real number t such that $t \in [a, b]$ and $|t - t_0| < s_0$ holds $|f(t) - f(t_0)| < e$. Set $s = \frac{s_0}{2}$. Reconsider $s_2 = \min(t_0 - a, b - t_0)$ as a real number. Reconsider $s_3 = \min(s, s_2)$ as a real number. Set $c = t_0 - s_3$. Set $d = t_0 + s_3$. Set $e_0 = \frac{f(t_0)}{2}$. For every real number t such that $t \in [c, d]$ holds $e_0 \leq f(t)$. \square

- (7) Let us consider a partial function f from \mathbb{R} to \mathbb{R} , and real numbers a, b . Suppose $a < b$ and $[a, b] \subseteq \text{dom } f$ and $f|_{[a, b]}$ is continuous. Then there exists a sequence I of real numbers such that

$$(i) \text{ for every natural number } n, I(n) = \int_{a+\frac{1}{n+1}}^{b-\frac{1}{n+1}} f(x)dx, \text{ and}$$

(ii) I is convergent, and

$$(iii) \lim I = \int_a^b f(x)dx.$$

PROOF: Define $\mathcal{M}(\text{natural number}) = (\int_{a+\frac{1}{s_1+1}}^{b-\frac{1}{s_1+1}} f(x)dx) \in \mathbb{R}$. Consider I

being a function from \mathbb{N} into \mathbb{R} such that for every element x of \mathbb{N} , $I(x) = \mathcal{M}(x)$. For every natural number n , $I(n) = \int_{a+\frac{1}{n+1}}^{b-\frac{1}{n+1}} f(x)dx$. Set $X = [a, b]$.

Consider t_1, t_2 being real numbers such that $t_1, t_2 \in \text{dom}(|f| \upharpoonright X)$ and $(|f| \upharpoonright X)(t_1) = \sup \text{rng}(|f| \upharpoonright X)$ and $(|f| \upharpoonright X)(t_2) = \inf \text{rng}(|f| \upharpoonright X)$. Set $K = (|f| \upharpoonright X)(t_1)$. For every real number t such that $t \in X$ holds $|f(t)| \leq K$.

Set $L = \int_a^b f(x)dx$. For every real number p such that $0 < p$ there exists a natural number n such that for every natural number m such that $n \leq m$ holds $|I(m) - L| < p$ by [7, (17)], [8, (10), (11)]. \square

2. DIFFERENTIATION RULES AND PROPERTIES

Now we state the propositions:

- (8) Let us consider an open subset Z of \mathbb{R} . Then
- (i) the function \sin is differentiable on Z , and
 - (ii) $(\text{the function } \sin)'|_Z = (\text{the function } \cos)|_Z$, and

(iii) the function \cos is differentiable on Z , and

(iv) $(\text{the function } \cos)'_{\upharpoonright Z} = -(\text{the function } \sin)_{\upharpoonright Z}$.

(9) Let us consider a partial function f from \mathbb{R} to \mathbb{R} . Then $f + f = 2 \cdot f$.

Let us consider a partial function f from \mathbb{R} to \mathbb{R} , a subset Z of \mathbb{R} , and a real number x . Now we state the propositions:

(10) If Z is open and $x \in Z$ and $Z \subseteq \text{dom } f$, then $f_{\upharpoonright Z}$ is differentiable in x iff f is differentiable in x .

(11) If Z is open and $x \in Z$ and $Z \subseteq \text{dom } f$ and f is differentiable in x , then $f'(x) = (f_{\upharpoonright Z})'(x)$. The theorem is a consequence of (10).

(12) Let us consider a partial function f from \mathbb{R} to \mathbb{R} , and subsets X, Z of \mathbb{R} . Suppose Z is open and $Z \subseteq X$ and f is differentiable on X . Then $f'_{\upharpoonright Z} = f'_{\upharpoonright X} \upharpoonright Z$.

PROOF: For every object x such that $x \in \text{dom}(f'_{\upharpoonright X} \upharpoonright Z)$ holds $(f'_{\upharpoonright X} \upharpoonright Z)(x) = f'_{\upharpoonright Z}(x)$. \square

(13) Let us consider real numbers a, b , and a partial function u from \mathbb{R} to \mathbb{R} . Suppose $a < b$ and u is differentiable on $]a, b[$ and $\text{dom } u = [a, b]$ and u is continuous and for every real number t such that $t \in]a, b[$ holds $u'_{\upharpoonright]a, b[}(t) = 0$. Then there exists a real number C such that for every real number t such that $t \in [a, b]$ holds $u(t) = C$. The theorem is a consequence of (1).

3. PROPERTIES OF PARAMETRIC CURVES AND AREA CALCULATIONS

Now we state the proposition:

(14) Let us consider partial functions x, y from \mathbb{R} to \mathbb{R} , and an open subset Z of \mathbb{R} . Suppose x is differentiable and y is differentiable and $[0, \pi] \subseteq Z \subseteq \text{dom } x$ and $Z \subseteq \text{dom } y$ and $y'_{\upharpoonright Z}$ is continuous and $x'_{\upharpoonright Z}$ is continuous and for every real number t such that $t \in Z$ holds $x'_{\upharpoonright Z}(t)^2 + y'_{\upharpoonright Z}(t)^2 = 1$ and $y(0) = 0$ and $y(\pi) = 0$.

Then there exists a partial function u from \mathbb{R} to \mathbb{R} and there exists a sequence F of real numbers such that u is differentiable on $]0, \pi[$ and $u'_{\upharpoonright]0, \pi[}$ is continuous and $\text{dom } u = [0, \pi]$ and u is continuous and $y_{\upharpoonright [0, \pi]} = (u \cdot (\text{the function } \sin))_{\upharpoonright [0, \pi]}$ and for every real number t such that $t \in]0, \pi[$ holds $y'(t) = u'(t) \cdot (\text{the function } \sin)(t) + u(t) \cdot (\text{the function } \cos)(t)$ and for

every natural number n , $F(n) = \int_{\frac{1}{n+1}}^{\pi - \frac{1}{n+1}} ((\text{AffineMap}(0, 1)) - ((u'_{\upharpoonright]0, \pi[} \cdot u'_{\upharpoonright]0, \pi[}) \cdot (\text{the function } \sin)) \cdot (\text{the function } \sin))(x) dx$ and F is convergent and

$$\begin{aligned}
& \int_0^\pi (y \cdot x'_{|Z})(x) dx \leq \frac{1}{2} \cdot \left(\int_0^\pi (y \cdot y + x'_{|Z} \cdot x'_{|Z})(x) dx \right) \text{ and } y \cdot y + x'_{|Z} \cdot x'_{|Z} = y \cdot y + \\
& (\text{AffineMap}(0, 1)) - y'_{|Z} \cdot y'_{|Z} \text{ and } \int_0^\pi (y \cdot y + x'_{|Z} \cdot x'_{|Z})(x) dx = \int_0^\pi (y \cdot y + (\text{Af-} \\
& \text{fineMap}(0, 1)) - y'_{|Z} \cdot y'_{|Z})(x) dx \text{ and } \int_0^\pi (y \cdot y + (\text{AffineMap}(0, 1)) - y'_{|Z} \cdot \\
& y'_{|Z})(x) dx = \lim F.
\end{aligned}$$

4. FORMALIZATION OF THE ISOPERIMETRIC THEOREM

Now we state the propositions:

- (15) Let us consider partial functions x, y from \mathbb{R} to \mathbb{R} , and an open subset Z of \mathbb{R} . Suppose x is differentiable and y is differentiable and $[0, \pi] \subseteq Z \subseteq \text{dom } x$ and $Z \subseteq \text{dom } y$ and $x'_{|Z}$ is continuous and $y'_{|Z}$ is continuous and for every real number t such that $t \in Z$ holds $x'_{|Z}(t)^2 + y'_{|Z}(t)^2 = 1$ and $y(0) = 0$ and $y(\pi) = 0$. Then

$$(i) \int_0^\pi (y \cdot x'_{|Z})(x) dx \leq \frac{1}{2} \cdot \pi, \text{ and}$$

$$\begin{aligned}
(ii) \int_0^\pi (y \cdot x'_{|Z})(x) dx &= \frac{1}{2} \cdot \pi \text{ iff for every real number } t \text{ such that } t \in [0, \pi] \\
&\text{holds } y(t) = (\text{the function sin})(t) \text{ and } x(t) = -(\text{the function cos})(t) + \\
&(\text{the function cos})(0) + x(0) \text{ or for every real number } t \text{ such that } t \in \\
&[0, \pi] \text{ holds } y(t) = -(\text{the function sin})(t) \text{ and } x(t) = (\text{the function} \\
&\text{cos})(t) - (\text{the function cos})(0) + x(0).
\end{aligned}$$

- (16) Let us consider partial functions x, y from \mathbb{R} to \mathbb{R} . Suppose x is differentiable and y is differentiable and $[0, \pi] \subseteq \text{dom } x$ and $[0, \pi] \subseteq \text{dom } y$ and $x'_{|\text{dom } x}$ is continuous and $y'_{|\text{dom } y}$ is continuous and for every real number t such that $t \in \text{dom } x \cap \text{dom } y$ holds $x'(t)^2 + y'(t)^2 = 1$ and $y(0) = 0$ and $y(\pi) = 0$. Then

$$(i) \int_0^\pi (y \cdot x'_{|\text{dom } x})(x) dx \leq \frac{1}{2} \cdot \pi, \text{ and}$$

$$\begin{aligned}
(ii) \int_0^\pi (y \cdot x'_{|\text{dom } x})(x) dx &= \frac{1}{2} \cdot \pi \text{ iff for every real number } t \text{ such that } t \in \\
&[0, \pi] \text{ holds } y(t) = (\text{the function sin})(t) \text{ and } x(t) = -(\text{the function cos})
\end{aligned}$$

$(t) + (\text{the function } \cos)(0) + x(0)$ or for every real number t such that $t \in [0, \pi]$ holds $y(t) = -(\text{the function } \sin)(t)$ and $x(t) = (\text{the function } \cos)(t) - (\text{the function } \cos)(0) + x(0)$.

The theorem is a consequence of (12).

- (17) Let us consider partial functions x, y from \mathbb{R} to \mathbb{R} , and a real number L . Suppose $0 < L$ and x is differentiable and y is differentiable and $[0, \pi] \subseteq \text{dom } x$ and $[0, \pi] \subseteq \text{dom } y$ and $x'_{|_{\text{dom } x}}$ is continuous and $y'_{|_{\text{dom } y}}$ is continuous and for every real number t such that $t \in \text{dom } x \cap \text{dom } y$ holds $x'(t)^2 + y'(t)^2 = \frac{L}{\pi}$ and $y(0) = 0$ and $y(\pi) = 0$. Then

$$(i) \int_0^\pi (y \cdot x'_{|_{\text{dom } x}})(x) dx \leq \frac{1}{2} \cdot L, \text{ and}$$

$$(ii) \int_0^\pi (y \cdot x'_{|_{\text{dom } x}})(x) dx = \frac{1}{2} \cdot L \text{ iff for every real number } t \text{ such that } t \in [0, \pi] \text{ holds } y(t) = \frac{(\text{the function } \sin)(t)}{\sqrt{\frac{\pi}{L}}} \text{ and } x(t) = -\frac{(\text{the function } \cos)(t)}{\sqrt{\frac{\pi}{L}}} + \frac{(\text{the function } \cos)(0)}{\sqrt{\frac{\pi}{L}}} + x(0) \text{ or for every real number } t \text{ such that } t \in [0, \pi] \text{ holds } y(t) = -\frac{(\text{the function } \sin)(t)}{\sqrt{\frac{\pi}{L}}} \text{ and } x(t) = \frac{(\text{the function } \cos)(t)}{\sqrt{\frac{\pi}{L}}} - \frac{(\text{the function } \cos)(0)}{\sqrt{\frac{\pi}{L}}} + x(0).$$

PROOF: Set $k = \sqrt{\frac{\pi}{L}}$. Set $x_1 = k \cdot x$. Set $y_1 = k \cdot y$. For every real number t such that $t \in \text{dom } x_1 \cap \text{dom } y_1$ holds $x_1'(t)^2 + y_1'(t)^2 = 1$.

$$\int_0^\pi (y_1 \cdot x_1'_{|_{\text{dom } x_1}})(x) dx \leq \frac{1}{2} \cdot \pi \text{ and } \left(\int_0^\pi (y_1 \cdot x_1'_{|_{\text{dom } x_1}})(x) dx = \frac{1}{2} \cdot \pi \text{ iff for every} \right.$$

real number t such that $t \in [0, \pi]$ holds $y_1(t) = (\text{the function } \sin)(t)$ and $x_1(t) = -(\text{the function } \cos)(t) + (\text{the function } \cos)(0) + x_1(0)$ or for every real number t such that $t \in [0, \pi]$ holds $y_1(t) = -(\text{the function } \sin)(t)$ and $x_1(t) = (\text{the function } \cos)(t) - (\text{the function } \cos)(0) + x_1(0)$.

$$\int_0^\pi (y_1 \cdot x_1'_{|_{\text{dom } x_1}})(x) dx = \frac{1}{2} \cdot \pi \text{ iff } \int_0^\pi (y \cdot x'_{|_{\text{dom } x}})(x) dx = \frac{1}{2} \cdot L. \text{ For every}$$

real number t such that $t \in [0, \pi]$ holds $y_1(t) = (\text{the function } \sin)(t)$ and $x_1(t) = -(\text{the function } \cos)(t) + (\text{the function } \cos)(0) + x_1(0)$ iff for every real number t such that $t \in [0, \pi]$ holds $y(t) = \frac{(\text{the function } \sin)(t)}{k}$ and $x(t) = -\frac{(\text{the function } \cos)(t)}{k} + \frac{(\text{the function } \cos)(0)}{k} + x(0)$. For every real number t such that $t \in [0, \pi]$ holds $y_1(t) = -(\text{the function } \sin)(t)$ and $x_1(t) = (\text{the function } \cos)(t) - (\text{the function } \cos)(0) + x_1(0)$ iff for every

real number t such that $t \in [0, \pi]$ holds $y(t) = -\frac{(\text{the function } \sin)(t)}{k}$ and $x(t) = \frac{(\text{the function } \cos)(t)}{k} - \frac{(\text{the function } \cos)(0)}{k} + x(0)$. \square

REFERENCES

- [1] Grzegorz Bancerek, Czesław Byliński, Adam Grabowski, Artur Korniłowicz, Roman Matuszewski, Adam Naumowicz, Karol Pąk, and Josef Urban. Mizar: State-of-the-art and beyond. In Manfred Kerber, Jacques Carette, Cezary Kaliszyk, Florian Rabe, and Volker Sorge, editors, *Intelligent Computer Mathematics*, volume 9150 of *Lecture Notes in Computer Science*, pages 261–279. Springer International Publishing, 2015. ISBN 978-3-319-20614-1. doi:10.1007/978-3-319-20615-8_17.
- [2] Grzegorz Bancerek, Czesław Byliński, Adam Grabowski, Artur Korniłowicz, Roman Matuszewski, Adam Naumowicz, and Karol Pąk. The role of the Mizar Mathematical Library for interactive proof development in Mizar. *Journal of Automated Reasoning*, 61(1):9–32, 2018. doi:10.1007/s10817-017-9440-6.
- [3] Viktor Blåsjö. The isoperimetric problem. *The American Mathematical Monthly*, 112(6):526–566, 2005.
- [4] Noboru Endou. Differentiation on interval. *Formalized Mathematics*, 31(1):9–21, 2023. doi:10.2478/forma-2023-0002.
- [5] Noboru Endou and Yasunari Shidama. Multidimensional measure space and integration. *Formalized Mathematics*, 31(1):181–192, 2023. doi:10.2478/forma-2023-0017.
- [6] Noboru Endou and Yasunari Shidama. Integral of continuous functions of two variables. *Formalized Mathematics*, 31(1):309–324, 2023. doi:10.2478/forma-2023-0025.
- [7] Noboru Endou, Katsumi Wasaki, and Yasunari Shidama. Definition of integrability for partial functions from \mathbb{R} to \mathbb{R} and integrability for continuous functions. *Formalized Mathematics*, 9(2):281–284, 2001.
- [8] Noboru Endou, Yasunari Shidama, and Masahiko Yamazaki. Integrability and the integral of partial functions from \mathbb{R} into \mathbb{R} . *Formalized Mathematics*, 14(4):207–212, 2006. doi:10.2478/v10037-006-0023-y.
- [9] John Harrison. The HOL Light system reference. 2023. <http://www.cl.cam.ac.uk/~jrh13/hol-light/reference.pdf>.
- [10] John Harrison. The isoperimetric inequality. 2023. Available online at <https://github.com/jrh13/hol-light/blob/master/100/isoperimetric.ml>.
- [11] Andreas Hehl. The isoperimetric inequality. *Proseminar Curves and Surfaces, Universitaet Tuebingen, Tuebingen*, 2013.
- [12] Peter David Lax. A short path to the shortest path. *The American Mathematical Monthly*, 102(2):158–159, 1995.
- [13] Robert Osserman. The isoperimetric inequality. *Bulletin of American Mathematical Monthly*, 6(84):1182–1238, 1978.
- [14] Alan Siegel. An isoperimetric theorem in plane geometry. *Discrete and Computational Geometry*, 29(2):239–255, 2003. doi:10.1007/s00454-002-2809-1.
- [15] Marten Straatsma. Towards formalising the isoperimetric theorem. BSc thesis, Radboud University Nijmegen, 2022.
- [16] Freek Wiedijk. Formalizing 100 theorems. Available online at <http://www.cs.ru.nl/~freek/100/>.

Accepted December 14, 2024