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About Path and Cycle Graphs

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Summary. In this article path and cycle graphs are formalized in the Mizar system.

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Introduction

We continue the development of the formalization of graphs in [12], as described in [13] (compare Isabelle graph library [14] or similar efforts in PVS [3]). For our recent encodings, including spanning subgraphs theorem or handshaking lemma, see [6] and [10]. Path and cycle graphs are two fundamental graph families (cf. [2], [15], [4]). In this paper both types are formalized in the Mizar system [5], [1] in a way that also includes the 1-cycle, 2-cycle, ray and double-ray graph in the definitions. It is shown how a finite path graph can be constructed successively and how to construct cycle graphs from finite path graphs. A maximal graph path is characterized for every path graph as well. Furthermore, the rather obvious fact that a graph circuit in a cycle graph covers all its vertices and edges is proven and constitutes the longest proof in this work.

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1. Preliminaries

One can verify that there exists a graph which is trivial, non-directed-multi, and loopfull. Let G be a non acyclic graph. One can verify that there exists a subgraph of G which is non acyclic. Now we state the propositions:

(1) Let us consider a graph G_1, a subgraph G_2 of G_1, a vertex v_1 of G_1, and a vertex v_2 of G_2. Suppose $v_1 = v_2$. Then

- (i) v_2.inDegree() $\subseteq$ v_1.inDegree(), and
- (ii) v_2.outDegree() $\subseteq$ v_1.outDegree(), and
- (iii) v_2.degree() $\subseteq$ v_1.degree().

(2) Let us consider a graph G, and a trail T of G. Then T.length() $= \overline{\overline{T.\text{edges}()}}$.

Let G be a non trivial, connected graph. One can verify that every vertex of G is non isolated.

Let G be a non acyclic graph. One can verify that there exists a walk of G which is cycle-like. Now we state the propositions:

(3) Let us consider a non trivial, tree-like graph T, a vertex v of T, and a subgraph F of T with vertex v removed. Then F.numComponents() $=$ v.degree().

Proof: Define $\mathcal{H}$(vertex of F) $=$ F.reachableFrom($\$_1$). Consider h' being a function from the vertices of F into F.componentSet() such that for every vertex w of F, $h'(w) = \mathcal{H}(w)$. □

(4) Let us consider a non trivial, finite, tree-like graph T, a vertex v of T, a subgraph F of T with vertex v removed, and a component C of F. Then there exists a vertex w of T such that

- (i) w is endvertex, and
- (ii) $w \in$ the vertices of C.

(5) Let us consider a graph G_2, objects v, e, w, a vertex v_2 of G_2, a supergraph G_1 of G_2 extended by v, w and e between them, and a vertex v_1 of G_1. Suppose $v_1 \neq v$ and $v_1 \neq w$ and $v_1 = v_2$. Then

- (i) v_1.edgesIn() $=$ v_2.edgesIn(), and
- (ii) v_1.inDegree() $=$ v_2.inDegree(), and
- (iii) v_1.edgesOut() $=$ v_2.edgesOut(), and
- (iv) v_1.outDegree() $=$ v_2.outDegree(), and
- (v) v_1.edgesInOut() $=$ v_2.edgesInOut(), and
- (vi) v_1.degree() $=$ v_2.degree().

(6) Let us consider a graph G_2, a vertex v of G_2, objects e, w, a supergraph G_1 of G_2 extended by v, w and e between them, and a vertex v_1 of G_1. Suppose $e \notin$ the edges of G_2 and $w \notin$ the vertices of G_2 and $v_1 = v$. Then

(i) v_1.edgesIn() $= v$.edgesIn(), and

(ii) v_1.inDegree() $= v$.inDegree(), and

(iii) v_1.edgesOut() $= v$.edgesOut() $\cup \{e\}$, and

(iv) v_1.outDegree() $= v$.outDegree() $+ 1$, and

(v) v_1.edgesInOut() $= v$.edgesInOut() $\cup \{e\}$, and

(vi) v_1.degree() $= v$.degree() $+ 1$.

(7) Let us consider a graph G_2, objects v, e, a vertex w of G_2, a supergraph G_1 of G_2 extended by v, w and e between them, and a vertex w_1 of G_1. Suppose $e \notin$ the edges of G_2 and $v \notin$ the vertices of G_2 and $w_1 = w$. Then

(i) w_1.edgesIn() $= w$.edgesIn() $\cup \{e\}$, and

(ii) w_1.inDegree() $= w$.inDegree() $+ 1$, and

(iii) w_1.edgesOut() $= w$.edgesOut(), and

(iv) w_1.outDegree() $= w$.outDegree(), and

(v) w_1.edgesInOut() $= w$.edgesInOut() $\cup \{e\}$, and

(vi) w_1.degree() $= w$.degree() $+ 1$.

(8) Let us consider a graph G, and a component C of G. Then C.endVertices() $\subseteq G$.endVertices().

Let G be an edgeless graph. Let us note that G.endVertices() is empty.

2. Path Graphs

Let G be a graph. We say that G is path-like if and only if

(Def. 1) G is tree-like and for every vertex v of G, v.degree() $\subseteq 2$.

Observe that every graph which is path-like is also tree-like, locally-finite, and with max degree and every graph which is trivial and edgeless is also path-like and every graph which is trivial and path-like is also edgeless and there exists a graph which is finite and path-like. Now we state the proposition:

(9) Let us consider a locally-finite graph G. Then G is path-like if and only if G is tree-like and for every vertex v of G, v.degree() $\leqslant 2$.

Let F be a graph-yielding function. We say that F is path-like if and only if

(Def. 2) for every object x such that $x \in \operatorname{dom} F$ there exists a graph G such that $F(x) = G$ and G is path-like.

Let P be a path-like graph. Observe that $\langle P \rangle$ is path-like and $\mathbb{N} \longmapsto P$ is path-like.

Let F be a non empty, graph-yielding function. Let us note that F is path-like if and only if the condition (Def. 3) is satisfied.

(Def. 3) for every element x of $\operatorname{dom} F$, $F(x)$ is path-like.

Let S be a graph sequence. Observe that S is path-like if and only if the condition (Def. 4) is satisfied.

(Def. 4) for every natural number n, $S(n)$ is path-like.

One can verify that every graph-yielding function which is empty is also path-like and every graph-yielding function which is trivial and edgeless is also path-like and every graph-yielding function which is path-like is also tree-like and there exists a graph sequence which is non empty and path-like.

Let F be a path-like, non empty, graph-yielding function and x be an element of $\operatorname{dom} F$. One can check that $F(x)$ is path-like. Let S be a path-like graph sequence and n be a natural number. One can check that $S(n)$ is path-like. Let p be a path-like, graph-yielding finite sequence. Note that $p{\restriction}n$ is path-like and $p_{\restriction n}$ is path-like. Let m be a natural number. Let us observe that $\operatorname{smid}(p, m, n)$ is path-like and $\langle p(m), \ldots, p(n) \rangle$ is path-like. Let p, q be path-like, graph-yielding finite sequences. Note that $p \frown q$ is path-like and $p \frown\!\frown q$ is path-like. Let P_1, P_2 be path-like graphs. One can verify that $\langle P_1, P_2 \rangle$ is path-like. Let P_3 be a path-like graph. One can check that $\langle P_1, P_2, P_3 \rangle$ is path-like.

Let S be a graph-membered set. We say that S is path-like if and only if

(Def. 5) for every graph G such that $G \in S$ holds G is path-like.

Observe that every graph-membered set which is empty is also path-like and every graph-membered set which is path-like is also tree-like. Let P_1 be a path-like graph. Let us note that $\{P_1\}$ is path-like. Let P_2 be a path-like graph. Let us observe that $\{P_1, P_2\}$ is path-like. Let F be a path-like, graph-yielding function. One can verify that $\operatorname{rng} F$ is path-like. Let X be a path-like, graph-membered set. Note that every subset of X is path-like. Let Y be a set. Observe that $X \cap Y$ is path-like and $X \setminus Y$ is path-like. Let X, Y be path-like, graph-membered sets. One can verify that $X \cup Y$ is path-like and $X \dot{-} Y$ is path-like and there exists a graph-membered set which is non empty and path-like.

Let S be a non empty, path-like, graph-membered set. Let us observe that every element of S is path-like. Now we state the propositions:

(10) Let us consider a path-like graph P_2, a vertex v_2 of P_2, objects e, w_2, and a supergraph P_1 of P_2 extended by v_2, w_2 and e between them. If v_2 is endvertex or P_2 is trivial, then P_1 is path-like. The theorem is a consequence of (6) and (5).

(11) Let us consider a path-like graph P_2, objects v_2, e, a vertex w_2 of P_2, and

a supergraph P_1 of P_2 extended by v_2, w_2 and e between them. If w_2 is endvertex or P_2 is trivial, then P_1 is path-like. The theorem is a consequence of (7) and (5).

Let n be a natural number. One can check that there exists a graph which is $(n+1)$-vertex, n-edge, and path-like.

Let n be a non zero natural number. Let us note that there exists a graph which is n-vertex, $(n -' 1)$-edge, and path-like and there exists a graph which is $(n+1)$-vertex, n-edge, path-like, and non trivial.

Let P be a path-like graph. Let us observe that every subgraph of P which is connected is also path-like. Now we state the propositions:

(12) Let us consider a graph G_2, objects v_1, e, v_2, and a supergraph G_1 of G_2 extended by v_1, v_2 and e between them. If G_1 is path-like, then G_2 is path-like.

(13) Let us consider a path-like graph P_1, a vertex v of P_1, and a subgraph P_2 of P_1 with vertex v removed. If v is endvertex or P_1 is trivial, then P_2 is path-like.

(14) Let us consider a finite, path-like graph G, and a connected subgraph H of G. Then there exists a non empty, finite, path-like, graph-yielding finite sequence p such that

(i) $p(1) \approx H$, and

(ii) $p(\operatorname{len} p) = G$, and

(iii) $\operatorname{len} p = G.\mathrm{order}() - H.\mathrm{order}() + 1$, and

(iv) for every element n of $\operatorname{dom} p$ such that $n \leqslant \operatorname{len} p - 1$ there exist vertices v_1, v_2 of G and there exists an object e such that $p(n+1)$ is a supergraph of $p(n)$ extended by v_1, v_2 and e between them and $e \in$ (the edges of G) $\setminus$ (the edges of $p(n)$) and ($v_1 \in$ the vertices of $p(n)$ and $v_2 \notin$ the vertices of $p(n)$ and if $p(n)$ is not trivial, then $v_1 \in p(n).\mathrm{endVertices}()$ or $v_1 \notin$ the vertices of $p(n)$ and $v_2 \in$ the vertices of $p(n)$ and if $p(n)$ is not trivial, then $v_2 \in p(n).\mathrm{endVertices}()$).

Proof: Define $\mathcal{P}$[natural number] $\equiv$ for every finite, path-like graph G for every connected subgraph H of G such that $\$_1 = G.\mathrm{order}() - H.\mathrm{order}()$ there exists a non empty, finite, path-like, graph-yielding finite sequence p such that $p(1) \approx H$ and $p(\operatorname{len} p) = G$ and $\operatorname{len} p = G.\mathrm{order}() - H.\mathrm{order}() + 1$ and for every element n of $\operatorname{dom} p$ such that $n \leqslant \operatorname{len} p - 1$ there exist vertices v_1, v_2 of G and there exists an object e such that $p(n+1)$ is a supergraph of $p(n)$ extended by v_1, v_2 and e between them and $e \in$ (the edges of G)$\setminus$(the edges of $p(n)$) and ($v_1 \in$ the vertices of $p(n)$ and $v_2 \notin$ the vertices of $p(n)$ and if $p(n)$ is not trivial, then $v_1 \in p(n).\mathrm{endVertices}()$ or $v_1 \notin$

the vertices of $p(n)$ and $v_2 \in$ the vertices of $p(n)$ and if $p(n)$ is not trivial, then $v_2 \in p(n)$.endVertices()). $\mathcal{P}[0]$ by [12, (117)], [8, (21)]. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [12, (117), (26)], [8, (31)], [12, (48), (47), (107)]. For every natural number k, $\mathcal{P}[k]$. □

(15) Let us consider a finite, path-like graph G. Then there exists a non empty, finite, path-like, graph-yielding finite sequence p such that

(i) $p(1)$ is trivial and edgeless, and

(ii) $p(\operatorname{len} p) = G$, and

(iii) $\operatorname{len} p = G$.order(), and

(iv) for every element n of $\operatorname{dom} p$ such that $n \leqslant \operatorname{len} p - 1$ there exist vertices v_1, v_2 of G and there exists an object e such that $p(n+1)$ is a supergraph of $p(n)$ extended by v_1, v_2 and e between them and $e \in$ (the edges of G) $\setminus$ (the edges of $p(n)$) and ($v_1 \in$ the vertices of $p(n)$ and if $n \geqslant 2$, then $v_1 \in p(n)$.endVertices() and $v_2 \notin$ the vertices of $p(n)$ or $v_1 \notin$ the vertices of $p(n)$ and $v_2 \in$ the vertices of $p(n)$ and if $n \geqslant 2$, then $v_2 \in p(n)$.endVertices()).

PROOF: Set $H =$ the trivial subgraph of G. Consider p being a non empty, finite, path-like, graph-yielding finite sequence such that $p(1) \approx H$ and $p(\operatorname{len} p) = G$ and $\operatorname{len} p = G$.order()$-H$.order()$+1$ and for every element n of $\operatorname{dom} p$ such that $n \leqslant \operatorname{len} p - 1$ there exist vertices v_1, v_2 of G and there exists an object e such that $p(n+1)$ is a supergraph of $p(n)$ extended by v_1, v_2 and e between them and $e \in$ (the edges of G) $\setminus$ (the edges of $p(n)$) and ($v_1 \in$ the vertices of $p(n)$ and $v_2 \notin$ the vertices of $p(n)$ and if $p(n)$ is not trivial, then $v_1 \in p(n)$.endVertices() or $v_1 \notin$ the vertices of $p(n)$ and $v_2 \in$ the vertices of $p(n)$ and if $p(n)$ is not trivial, then $v_2 \in p(n)$.endVertices()). Consider v_1, v_2 being vertices of G, e being an object such that $p(n+1)$ is a supergraph of $p(n)$ extended by v_1, v_2 and e between them and $e \in$ (the edges of G) $\setminus$ (the edges of $p(n)$) and ($v_1 \in$ the vertices of $p(n)$ and $v_2 \notin$ the vertices of $p(n)$ and if $p(n)$ is not trivial, then $v_1 \in p(n)$.endVertices() or $v_1 \notin$ the vertices of $p(n)$ and $v_2 \in$ the vertices of $p(n)$ and if $p(n)$ is not trivial, then $v_2 \in p(n)$.endVertices()). If $n \geqslant 2$, then $p(n)$ is not trivial. □

(16) Let us consider a non empty, graph-yielding finite sequence p. Suppose $p(1)$ is path-like and for every element n of $\operatorname{dom} p$ such that $n \leqslant \operatorname{len} p - 1$ there exist objects v_1, e, v_2 such that $p(n+1)$ is a supergraph of $p(n)$ extended by v_1, v_2 and e between them and ($p(n)$ is trivial or $v_1 \in p(n)$.endVertices() or $v_2 \in p(n)$.endVertices()). Then $p(\operatorname{len} p)$ is path-like.
PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ if $\$_1 \leqslant \operatorname{len} p - 1$, then $p(\$_1 + 1)$ is a path-like graph. For every natural number n such that $\mathcal{P}[n]$ holds

$\mathcal{P}[n+1]$. For every natural number n, $\mathcal{P}[n]$. □

(17) Let us consider a non trivial, finite, path-like graph G. Then there exists a non empty, finite, path-like, graph-yielding finite sequence p such that

(i) $p(1)$ is 2-vertex and path-like, and

(ii) $p(\operatorname{len} p) = G$, and

(iii) $\operatorname{len} p + 1 = G.\mathrm{order}()$, and

(iv) for every element n of $\operatorname{dom} p$ such that $n \leqslant \operatorname{len} p - 1$ there exist vertices v_1, v_2 of G and there exists an object e such that $p(n+1)$ is a supergraph of $p(n)$ extended by v_1, v_2 and e between them and $e \in (\text{the edges of } G) \setminus (\text{the edges of } p(n))$ and ($v_1 \in p(n).\mathrm{endVertices}()$ and $v_2 \notin$ the vertices of $p(n)$ or $v_1 \notin$ the vertices of $p(n)$ and $v_2 \in p(n).\mathrm{endVertices}()$).

The theorem is a consequence of (15), (10), and (11).

(18) Let us consider a non empty, graph-yielding finite sequence p. Suppose $p(1)$ is non trivial and path-like and for every element n of $\operatorname{dom} p$ such that $n \leqslant \operatorname{len} p - 1$ there exist objects v_1, e, v_2 such that $p(n+1)$ is a supergraph of $p(n)$ extended by v_1, v_2 and e between them and ($v_1 \in p(n).\mathrm{endVertices}()$ or $v_2 \in p(n).\mathrm{endVertices}()$). Then $p(\operatorname{len} p)$ is path-like.

Proof: Define $\mathcal{P}[\text{natural number}] \equiv$ if $\$_1 \leqslant \operatorname{len} p - 1$, then $p(\$_1 + 1)$ is a path-like graph. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every natural number n, $\mathcal{P}[n]$. □

(19) Let us consider graphs G_1, G_2, and a partial graph mapping F from G_1 to G_2. If F is isomorphism, then G_1 is path-like iff G_2 is path-like.

(20) Let us consider graphs G_1, G_2. If $G_1 \approx G_2$, then if G_1 is path-like, then G_2 is path-like.

(21) Let us consider a graph G_1, a set E, and a graph G_2 given by reversing directions of the edges E of G_1. Then G_1 is path-like if and only if G_2 is path-like. The theorem is a consequence of (19).

Let P_2 be a 2-vertex, path-like graph. One can verify that every vertex of P_2 is endvertex. Now we state the propositions:

(22) Let us consider a finite, non trivial, path-like graph P. Then $\delta(P) = 1$.

Proof: Consider p being a non empty, finite, path-like, graph-yielding finite sequence such that $p(1)$ is 2-vertex and path-like and $p(\operatorname{len} p) = P$ and $\operatorname{len} p + 1 = P.\mathrm{order}()$ and for every element n of $\operatorname{dom} p$ such that $n \leqslant \operatorname{len} p - 1$ there exist vertices v_1, v_2 of P and there exists an object e such that $p(n+1)$ is a supergraph of $p(n)$ extended by v_1, v_2 and e between them and $e \in (\text{the edges of } P) \setminus (\text{the edges of } p(n))$ and ($v_1 \in p(n).\mathrm{endVertices}()$ and $v_2 \notin$ the vertices of $p(n)$ or $v_1 \notin$ the vertices of

$p(n)$ and $v_2 \in p(n)$.endVertices()). Define $\mathcal{P}$[natural number] $\equiv$ for every graph H such that $H = p(\$_1 + 1)$ and $\$_1 \leqslant \operatorname{len} p - 1$ holds $\delta(H) = 1$. $\mathcal{P}[0]$ by [12, (174)], [9, (36)]. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [7, (141)], [12, (174)], [9, (35)]. For every natural number k, $\mathcal{P}[k]$. □

(23) Let us consider a finite, path-like graph P. Then there exists a vertex-distinct path P_0 of P such that

(i) P_0.vertices() = the vertices of P, and

(ii) P_0.edges() = the edges of P, and

(iii) P.endVertices() = $\{P_0.\text{first}(), P_0.\text{last}()\}$ iff P is not trivial, and

(iv) P_0 is trivial iff P is trivial, and

(v) P_0 is closed iff P is trivial, and

(vi) P_0 is minimum length.

Proof: Define $\mathcal{P}$[natural number] $\equiv$ for every finite, path-like graph P such that P.order() $= \$_1 + 1$ there exists a vertex-distinct path P_0 of P such that P_0.vertices() = the vertices of P and P_0.edges() = the edges of P and (P.endVertices() = $\{P_0.\text{first}(), P_0.\text{last}()\}$ iff P is not trivial) and (P_0 is closed iff P is trivial) and P_0 is minimum length. $\mathcal{P}[0]$ by [12, (26), (22)], [11, (90)]. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$ by [12, (26)], (22), [12, (174)], (13). For every natural number n, $\mathcal{P}[n]$. Consider n being a natural number such that P.order() $= n + 1$. Consider P_0 being a vertex-distinct path of P such that P_0.vertices() = the vertices of P and P_0.edges() = the edges of P and (P.endVertices() = $\{P_0.\text{first}(), P_0.\text{last}()\}$ iff P is not trivial) and (P_0 is closed iff P is trivial) and P_0 is minimum length. □

(24) Let us consider a non zero natural number n, and n-vertex, path-like graphs P_1, P_2. Then P_2 is P_1-isomorphic. The theorem is a consequence of (23).

(25) Let us consider a natural number n, and n-edge, path-like graphs P_1, P_2. Then P_2 is P_1-isomorphic. The theorem is a consequence of (24).

(26) Let us consider a non trivial, path-like graph P. Then

(i) P.order() $= 2$ iff $\Delta(P) = 1$, and

(ii) P.order() $\neq 2$ iff $\Delta(P) = 2$.

(27) Let us consider a non trivial, path-like graph P, and a vertex v of P. If v is not endvertex, then v.degree() $= 2$.

Let us consider a finite, non trivial, path-like graph P. Now we state the propositions:

(28) There exist vertices v_1, v_2 of P such that

(i) $v_1 \neq v_2$, and

(ii) P.endVertices() $= \{v_1, v_2\}$.

The theorem is a consequence of (23).

(29) $\overline{\overline{P.\text{endVertices}()}} = 2$. The theorem is a consequence of (28).

(30) Let us consider a finite, non trivial graph G. Suppose G is acyclic and $\delta(G) = 1$ and $\overline{\overline{G.\text{endVertices}()}} = 2$. Then G is path-like.

Proof: Set $F =$ the subgraph of G with vertex v removed. $3 \subseteq F$.numComponents(). Consider c_1, c_2, c_3 being objects such that $c_1, c_2 \in F$.componentSet() and $c_3 \in F$.componentSet() and $c_1 \neq c_2$ and $c_1 \neq c_3$ and $c_2 \neq c_3$. Consider v_1 being a vertex of F such that $c_1 = F$.reachableFrom(v_1). Consider v_2 being a vertex of F such that $c_2 = F$.reachableFrom(v_2). Consider v_3 being a vertex of F such that $c_3 = F$.reachableFrom(v_3). Set $C_1 =$ the subgraph of F induced by F.reachableFrom(v_1). Set $C_2 =$ the subgraph of F induced by F.reachableFrom(v_2). Set $C_3 =$ the subgraph of F induced by F.reachableFrom(v_3). Consider w_1 being a vertex of G such that w_1 is endvertex and $w_1 \in$ the vertices of C_1. Consider w_2 being a vertex of G such that w_2 is endvertex and $w_2 \in$ the vertices of C_2. Consider w_3 being a vertex of G such that w_3 is endvertex and $w_3 \in$ the vertices of C_3. $w_1 \neq w_2$. $w_2 \neq w_3$. $w_3 \neq w_1$. $\square$

One can verify that every graph which is 2-vertex, simple, and connected is also path-like and every graph which is 2-vertex and path-like is also complete.

Let n be a natural number. Let us observe that every graph which is $(n+3)$-vertex and path-like is also non complete.

3. Cycle Graphs

Let G be a graph. We say that G is cycle-like if and only if

(Def. 6) G is connected, non acyclic, and 2-regular.

One can verify that there exists a graph which is non trivial and cycle-like and every graph which is connected, non acyclic, and 2-regular is also cycle-like and every graph which is cycle-like is also connected, non acyclic, and 2-regular.

Now we state the proposition:

(31) Let us consider a cycle-like graph G, and a circuit-like walk C of G. Then

(i) C.vertices() $=$ the vertices of G, and

(ii) C.edges() $=$ the edges of G.

Note that every graph which is cycle-like is also non edgeless, finite, and with max degree. Now we state the proposition:

(32) Let us consider a cycle-like graph G. Then G.order() $=$ G.size().

One can check that every graph which is trivial, non-directed-multi, and loopfull is also cycle-like and every graph which is trivial and cycle-like is also non-multi and loopfull and every graph which is non trivial and cycle-like is also loopless and there exists a graph which is trivial and cycle-like.

Let F be a graph-yielding function. We say that F is cycle-like if and only if

(Def. 7) for every object x such that $x \in \operatorname{dom} F$ there exists a graph G such that $F(x) = G$ and G is cycle-like.

Let C be a cycle-like graph. Observe that $\langle C \rangle$ is cycle-like and $\mathbb{N} \longmapsto C$ is cycle-like.

Let F be a non empty, graph-yielding function. Let us note that F is cycle-like if and only if the condition (Def. 8) is satisfied.

(Def. 8) for every element x of $\operatorname{dom} F$, $F(x)$ is cycle-like.

Let S be a graph sequence. Observe that S is cycle-like if and only if the condition (Def. 9) is satisfied.

(Def. 9) for every natural number n, $S(n)$ is cycle-like.

One can verify that every graph-yielding function which is empty is also cycle-like and every graph-yielding function which is trivial, non-directed-multi, and loopfull is also cycle-like and every graph-yielding function which is cycle-like is also connected and there exists a graph sequence which is non empty and cycle-like.

Let F be a cycle-like, non empty, graph-yielding function and x be an element of $\operatorname{dom} F$. Let us observe that $F(x)$ is cycle-like.

Let S be a cycle-like graph sequence and n be a natural number. Let us observe that $S(n)$ is cycle-like.

Let p be a cycle-like, graph-yielding finite sequence. One can verify that $p{\restriction}n$ is cycle-like and $p_{\lfloor n}$ is cycle-like.

Let m be a natural number. Let us note that $\operatorname{smid}(p, m, n)$ is cycle-like and $\langle p(m), \ldots, p(n) \rangle$ is cycle-like.

Let p, q be cycle-like, graph-yielding finite sequences. One can verify that $p \mathbin{^\frown} q$ is cycle-like and $p \mathbin{^\frown\!\!^\frown} q$ is cycle-like.

Let C_1, C_2 be cycle-like graphs. Observe that $\langle C_1, C_2 \rangle$ is cycle-like.

Let C_3 be a cycle-like graph. Let us observe that $\langle C_1, C_2, C_3 \rangle$ is cycle-like.

Let S be a graph-membered set. We say that S is cycle-like if and only if

(Def. 10) for every graph G such that $G \in S$ holds G is cycle-like.

Note that every graph-membered set which is empty is also cycle-like and every graph-membered set which is cycle-like is also connected.

Let C_1 be a cycle-like graph. One can check that $\{C_1\}$ is cycle-like.

Let C_2 be a cycle-like graph. Let us note that $\{C_1, C_2\}$ is cycle-like.

Let F be a cycle-like, graph-yielding function. Observe that $\operatorname{rng} F$ is cycle-like.

Let X be a cycle-like, graph-membered set. One can verify that every subset of X is cycle-like.

Let Y be a set. Note that $X \cap Y$ is cycle-like and $X \setminus Y$ is cycle-like.

Let X, Y be cycle-like, graph-membered sets. Observe that $X \cup Y$ is cycle-like and $X \dot{-} Y$ is cycle-like and there exists a graph-membered set which is non empty and cycle-like.

Let S be a non empty, cycle-like, graph-membered set. Let us note that every element of S is cycle-like.

4. Properties of Cycle Graphs

Now we state the propositions:

(33) Let us consider a trivial, edgeless graph G_2, a vertex v of G_2, and an object e. Then every supergraph of G_2 extended by e between vertices v and v is cycle-like.

(34) Let us consider a finite, non trivial, path-like graph P, elements v_1, v_2 of P.endVertices(), an object e, and a supergraph C of P extended by e between vertices v_1 and v_2. Suppose $v_1 \neq v_2$ and $e \notin$ the edges of P. Then C is cycle-like. The theorem is a consequence of (29), (27), and (23).

(35) Let us consider a cycle-like graph C, and an edge e of C. Then every subgraph of C with edge e removed is finite and path-like. The theorem is a consequence of (31).

Let C be a cycle-like graph and e be an edge of C. One can verify that every subgraph of C with edge e removed is finite and path-like. Now we state the propositions:

(36) Let us consider a trivial, cycle-like graph G_1, a vertex v of G_1, and an edge e of G_1. Then there exists a trivial, edgeless graph G_2 such that G_1 is a supergraph of G_2 extended by e between vertices v and v.

(37) Let us consider a non trivial, cycle-like graph C, vertices v_1, v_2 of C, and an edge e of C. Suppose e joins v_1 to v_2 in C. Then there exists a non trivial, finite, path-like graph P such that

    (i) $e \notin$ the edges of P, and

(ii) C is a supergraph of P extended by e between vertices v_1 and v_2, and

(iii) P.endVertices() $= \{v_1, v_2\}$.

The theorem is a consequence of (28).

(38) Let us consider a cycle-like graph C. Then C.order() $= 2$ if and only if C is not non-multi.

PROOF: Consider e_1, e_2, v_1, v_2 being objects such that e_1 joins v_1 and v_2 in C and e_2 joins v_1 and v_2 in C and $e_1 \neq e_2$. Set $W_1 = C$.walkOf(v_1, e_1, v_2). Set $W_2 = W_1$.addEdge(e_2). $v_1 \neq v_2$. The vertices of $C = W_2$.vertices(). □

Let n be a natural number. Let us note that every graph which is n-vertex and cycle-like is also n-edge and every graph which is n-edge and cycle-like is also n-vertex and there exists a graph which is $(n+1)$-vertex, $(n+1)$-edge, and cycle-like and every graph which is $(n+2)$-vertex and cycle-like is also loopless and every graph which is $(n+3)$-vertex and cycle-like is also simple and there exists a graph which is $(n+2)$-vertex, $(n+2)$-edge, loopless, and cycle-like and there exists a graph which is $(n+3)$-vertex, $(n+3)$-edge, simple, and cycle-like.

Let n be a non zero natural number. Observe that there exists a graph which is n-vertex, n-edge, and cycle-like and every graph which is $(n+1)$-vertex and cycle-like is also loopless and every graph which is $(n+2)$-vertex and cycle-like is also simple and there exists a graph which is $(n+1)$-vertex, $(n+1)$-edge, cycle-like, and loopless and there exists a graph which is $(n+2)$-vertex, $(n+2)$-edge, cycle-like, and simple. Now we state the propositions:

(39) Let us consider a cycle-like graph C_1, and a non acyclic subgraph C_2 of C_1. Then $C_1 \approx C_2$. The theorem is a consequence of (31).

(40) Let us consider graphs G_1, G_2, and a partial graph mapping F from G_1 to G_2. Suppose F is isomorphism. Then G_1 is cycle-like if and only if G_2 is cycle-like.

(41) Let us consider graphs G_1, G_2. Suppose $G_1 \approx G_2$. If G_1 is cycle-like, then G_2 is cycle-like. The theorem is a consequence of (40).

(42) Let us consider a graph G_1, a set E, and a graph G_2 given by reversing directions of the edges E of G_1. Then G_1 is cycle-like if and only if G_2 is cycle-like. The theorem is a consequence of (40).

(43) Let us consider a non zero natural number n, and n-vertex, cycle-like graphs C_1, C_2. Then C_2 is C_1-isomorphic. The theorem is a consequence of (37), (24), and (29).

(44) Let us consider a non zero natural number n, and n-edge, cycle-like graphs C_1, C_2. Then C_2 is C_1-isomorphic.

(45) Let us consider a finite, non trivial, path-like graph P, an object v,

and a supergraph C of P extended by vertex v and edges between v and P.endVertices() of P. Suppose $v \notin$ the vertices of P. Then C is simple and cycle-like.

Proof: $\overline{\overline{P.\text{endVertices}()}} \neq 0$. Consider w_1, w_2 being vertices of P such that $w_1 \neq w_2$ and P.endVertices() $= \{w_1, w_2\}$. There exists a component G_3 of P and there exist vertices w_1, w_2 of G_3 such that w_1, $w_2 \in P$.endVertices() and $w_1 \neq w_2$. □

(46) Let us consider a non trivial, cycle-like graph C, and a vertex v of C. Then every subgraph of C with vertex v removed is finite and path-like. The theorem is a consequence of (31).

(47) Let us consider a simple, cycle-like graph C, and a vertex v of C. Then there exists a non trivial, path-like graph P such that

(i) $v \notin$ the vertices of P, and

(ii) C is a supergraph of P extended by vertex v and edges between v and P.endVertices() of P.

Proof: Set $P =$ the subgraph of C with vertex v removed. P is path-like. P is not trivial. □

Let us observe that every graph which is 3-vertex, simple, and complete is also cycle-like and every graph which is 3-vertex and cycle-like is also simple, complete, and chordal.

Let n be a natural number. Observe that every graph which is $(n+4)$-vertex and cycle-like is also non chordal and non complete.

Let n be a non zero natural number. One can verify that every graph which is $(n+3)$-vertex and cycle-like is also non chordal and non complete and there exists a graph which is cycle-like, non complete, and non chordal.

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