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Formal Proof of Transcendence of the Number e. Part II

Yasushige Watase
Suginami-ku Matsunoki 6, 3-21 Tokyo
Japan

Summary. In this article we formalize the main part of Hurwitz's proof of the transcendence of the number e in the Mizar language. The previous article prepared the necessary definitions and lemmas. Here we deal with main crucial steps of the proof.

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Introduction

In this article, which is a continuation of [19], we formalize the main part of Hurwitz's proof [8] that the number e is transcendental [12] using the Mizar formalism [2], [3]. For related proof developments of the transcendence of e (which is one of the items (#67) in Freek Wiedijk's "Top 100 Mathematical Theorems" list [21]) in Coq or HOL Light, see [4] and [5], respectively (although the formalization is available also in the number of proof assistants, such as Isabelle [6], Lean [15], or Metamath [13]). This is also a small step towards developing transcendental number theory [1], started in Mizar with Liouville numbers [11] as well as the theory of formal polynomials and their derivation [18]. The following is a summary of the formalized proof (see also [7]).

The core idea of Hurwitz's proof can be expressed as a lemma about algebraic numbers, namely if we assume e is algebraic, a polynomial over $\mathbb{Z}$ admits e as a root (see `E_TRANS2:41`). It corresponds to the equation (3) of [8]. This theorem,

which is a base for a proof by contradiction, is technically preserved at the end of the Mizar article. However it is actually redundant, after we complete the proof of the main result.

In the first section we define a polynomial f_0 over $\mathbb{Z}$ and observe properties of f_0. It is defined by $f_0(x) = x^{p-1}(x-1)^p(x-2)^p \cdots (x-m)^p$, where p is an odd prime number and $m+1$ is the number of component of the products. The f_0 is defined as `E_TRANS2:def 5`. The components $(x-j)_{j=0,1,\ldots,m}$ are represented by $\tau(j)$ in the article and obtain:

$$f_0 = \tau(0)^{p-1} \prod_{j=1}^{m} \tau(j)^p$$

The third section is about properties of f_0 and $\mathcal{F}(f_0)$ where $\mathcal{F}$ is introduced in [19], the transformation $\mathcal{F}(f) = f + f' + f'' + \cdots + f^{(deg\ f)}$.

We deal with k^{th} differentiation of the f_0 and evaluate by a number j. The following number-theoretical properties are formally proven:

1. $\prod_{j=1}^{m} \tau(j)^p(0) = (((-1)^m) \cdot (m!))^p$ (`E_TRANS2:17`)
2. $f_0^{(k)}(0) = 0$, if $0 \leqslant k \leqslant p-2$ (`E_TRANS2:18`)
3. $f_0^{(k)}(0) = k!(\prod_{j=1}^{m} \tau(j))(k-p+1)$, if $p \leqslant k$ (`E_TRANS2:21`)
4. $f_0^{(k)}(j) = 0$, if $k \leqslant p, 1 \leqslant j \leqslant m$ (`E_TRANS2:23`)
5. $f_0^{(k)} = \tau(j)u + p!v$ for some $u, v \in \mathbb{Z}[X]$, if $p \leqslant k, 1 \leqslant j \leqslant m$ (`E_TRANS2:27`)
6. $f_0^{(k)}(j) \in (p!)$, if $p \leqslant k, 1 \leqslant j \leqslant m$ (`E_TRANS2:29`)

We denote $\mathbf{F}$ for $\mathcal{F}(f_0)$ for simplicity.

7. $\mathbf{F}(0) = (p-1)!(((-1)|^m) * (m!))^p + p!u$ for some $u \in \mathbb{Z}[X]$ (`E_TRANS2:30`)
8. $\mathbf{F}(j) \in (p!)$, if $1 \leqslant j \leqslant m$ (`E_TRANS2:31`)

We then obtain an equation system shown as below, where C_i stands for coefficient of the i^{th} coefficient of g_0. This is based on the equation system numbered (4) in Hurwitz's proof [8].

$$\begin{cases} \frac{1}{(p-1)!}C_0\mathbf{F}(0) & - & \frac{1}{(p-1)!}C_0e^0\mathbf{F}(0) & = \frac{1}{(p-1)!}C_0\varepsilon_0 \\ \frac{1}{(p-1)!}C_1\mathbf{F}(1) & - & \frac{1}{(p-1)!}C_1e^1\mathbf{F}(0) & = \frac{1}{(p-1)!}C_1\varepsilon_1 \\ \quad\vdots & & \quad\vdots & \quad\vdots \\ \frac{1}{(p-1)!}C_m\mathbf{F}(m) & - & \frac{1}{(p-1)!}C_me^m\mathbf{F}(0) & = \frac{1}{(p-1)!}C_m\varepsilon_m \end{cases}$$

where each equation is a product of i^{th} coefficient of g_0 and $\mathbf{F}(i) - e^x\mathbf{F}(i)(= -ie^{(i-\vartheta)i}f_0(\vartheta i))$ which is from the result of the mean value theorem to $e^x\mathbf{F}(x)$. In actual coding the sequence $C_m\mathbf{F}(m)$ and $(p-1)!C_me^m\mathbf{F}(0)$ are defined as δ_1, δ_2 (`delta_1` and `delta_2` in the Mizar source), respectively.

We have new equation by adding each term of the equation system vertically:

$$\frac{1}{(p-1)!}\sum_{i=1}^{m} C_i\mathbf{F}(i) \;-\; \frac{1}{(p-1)!}\sum_{i=1}^{m} C_ie^i\mathbf{F}(0) = \frac{1}{(p-1)!}\sum_{i=1}^{m} C_i\varepsilon_i$$

One can verify formally that the left hand side is not divisible by p, because p divides the first term $\frac{1}{(p-1)!}\Sigma C_i\mathbf{F}(i)$, but $p \nmid \frac{1}{(p-1)!}\Sigma e^iC_i\mathbf{F}(0)$. The right-hand side is a member of $\mathbb{Z}$ and bounded by $1/2$ by choosing sufficiently large p, this means it is 0, which contradicts the left-hand side nature. Therefore e is transcendental number.

1. Preliminaries

From now on R denotes an integral domain, p denotes an odd, prime natural number, and m denotes a positive natural number. Now we state the propositions:

(1) Let us consider a natural number i, and an element r of $\mathbb{R}_{\mathrm{F}}$. Then $\sum(i \mapsto r) = i \cdot r$.

PROOF: Define $\mathcal{P}$[natural number] $\equiv \sum(\$_1 \mapsto r) = \$_1 \cdot r$. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$. For every natural number i, $\mathcal{P}[i]$. □

(2) Let us consider sequences p_1, q_1 of $\mathbb{Z}^{\mathrm{R}}$. Then $(p_1 * q_1)(0) = p_1(0) \cdot q_1(0)$.

2. On the Ring of Polynomials

Now we state the propositions:

(3) Let us consider an element f of the carrier of Polynom-Ring $\mathbb{Z}^{\mathrm{R}}$, and a natural number n. Then ${}^{@}f^n = ({}^{@}f)^n$.

PROOF: Define $\mathcal{P}$[natural number] $\equiv {}^{@}f^{\$_1} = ({}^{@}f)^{\$_1}$. $\mathcal{P}[0]$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$. □

(4) Let us consider an element f of the carrier of Polynom-Ring R, and a natural number n. Then $\mathrm{R2P}(f^n) = (\mathrm{R2P}(f))^n$.

PROOF: Define $\mathcal{P}$[natural number] $\equiv \mathrm{R2P}(f^{\$_1}) = (\mathrm{R2P}(f))^{\$_1}$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$. □

(5) Let us consider a natural number n, and an element f of the carrier of Polynom-Ring $\mathbb{Z}^{\mathrm{R}}$. Then $n \cdot f = n(\in \mathbb{Z}^{\mathrm{R}}) \cdot f$.
PROOF: Define $\mathcal{P}$[natural number] $\equiv$ $\$_1 \cdot f = \$_1(\in \mathbb{Z}^{\mathrm{R}}) \cdot f$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$. □

(6) Let us consider an element M of $\mathbb{R}_{\mathrm{F}}$, and a finite sequence F of elements of $\mathbb{R}_{\mathrm{F}}$. Suppose for every natural number i such that $i \in \operatorname{dom} F$ holds $|F(i)| \leqslant M$. Then $|\prod F| \leqslant M^{\operatorname{len} F}$.
PROOF: Define $\mathcal{P}$[natural number] $\equiv$ for every finite sequence F of elements of $\mathbb{R}_{\mathrm{F}}$ such that $\operatorname{len} F = \$_1$ and for every natural number i such that $i \in \operatorname{dom} F$ holds $|F(i)| \leqslant M$ holds $|\prod F| \leqslant M^{\operatorname{len} F}$. $\mathcal{P}[0]$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every natural number n, $\mathcal{P}[n]$. □

Let p be a polynomial over $\mathbb{Z}^{\mathrm{R}}$. Observe that the functor $|p|$ yields a sequence of $\mathbb{Z}^{\mathrm{R}}$ and is defined by

(Def. 1) for every natural number n, $it(n) = |p(n)|$.

Note that $|p|$ is finite-Support as a (the carrier of $\mathbb{Z}^{\mathrm{R}}$)-valued function. In the sequel g denotes a non zero polynomial over $\mathbb{Z}^{\mathrm{R}}$. Let us consider g. One can verify that $\operatorname{rng}|g|$ is finite. Now we state the proposition:

(7) Let us consider a non zero polynomial g over $\mathbb{Z}^{\mathrm{R}}$. Then there exists a natural number M such that for every natural number i, $|g(i)| \leqslant M$.
PROOF: $\operatorname{rng}|g| \subseteq \mathbb{N}$. Reconsider $S = \operatorname{rng}|g|$ as a finite, non empty, natural-membered set. Reconsider $M = \max S$ as a natural number. For every natural number i, $|g(i)| \leqslant M$. □

3. The Polynomial f_0 and Its Properties

Let i be a natural number. The functor $\tau(i)$ yielding an element of the carrier of Polynom-Ring $\mathbb{Z}^{\mathrm{R}}$ is defined by the term

(Def. 2) $\langle (-i)(\in \mathbb{Z}^{\mathrm{R}}), 1_{\mathbb{Z}^{\mathrm{R}}} \rangle$.

Let p be a non zero natural number and m be a natural number. The functor x.(m,p) yielding a finite sequence of elements of the carrier of Polynom-Ring $\mathbb{Z}^{\mathrm{R}}$ is defined by

(Def. 3) $\operatorname{len} it = m$ and for every natural number i such that $i \in \operatorname{dom} it$ holds $it(i) = (\tau(i))^p$.

Let p be an odd, prime natural number and m be a positive natural number. The functor $f_0^{\mathrm{seq}}(m,p)$ yielding a finite sequence of elements of the carrier of Polynom-Ring $\mathbb{Z}^{\mathrm{R}}$ is defined by the term

(Def. 4) $\mathrm{x}.(m,p) \frown \langle (\tau(0))^{p-'1} \rangle$.

The functor $f_0(m,p)$ yielding an element of the carrier of Polynom-Ring $\mathbb{Z}^{\mathrm{R}}$ is defined by the term

(Def. 5) $\prod f_0^{\mathrm{seq}}(m,p)$.

Now we state the propositions:

(8) Let us consider natural numbers i, n. Then $\operatorname{len} \mathrm{R2P}((\tau(i))^n) = n+1$.

(9) Let us consider elements f, g of the carrier of Polynom-Ring $\mathbb{Z}^{\mathrm{R}}$. Suppose $(\operatorname{len} \mathrm{R2P}(f)) \cdot (\operatorname{len} \mathrm{R2P}(g)) \neq 0$. Then $\operatorname{len} \mathrm{R2P}(f \cdot g) = \operatorname{len} \mathrm{R2P}(f) + \operatorname{len} \mathrm{R2P}(g) - 1$.

(10) Let us consider a non zero natural number k, and an odd, prime natural number p. Then

(i) $\mathrm{x}.(k,p) \frown \langle (\tau(k+1))^p \rangle = \mathrm{x}.(k+1,p)$, and

(ii) $\prod \mathrm{x}.(k+1,p) = (\prod \mathrm{x}.(k,p)) \cdot (\tau(k+1))^p$.

PROOF: $\mathrm{x}.(k,p) \frown \langle (\tau(k+1))^p \rangle = \mathrm{x}.(k+1,p)$. □

Let us consider an odd, prime natural number p and a positive natural number m. Now we state the propositions:

(11) $\operatorname{len} \mathrm{R2P}(\prod \mathrm{x}.(m,p)) = m \cdot p + 1$.

PROOF: Define $\mathcal{P}[\text{non zero natural number}] \equiv \operatorname{len} \mathrm{R2P}(\prod \mathrm{x}.(\$_1,p)) = \$_1 \cdot p + 1$. $\mathcal{P}[1]$. For every non zero natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every non zero natural number k, $\mathcal{P}[k]$. □

(12) $\operatorname{len} \mathrm{R2P}(f_0(m,p)) = m \cdot p + p$. The theorem is a consequence of (11), (8), and (9).

(13) Let us consider a natural number i.
Then $(\mathrm{Der1}(\mathbb{Z}^{\mathrm{R}}))(\tau(i)) = 1_{\text{Polynom-Ring } \mathbb{Z}^{\mathrm{R}}}$.

(14) Let us consider an element f of the carrier of Polynom-Ring $\mathbb{Z}^{\mathrm{R}}$, and a natural number i. Then

(i) $(\tau(0) * f)(i+1) = f(i)$, and

(ii) $(\tau(0) * f)(0) = 0_{\mathbb{Z}^{\mathrm{R}}}$.

PROOF: For every natural number i, $(\tau(0) * f)(i+1) = f(i)$ and $(\tau(0) * f)(0) = 0_{\mathbb{Z}^{\mathrm{R}}}$ by [14, (16)], [17, (12)], [20, (31)]. □

From now on f denotes an element of the carrier of Polynom-Ring $\mathbb{Z}^{\mathrm{R}}$. Now we state the propositions:

(15) Let us consider an odd, prime natural number p, and a positive natural number m. Then

(i) $\operatorname{len} \mathrm{x}.(m,p) = m$, and

(ii) $\operatorname{len} f_0^{\mathrm{seq}}(m,p) = m+1$, and

(iii) $(f_0^{\mathrm{seq}}(m,p))(\operatorname{len}\mathrm{x}.(m,p)+1) = (\tau(0))^{p-'1}$.

(16) Let us consider an odd, prime natural number p, a positive natural number m, and a natural number k. Suppose $0 \leqslant k \leqslant p-1$. Let us consider natural numbers i, j. Suppose $i \in \operatorname{Seg}(k+1)$. Then $\tau(j) \mid (\mathrm{LBZ}(\mathrm{Der1}(\mathbb{Z}^{\mathrm{R}}), k, \prod(f_0^{\mathrm{seq}}(m,p))_{\restriction j}, (\tau(j))^p))_{/i}$.
Proof: Set $D = \mathrm{Der1}(\mathbb{Z}^{\mathrm{R}})$. For every natural numbers i, j such that $i \in \operatorname{Seg}(k+1)$ holds $\tau(j) \mid (\mathrm{LBZ}(D, k, \prod(f_0^{\mathrm{seq}}(m,p))_{\restriction j}, (\tau(j))^p))_{/i}$ by (13), [16, (19)]. □

4. Some Number-theoretical Properties

Now we state the proposition:

(17) Let us consider an odd, prime natural number p, and a positive natural number m. Then $(\mathrm{R2P}(\prod \mathrm{x}.(m,p)))(0) = ((-1)^m \cdot (m!))^p$.
Proof: Define $\mathcal{P}[\text{natural number}] \equiv (\mathrm{R2P}(\prod \mathrm{x}.(\$_1,p)))(0) = ((-1)^{\$_1} \cdot (\$_1!))^p$. $\mathcal{P}[1]$. For every non zero natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every non zero natural number k, $\mathcal{P}[k]$. □

Let us consider an odd, prime natural number p, a positive natural number m, and a natural number k. Now we state the propositions:

(18) If $0 \leqslant k \leqslant p -' 2$, then $(\mathrm{Der1}(\mathbb{Z}^{\mathrm{R}}))^k(f_0(m,p))(0) = 0_{\mathbb{Z}^{\mathrm{R}}}$.

(19) Suppose $0 \leqslant k \leqslant p -' 2$. Then $\mathrm{eval}(\mathrm{R2P}((\mathrm{Der1}(\mathbb{Z}^{\mathrm{R}}))^k(f_0(m,p))), 0_{\mathbb{Z}^{\mathrm{R}}}) = 0_{\mathbb{Z}^{\mathrm{R}}}$. The theorem is a consequence of (18).

(20) Let us consider an odd, prime natural number p, and a positive natural number m. Then $\mathrm{eval}(\mathrm{R2P}((\mathrm{Der1}(\mathbb{Z}^{\mathrm{R}}))^{p-'1}(f_0(m,p))), 0_{\mathbb{Z}^{\mathrm{R}}}) = (p -' 1)! \cdot (((-1)^m \cdot (m!))^p (\in \mathbb{Z}^{\mathrm{R}}))$. The theorem is a consequence of (17).

(21) Let us consider an odd, prime natural number p, a positive natural number m, and a non zero natural number k. Suppose $p \leqslant k$. Then $\mathrm{eval}(\mathrm{R2P}((\mathrm{Der1}(\mathbb{Z}^{\mathrm{R}}))^k(f_0(m,p))), 0_{\mathbb{Z}^{\mathrm{R}}}) = k! \cdot (\mathrm{R2P}(\prod \mathrm{x}.(m,p)))(k -' (p -' 1))$.

(22) Let us consider a natural number j, and an element u of the carrier of Polynom-Ring $\mathbb{Z}^{\mathrm{R}}$. Then $\mathrm{eval}(\mathrm{R2P}((\tau(j)) \cdot u), j(\in \mathbb{Z}^{\mathrm{R}})) = 0_{\mathbb{Z}^{\mathrm{R}}}$.

(23) Let us consider an odd, prime natural number p, a positive natural number m, and natural numbers k, j. Suppose $k < p$ and $j \in \operatorname{Seg} m$. Then $\mathrm{eval}(\mathrm{R2P}((\mathrm{Der1}(\mathbb{Z}^{\mathrm{R}}))^k(f_0(m,p))), j(\in \mathbb{Z}^{\mathrm{R}})) = 0_{\mathbb{Z}^{\mathrm{R}}}$. The theorem is a consequence of (16) and (22).

(24) Let us consider a natural number i.
Then $(\mathrm{Der1}(\mathbb{Z}^{\mathrm{R}}))(\tau(i)) = 1_{\text{Polynom-Ring } \mathbb{Z}^{\mathrm{R}}}$.

(25) Let us consider an odd, prime natural number p, a positive natural number m, and natural numbers j, k. Suppose $j \in \operatorname{Seg} m$ and $p \leqslant k$. Let us consider a natural number i. Suppose $i \in \operatorname{Seg} p$. Then $\tau(j) \mid (\operatorname{LBZ}(\operatorname{Der1}(\mathbb{Z}^{\mathrm{R}}), k, \prod(f_0^{\mathrm{seq}}(m,p))_{\restriction j}, (\tau(j))^p))_{/i}$.

(26) Let us consider an odd, prime natural number p, a positive natural number m, natural numbers k, j, and a natural number i. Suppose $p+1 < i$ and $i \in \operatorname{dom}(\operatorname{LBZ}(\operatorname{Der1}(\mathbb{Z}^{\mathrm{R}}), k, \prod(f_0^{\mathrm{seq}}(m,p))_{\restriction j}, (\tau(j))^p))$. Then $(\operatorname{LBZ}(\operatorname{Der1}(\mathbb{Z}^{\mathrm{R}}), k, \prod(f_0^{\mathrm{seq}}(m,p))_{\restriction j}, (\tau(j))^p))_{/i} = 0_{\text{Polynom-Ring } \mathbb{Z}^{\mathrm{R}}}$.
PROOF: Set $D = \operatorname{Der1}(\mathbb{Z}^{\mathrm{R}})$. Set $\mathbb{P} =$ Polynom-Ring $\mathbb{Z}^{\mathrm{R}}$. Set $x_1 = \tau(j)$. Set $y_1 = \prod(f_0^{\mathrm{seq}}(m,p))_{\restriction j}$. $1_{\mathbb{P}} = D(x_1)$. For every natural number i such that $p+1 < i$ and $i \in \operatorname{dom}(\operatorname{LBZ}(D, k, y_1, {x_1}^p))$ holds $(\operatorname{LBZ}(D, k, y_1, {x_1}^p))_{/i} = 0_{\mathbb{P}}$. $\square$

(27) Let us consider an odd, prime natural number p, a positive natural number m, and natural numbers k, j. Suppose $j \in \operatorname{Seg} m$ and $p \leqslant k$. Then there exist elements u, v of the carrier of Polynom-Ring $\mathbb{Z}^{\mathrm{R}}$ such that $(\operatorname{Der1}(\mathbb{Z}^{\mathrm{R}}))^k(f_0(m,p)) = (\tau(j)) \cdot u + p! \cdot v$.
PROOF: Set $D = \operatorname{Der1}(\mathbb{Z}^{\mathrm{R}})$. Set $\mathbb{P} =$ Polynom-Ring $\mathbb{Z}^{\mathrm{R}}$. Set $t_1 = \tau(j)$. Set $\mathfrak{j} = \prod(f_0^{\mathrm{seq}}(m,p))_{\restriction j}$. $1_{\mathbb{P}} = D(t_1)$. Reconsider $l_3 = \operatorname{LBZ}(D, k, \mathfrak{j}, {t_1}^p)$ as a non empty finite sequence of elements of the carrier of $\mathbb{P}$. Set $l_4 = l_3 \restriction p$. For every natural number i such that $i \in \operatorname{Seg} p$ holds $\tau(j) \mid l_{4/i}$. Consider u being an element of $\mathbb{P}$ such that $\sum l_4 = (\tau(j)) \cdot u$. Set $k_2 = k + 1 -' (p+1)$. For every natural number i_1 such that $i_1 \in \operatorname{dom}(l_{3\,\restriction p+1})$ holds $(l_{3\,\restriction p+1})_{/i_1} = 0_{\mathbb{P}}$. $l_{3\,\restriction p+1} = k_2 \mapsto 0_{\mathbb{P}}$. $\square$

(28) Let us consider an element u of the carrier of Polynom-Ring $\mathbb{Z}^{\mathrm{R}}$, and elements a, b of $\mathbb{Z}^{\mathrm{R}}$. Then $\operatorname{eval}(a \cdot (\operatorname{R2P}(u)), b) \in \{a\}$–ideal.

(29) Let us consider an odd, prime natural number p, a positive natural number m, and natural numbers k, j. Suppose $j \in \operatorname{Seg} m$ and $p \leqslant k$. Then $\operatorname{eval}(\operatorname{R2P}((\operatorname{Der1}(\mathbb{Z}^{\mathrm{R}}))^k(f_0(m,p))), j(\in \mathbb{Z}^{\mathrm{R}})) \in \{p!(\in \mathbb{Z}^{\mathrm{R}})\}$–ideal. The theorem is a consequence of (27), (22), (5), and (28).

5. PROPERTIES OF THE POLYNOMIAL TRANSFORMATION $\mathcal{F}$

Now we state the propositions:

(30) Let us consider an odd, prime natural number p, and a positive natural number m. Then there exists an element u of $\mathbb{Z}^{\mathrm{R}}$ such that $(\mathcal{F}(f_0(m,p)))(0) = (p -' 1)! \cdot (((-1)^m \cdot (m!))^p (\in \mathbb{Z}^{\mathrm{R}})) + p!(\in \mathbb{Z}^{\mathrm{R}}) \cdot u$.
PROOF: Set $G_3 = \mathcal{G}(f_0(m,p))$. Set $p_1 = p -' 1$. $\operatorname{eval}(G_3 \restriction (p -' 1), 0_{\mathbb{Z}^{\mathrm{R}}}) = p_1 \mapsto 0_{\mathbb{Z}^{\mathrm{R}}}$. For every natural number j such that $j \in \operatorname{dom}(\operatorname{eval}(G_{3\,\restriction p}, 0_{\mathbb{Z}^{\mathrm{R}}}))$ holds $(\operatorname{eval}(G_{3\,\restriction p}, 0_{\mathbb{Z}^{\mathrm{R}}}))(j) \in \{p!(\in \mathbb{Z}^{\mathrm{R}})\}$–ideal by [9, (6)], (21), [10, (18),(19)].

Consider u being an element of $\mathbb{Z}^{\mathrm{R}}$ such that $(\mathrm{Eval}(\mathrm{R2P}(^{@}\sum G_{3\restriction p})))(0) = p!(\in \mathbb{Z}^{\mathrm{R}}) \cdot u$. $\square$

(31) Let us consider an odd, prime natural number p, a positive natural number m, and a natural number j. Suppose $j \in \mathrm{Seg}\, m$. Then $(\mathcal{F}(f_0(m,p)))(j(\in \mathbb{R}_{\mathrm{F}})) \in \{p!(\in \mathbb{Z}^{\mathrm{R}})\}$–ideal.

Proof: Set $G_3 = \mathcal{G}(f_0(m,p))$. $\mathrm{eval}(G_3\restriction p, j(\in \mathbb{Z}^{\mathrm{R}})) = p \mapsto 0_{\mathbb{Z}^{\mathrm{R}}}$. For every natural number k such that $k \in \mathrm{dom}(\mathrm{eval}(G_{3\restriction p}, j(\in \mathbb{Z}^{\mathrm{R}})))$ holds $(\mathrm{eval}(G_{3\restriction p}, j(\in \mathbb{Z}^{\mathrm{R}})))(k) \in \{p!(\in \mathbb{Z}^{\mathrm{R}})\}$–ideal. $\square$

6. The Main Part of the Proof

Now we state the proposition:

(32) Let us consider an element x of $\mathbb{R}_{\mathrm{F}}$. Then $(\mathrm{Eval}(\mathrm{R2P}(^{@}f_0(m,p))))(x) = (\mathrm{eval}(\mathrm{R2P}(^{@}\prod \mathrm{x}.(m,p)), x)) \cdot (\mathrm{eval}(\mathrm{R2P}(^{@}(\tau(0))^{p-'1}), x))$.

Let us consider m, p, and g. The functor $\delta_1(m,p,g)$ yielding a finite sequence of elements of $\mathbb{R}_{\mathrm{F}}$ is defined by

(Def. 6) $\mathrm{len}\, it = m$ and for every natural number i such that $i \in \mathrm{dom}\, it$ holds $it(i) = g(i) \cdot (\mathcal{F}(f_0(m,p)))(i(\in \mathbb{R}_{\mathrm{F}}))$.

In the sequel z_0 denotes a non zero element of $\mathbb{R}_{\mathrm{F}}$.

Let us consider m, p, g, and z_0. The functor $\delta_2(m,p,g,z_0)$ yielding a finite sequence of elements of $\mathbb{R}_{\mathrm{F}}$ is defined by

(Def. 7) $\mathrm{len}\, it = m$ and for every natural number i such that $i \in \mathrm{dom}\, it$ holds $it(i) = -g(i) \cdot (\mathrm{power}_{\mathbb{R}_{\mathrm{F}}}(z_0, i) \cdot (\mathcal{F}(f_0(m,p)))(0))$.

The functor $\delta(m,p,g,z_0)$ yielding a finite sequence of elements of $\mathbb{R}_{\mathrm{F}}$ is defined by the term

(Def. 8) $\delta_1(m,p,g) + \delta_2(m,p,g,z_0)$.

The functor $\delta_1^{\mathbb{Z}}(m,p,g)$ yielding a finite sequence of elements of $\mathbb{Z}^{\mathrm{R}}$ is defined by the term

(Def. 9) $\delta_1(m,p,g)$.

Now we state the propositions:

(33) $\sum \delta_1(m,p,g) \in \mathbb{Z}^{\mathrm{R}}$.

Proof: For every natural number i such that $i \in \mathrm{dom}(\delta_1(m,p,g))$ holds $(\delta_1(m,p,g))(i) \in \mathbb{Z}$. $\square$

(34) Let us consider a non zero polynomial g over $\mathbb{Z}^{\mathrm{R}}$. Suppose $\deg(g) = m$. Let us consider a non zero element x of $\mathbb{R}_{\mathrm{F}}$. Then $\sum \delta_2(m,p,g,x) = g(0) \cdot (\mathcal{F}(f_0(m,p)))(0) - (\mathrm{ExtEval}(g,x)) \cdot (\mathcal{F}(f_0(m,p)))(0)$.

Proof: For every non zero element x of $\mathbb{R}_{\mathrm{F}}$, $\sum \delta_2(m,p,g,x) = g(0) \cdot (\mathcal{F}(f_0(m,p)))(0) - (\mathrm{ExtEval}(g,x)) \cdot (\mathcal{F}(f_0(m,p)))(0)$. $\square$

(35) $\sum \delta_1(m,p,g) \in \{p!(\in \mathbb{Z}^{\mathrm{R}})\}$–ideal. The theorem is a consequence of (31).

(36) Let us consider an element x of $\mathbb{R}_{\mathrm{F}}$. Suppose $0 < x \leqslant m$. Let us consider a natural number i. Suppose $i \in \operatorname{Seg} m$.
Then $|\operatorname{eval}(\mathrm{R2P}(^{@}(\mathrm{x}.(m,p))_{/i}), x)| \leqslant m^p$.
PROOF: Set $F_1 = \mathbb{R}_{\mathrm{F}}$. Reconsider $z_0 = -i$ as an element of F_1. $|(z_0 + x)^p| \leqslant m^p$. □

(37) Let us consider an element x of $\mathbb{R}_{\mathrm{F}}$. Then $\operatorname{eval}(\mathrm{R2P}(^{@}(\tau(0))^{p-'1}), x) = x^{p-'1}$. The theorem is a consequence of (3) and (4).

(38) (i) $m^{m+1} \operatorname{ExpSeq}_{\mathbb{R}}$ is convergent, and
(ii) $\lim m^{m+1} \operatorname{ExpSeq}_{\mathbb{R}} = 0$.

(39) Let us consider a non zero natural number M, and a non zero element z_0 of $\mathbb{R}_{\mathrm{F}}$. Suppose $z_0 = e$. Then there exists a natural number n_1 such that for every natural number n such that $n_1 \leqslant n$ holds $|\frac{(m^{m+1})^n}{n!} - 0| < \frac{1}{2 \cdot (M \cdot (z_0{}^m))}$.
The theorem is a consequence of (38).

(40) Every $\mathbb{Z}$-valued polynomial over $\mathbb{F}_{\mathbb{Q}}$ is a polynomial over $\mathbb{Z}^{\mathrm{R}}$.

7. CONCLUSION OF THE PROOF

Now we state the propositions:

(41) FORMAL COUNTERPART OF (3) IN [8]:
Suppose e is algebraic. Then there exists a $\mathbb{Z}$-valued polynomial g over $\mathbb{F}_{\mathbb{Q}}$ such that
(i) $\mathrm{P2R}(g)$ is irreducible, and
(ii) $\operatorname{ExtEval}(g, e(\in \mathbb{R}_{\mathrm{F}})) = 0$, and
(iii) $\deg(g) \geqslant 2$, and
(iv) $g(0) \neq 0_{\mathbb{F}_{\mathbb{Q}}}$.
PROOF: Consider x being an element of $\mathbb{C}_{\mathrm{F}}$ such that $x = e$ and x is integral over $\mathbb{F}_{\mathbb{Q}}$. Consider f_0 being an element of Polynom-Ring $\mathbb{F}_{\mathbb{Q}}$ such that $f_0 \neq \mathbf{0}.\mathbb{F}_{\mathbb{Q}}$ and $\{f_0\}$–ideal $= \operatorname{AnnPoly}(x, \mathbb{F}_{\mathbb{Q}})$ and $f_0 = \operatorname{NormPoly} f_0$. Consider f being a polynomial over $\mathbb{F}_{\mathbb{Q}}$ such that $f_0 = f$ and $\operatorname{ExtEval}(f, x) = 0_{\mathbb{C}_{\mathrm{F}}}$. Reconsider $m = \prod \operatorname{denomi}_{\mathrm{seq}}(f_0)$ as a non zero natural number. Reconsider $\mathfrak{O}_0 = m \cdot f_0$ as an element of the carrier of Polynom-Ring $\mathbb{F}_{\mathbb{Q}}$. $\operatorname{rng} \mathfrak{O}_0 \subseteq \mathbb{Z}$. □

(42) e is transcendental.
PROOF: Consider g being a $\mathbb{Z}$-valued polynomial over $\mathbb{F}_{\mathbb{Q}}$ such that $\mathrm{P2R}(g)$ is irreducible and $\operatorname{ExtEval}(g, e(\in \mathbb{R}_{\mathrm{F}})) = 0$ and $\deg(g) \geqslant 2$ and $g(0) \neq 0_{\mathbb{F}_{\mathbb{Q}}}$. Reconsider $g_0 = g$ as a polynomial over $\mathbb{Z}^{\mathrm{R}}$. Reconsider $g_0 = g$ as a non

zero polynomial over $\mathbb{Z}^{\mathrm{R}}$. Reconsider $m_0 = \deg(g_0)$ as a positive natural number. Reconsider $z_0 = e$ as a non zero element of $\mathbb{R}_{\mathrm{F}}$. Consider M_0 being a natural number such that for every natural number i, $|g_0(i)| \leqslant M_0$. Consider n_1 being a natural number such that for every natural number n such that $n_1 \leqslant n$ holds $|\frac{({m_0}^{m_0+1})^n}{n!} - 0| < \frac{1}{2\cdot(m_0\cdot M_0\cdot {m_0}^{m_0+1}\cdot({z_0}^{m_0}))}$. Consider p_1 being a prime number such that $n_1+m_0+M_0 < p_1$. $\sum \delta(m_0, p_1, g_0, z_0) = \sum \delta_1(m_0, p_1, g_0) + \sum \delta_2(m_0, p_1, g_0, z_0)$. $\sum \delta_1(m_0, p_1, g_0) \in \mathbb{Z}^{\mathrm{R}}$. Consider u being an element of $\mathbb{Z}^{\mathrm{R}}$ such that $(\mathcal{F}(f_0(m_0, p_1)))(0) = (p_1 -' 1)! \cdot (((-1)^{m_0} \cdot (m_0!))^{p_1} (\in \mathbb{Z}^{\mathrm{R}})) + p_1!(\in \mathbb{Z}^{\mathrm{R}}) \cdot u$. $\frac{\sum \delta_2(m_0,p_1,g_0,z_0)}{(p_1-'1)!}$ is an element of $\mathbb{Z}^{\mathrm{R}}$ and $\frac{\sum \delta_2(m_0,p_1,g_0,z_0)}{(p_1-'1)!} = (((-1)^{m_0} \cdot (m_0!))^{p_1} (\in \mathbb{Z}^{\mathrm{R}}) + p_1 \cdot u) \cdot g_0(0)$. $\sum \delta_1(m_0, p_1, g_0) \in \{p_1!(\in \mathbb{Z}^{\mathrm{R}})\}$–ideal. Consider v being an element of $\mathbb{Z}^{\mathrm{R}}$ such that $\sum \delta_1(m_0, p_1, g_0) = p_1!(\in \mathbb{Z}^{\mathrm{R}}) \cdot v$. $\frac{\sum \delta_1(m_0,p_1,g_0)}{(p_1-'1)!} = p_1 \cdot v$. $\frac{\sum \delta(m_0,p_1,g_0,z_0)}{(p_1-'1)!} \in \mathbb{Z}^{\mathrm{R}}$ and $\frac{\sum \delta(m_0,p_1,g_0,z_0)}{(p_1-'1)!} = \frac{\sum \delta_1(m_0,p_1,g_0)}{(p_1-'1)!} + \frac{\sum \delta_2(m_0,p_1,g_0,z_0)}{(p_1-'1)!}$. $|\frac{\sum \delta(m_0,p_1,g_0,z_0)}{(p_1-'1)!}| \leqslant \frac{1}{2}$. $\frac{\sum \delta(m_0,p_1,g_0,z_0)}{(p_1-'1)!} = 0$. □

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