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U-Small and U-Locally Small Categories[1]

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Summary. This paper deals with the notions of U-small set, U-small category, and U-locally small category (U is non-empty Grothendieck universe). We reuse the first Mizar formalization of categories contained in CAT_* series of Mizar articles in order to show the expressive power of the Tarski-Grothendieck set theory (which is the base for the Mizar Mathematical Library) in this area. We encode parts of SGA 4 by Nicolas Bourbaki.

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Introduction

Category theory is developed from the beginning of the Mizar Mathematical Library [2]. Gradually it was expanded, with several different definitions for a category: `Category` [4] and `category` (in [9] or with another definition in [14]) based on [11] and [7]. In the following, we will only use, among these three definitions, the first, as well as the notion $\mathcal{U}$ for Grothendieck's non-empty Universe (see the Mizar formalisation of facts from [15] in [8] described in [3]). This choice is by no means accidental as the base set theory within the repository of Mizar texts is Tarski-Grothendieck set theory [12], [13].

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The first part of this work is devoted to the definitions of $\mathcal{U}$`-set` and proper classes $\mathcal{U}$`-class`.

The second part is largely influenced by the number 0 *Univers* of the first presentation of SGA 4 [1]: we define the notion of a $\mathcal{U}$`-small set` (and of $\mathcal{U}$`-small` group as well as of $\mathcal{U}$`-small` Category).

> 1.1.1. Soient C et D deux catégories et Fonct(C, D) la catégorie des foncteurs de C dans D. a) Si C et D sont éléments d'un univers $\underline{U}$ [...], la catégorie Fonct(C, D) est un élément de $\underline{U}$ [...].

is reflected in theorems (70) and (91).

Our approach allows us to access the formalization of the definition of $\mathcal{U}$`-Category` (see a wider general insight into the topic in less formal sense offered by [6]).

Finally, the notions of $\mathcal{U}$`-small Category` and $\mathcal{U}$`-locally small Category` are introduced. We conclude with some classic examples (adapted from "Example 1.1.4" by Emily Riehl in "Category theory in Context" [10]).

1. Preliminaries

Now we state the propositions:

(1) Let us consider a non empty set X. Then $\{\langle A, B\rangle$, where A, B are elements of X : if B $= \emptyset$, then $A = \emptyset\} = (X \times X) \setminus (X \setminus \{\emptyset\} \times \{\emptyset\})$.

(2) Let us consider a non empty set X. Suppose $\{\emptyset\}$ is an element of X. Then $\{\emptyset\} \notin \mathrm{Funcs}\, X$.

(3) $\mathbb{N}_{\mathrm{even}}$ is denumerable.

(4) $\mathbb{N}_{\mathrm{odd}}$ is denumerable.

(5) Let us consider non empty sets X, Y, and an element y of Y. Then $X \times \{y\} \subseteq \bigcup Y^X$.

(6) Let us consider a non empty set X, and a non zero natural number n. If X^n is finite, then X is finite.

Let us consider a non empty set X. Now we state the propositions:

(7) $\bigcup \mathrm{SmallestPartition}(X) = X$.

(8) $X \approx \mathrm{SmallestPartition}(X)$.

(9) Let us consider a strict object-category $\mathcal{C}$. Then $(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{C}$.

Let x_1, x_2, x_3, x_4, x_5 be objects. The functor $\langle x_1, x_2, x_3, x_4, x_5\rangle$ yielding an object is defined by the term

(Def. 1) $\langle\langle x_1, x_2, x_3, x_4\rangle, x_5\rangle$.

Let x be an object. We say that x is quintuple if and only if

(Def. 2) there exist objects x_1, x_2, x_3, x_4, x_5 such that $x = \langle x_1, x_2, x_3, x_4, x_5 \rangle$.

Let x_1, x_2, x_3, x_4, x_5 be objects. Let us note that $\langle x_1, x_2, x_3, x_4, x_5 \rangle$ is quintuple.

Now we state the proposition:

(10) Let us consider objects x_1, x_2, x_3, x_4, x_5, y_1, y_2, y_3, y_4, y_5. Suppose $\langle x_1, x_2, x_3, x_4, x_5 \rangle = \langle y_1, y_2, y_3, y_4, y_5 \rangle$. Then

(i) $x_1 = y_1$, and

(ii) $x_2 = y_2$, and

(iii) $x_3 = y_3$, and

(iv) $x_4 = y_4$, and

(v) $x_5 = y_5$.

One can verify that there exists an object which is quintuple and there exists a set which is quintuple.

Let x be an object. Assume x is quintuple. The functor $(x)_{\mathbf{1}}$ yielding an object is defined by

(Def. 3) for every objects y_1, y_2, y_3, y_4, y_5 such that $x = \langle y_1, y_2, y_3, y_4, y_5 \rangle$ holds $it = y_1$.

Assume x is quintuple. The functor $(x)_{\mathbf{2}}$ yielding an object is defined by

(Def. 4) for every objects y_1, y_2, y_3, y_4, y_5 such that $x = \langle y_1, y_2, y_3, y_4, y_5 \rangle$ holds $it = y_2$.

Assume x is quintuple. The functor $(x)_{\mathbf{3}}$ yielding an object is defined by

(Def. 5) for every objects y_1, y_2, y_3, y_4, y_5 such that $x = \langle y_1, y_2, y_3, y_4, y_5 \rangle$ holds $it = y_3$.

Assume x is quintuple. The functor $(x)_{\mathbf{4}}$ yielding an object is defined by

(Def. 6) for every objects y_1, y_2, y_3, y_4, y_5 such that $x = \langle y_1, y_2, y_3, y_4, y_5 \rangle$ holds $it = y_4$.

Assume x is quintuple. The functor $(x)_{\mathbf{5}}$ yielding an object is defined by

(Def. 7) for every objects y_1, y_2, y_3, y_4, y_5 such that $x = \langle y_1, y_2, y_3, y_4, y_5 \rangle$ holds $it = y_5$.

Let x_1, x_2, x_3, x_4, x_5 be objects. Observe that $(\langle x_1, x_2, x_3, x_4, x_5 \rangle)_{\mathbf{1}}$ reduces to x_1 and $(\langle x_1, x_2, x_3, x_4, x_5 \rangle)_{\mathbf{2}}$ reduces to x_2 and $(\langle x_1, x_2, x_3, x_4, x_5 \rangle)_{\mathbf{3}}$ reduces to x_3 and $(\langle x_1, x_2, x_3, x_4, x_5 \rangle)_{\mathbf{4}}$ reduces to x_4 and $(\langle x_1, x_2, x_3, x_4, x_5 \rangle)_{\mathbf{5}}$ reduces to x_5.

Let x be a quintuple object. Observe that $\langle (x)_{\mathbf{1}}, (x)_{\mathbf{2}}, (x)_{\mathbf{3}}, (x)_{\mathbf{4}}, (x)_{\mathbf{5}} \rangle$ reduces to x.

2. Some Elementary Properties

From now on $\mathcal{U}$, $\mathcal{V}$ denote universal classes and x denotes an element of $\mathcal{U}$. Now we state the propositions:

(11) Let us consider objects x_1, x_2, x_3. Suppose $x = \langle x_1, x_2, x_3\rangle$. Then

(i) x_1 is an element of $\mathcal{U}$, and

(ii) x_2 is an element of $\mathcal{U}$, and

(iii) x_3 is an element of $\mathcal{U}$.

(12) Let us consider objects x_1, x_2, x_3, x_4. Suppose $x = \langle x_1, x_2, x_3, x_4\rangle$. Then

(i) x_1 is an element of $\mathcal{U}$, and

(ii) x_2 is an element of $\mathcal{U}$, and

(iii) x_3 is an element of $\mathcal{U}$, and

(iv) x_4 is an element of $\mathcal{U}$.

The theorem is a consequence of (11).

(13) Let us consider elements x_1, x_2, x_3, x_4, x_5 of $\mathcal{U}$. Then $\langle x_1, x_2, x_3, x_4, x_5\rangle$ is an element of $\mathcal{U}$.

(14) Let us consider objects x_1, x_2, x_3, x_4, x_5. Suppose $x = \langle x_1, x_2, x_3, x_4, x_5\rangle$. Then

(i) x_1 is an element of $\mathcal{U}$, and

(ii) x_2 is an element of $\mathcal{U}$, and

(iii) x_3 is an element of $\mathcal{U}$, and

(iv) x_4 is an element of $\mathcal{U}$, and

(v) x_5 is an element of $\mathcal{U}$.

The theorem is a consequence of (12).

Let $\mathcal{U}$ be a universal class and u_1, u_2, u_3 be elements of $\mathcal{U}$. Let us note that the functor $\langle u_1, u_2, u_3\rangle$ yields an element of $\mathcal{U}$. Let u_4 be an element of $\mathcal{U}$. Observe that the functor $\langle u_1, u_2, u_3, u_4\rangle$ yields an element of $\mathcal{U}$. Let u_5 be an element of $\mathcal{U}$. Let us note that the functor $\langle u_1, u_2, u_3, u_4, u_5\rangle$ yields an element of $\mathcal{U}$. Now we state the propositions:

(15) Let us consider a subset x of $\mathbf{U}_0$. If x is finite, then x is an element of $\mathbf{U}_0$.

(16) Let us consider a finite set X. If $X \subseteq \mathbf{U}_0$, then $X \in \mathbf{U}_0$.
Proof: Consider p being a function such that $\operatorname{rng} p = X$ and $\operatorname{dom} p \in \omega$. Define $\mathcal{P}[\text{object}, \text{object}] \equiv \$_2 = \{p(\$_1)\}$. Consider g being a function such that $\operatorname{dom} g = \operatorname{dom} p$ and for every object x such that $x \in \operatorname{dom} p$ holds $\mathcal{P}[x, g(x)]$. $\operatorname{rng} g \subseteq \mathbf{U}_0$. $\bigcup \operatorname{rng} g = X$. $\square$

(17) (i) $\bigcup\{\mathbb{N}\} \subseteq \mathbf{U}_0$, and

(ii) $\bigcup\{\mathbb{N}\} \notin \mathbf{U}_0$, and

(iii) $\{\mathbb{N}\} \not\subseteq \mathbf{U}_0$, and

(iv) $\{\mathbb{N}\} \notin \mathbf{U}_0$.

(18) Let us consider an object x. Then $x \in \mathcal{U}$ if and only if $\{x\} \in \mathcal{U}$.

Let us consider a set X and a non zero natural number n. Now we state the propositions:

(19) If $\{X\}^{\mathrm{Seg}\, n}$ is an element of $\mathcal{U}$, then X is an element of $\mathcal{U}$. The theorem is a consequence of (18).

(20) If $\{X\}^n$ is an element of $\mathcal{U}$, then X is an element of $\mathcal{U}$. The theorem is a consequence of (19).

(21) Let us consider a set X. If $\bigcup X \in \mathcal{U}$, then $X \in \mathcal{U}$.

3. Set and Class

Let X be a non empty set and x be an object. We say that x is X-Set if and only if

(Def. 8) $x \in X$.

A Set of X is a set defined by

(Def. 9) *it* is X-Set.

Now we state the propositions:

(22) Let us consider universal classes $\mathcal{U}$, $\mathcal{V}$. Suppose $\mathcal{U} \in \mathcal{V}$. Let us consider an object x. If x is $\mathcal{U}$-Set, then x is $\mathcal{V}$-Set.

(23) Let us consider universal classes $\mathcal{U}$, $\mathcal{V}$. Suppose $\mathcal{U} \in \mathcal{V}$. Then every Set of $\mathcal{U}$ is a Set of $\mathcal{V}$.

(24) Every Set of $\mathbf{U}_0$ is finite.

(25) Let us consider a subset x of $\mathbf{U}_0$. If x is finite, then x is a Set of $\mathbf{U}_0$. The theorem is a consequence of (15).

(26) Let us consider an object x. Then x is a Set of $\mathbf{U}_0$ if and only if x is a set of a finite rank.

Let $\mathcal{U}$ be a universal class and x be an object. We say that x is $\mathcal{U}$-Class if and only if

(Def. 10) $x \in 2^{\mathcal{U}}$ and $x \notin \mathcal{U}$.

Now we state the proposition:

(27) Let us consider a set x. If x is $\mathcal{U}$-Class, then x is not empty.

Let $\mathcal{U}$ be a universal class.

A Class of $\mathcal{U}$ is a non empty set defined by

(Def. 11) *it* is $\mathcal{U}$-Class.

Now we state the propositions:

(28) Let us consider a finite subset X of $\mathcal{U}$. Then $X \in \mathcal{U}$. The theorem is a consequence of (18) and (7).

(29) Every Class of $\mathcal{U}$ is not finite. The theorem is a consequence of (28).

(30) Let us consider a Set X of $\mathcal{U}$. Then $\mathcal{U} \setminus X$ is a Class of $\mathcal{U}$.

(31) Every non finite subset of $\mathbf{U}_0$ is a Class of $\mathbf{U}_0$.

(32) $\mathbb{N}$ is a Class of $\mathbf{U}_0$.

(33) $\mathbb{N}_{\rm even}$ is a Class of $\mathbf{U}_0$. The theorem is a consequence of (3) and (31).

(34) $\mathbb{N}_{\rm odd}$ is a Class of $\mathbf{U}_0$. The theorem is a consequence of (4) and (31).

(35) Let us consider an object x. Then

(i) x is not $\mathcal{U}$-Class, or

(ii) x is not $\mathcal{U}$-Set.

(36) Let us consider universal classes $\mathcal{U}$, $\mathcal{V}$. Suppose $\mathcal{U} \in \mathcal{V}$. Let us consider an object x. If x is $\mathcal{U}$-Class, then x is $\mathcal{V}$-Set.

(37) (i) $\bigcup\{\mathbb{N}\}$ is $\mathbf{U}_0$-Class, and

(ii) $\{\mathbb{N}\}$ is not $\mathbf{U}_0$-Class, and

(iii) $\{\mathbb{N}\}$ is not $\mathbf{U}_0$-Set.

4. Some Operations on Sets and Classes

Now we state the propositions:

(38) Let us consider an object x. Then there exists $\mathcal{U}$ such that x is $\mathcal{U}$-Set.

(39) Every set is (GrothendieckUniverse(x))-Set.

Let $\mathcal{U}$, $\mathcal{V}$ be universal classes. The functor $\sup(\mathcal{U}, \mathcal{V})$ yielding a universal class is defined by the term

(Def. 12) $\begin{cases} \mathcal{U}, & \textbf{if } \mathcal{V} \in \mathcal{U}, \\ \mathcal{V}, & \textbf{otherwise}. \end{cases}$

Now we state the propositions:

(40) Let us consider universal classes $\mathcal{U}$, $\mathcal{V}$, a Set x of $\mathcal{U}$, and a Set y of $\mathcal{V}$. Then there exists a Set z of $\sup(\mathcal{U}, \mathcal{V})$ such that for every object a, $a \in z$ iff $a = x$ or $a = y$.

(41) Let us consider a Set X of $\mathcal{U}$. Then $\bigcup X$ is a Set of $\mathcal{U}$.

Let us consider a set X. Now we state the propositions:

(42) If $\bigcup X$ is a Set of $\mathcal{U}$, then X is a Set of $\mathcal{U}$. The theorem is a consequence of (21).

(43) If $\bigcup X$ is empty, then X is $\mathcal{U}$-Set.

(44) Let us consider a Class X of $\mathcal{U}$. Then $\bigcup X$ is a Class of $\mathcal{U}$. The theorem is a consequence of (43) and (21).

(45) There exists a set X such that

(i) $\bigcup X$ is a Class of $\mathbf{U}_0$, and

(ii) X is not a Class of $\mathbf{U}_0$, and

(iii) X is not a Set of $\mathbf{U}_0$, and

(iv) X is a Set of $\mathbf{U}_1$.

The theorem is a consequence of (17).

(46) Let us consider a Set X of $\mathcal{U}$, and a set Y. If $Y \in X$, then Y is a Set of $\mathcal{U}$.

(47) Let us consider a Class X of $\mathcal{U}$, and a set Y. If $Y \in X$, then Y is a Set of $\mathcal{U}$.

5. U-petit

Let $\mathcal{U}$ be a universal class and x be a set. We say that x is $\mathcal{U}$-petit if and only if

(Def. 13) there exists an element u of $\mathcal{U}$ such that $u \approx x$.

Now we state the proposition:

(48) Every element of $\mathcal{U}$ is $\mathcal{U}$-petit.

Let us consider a set x. Now we state the propositions:

(49) x is $\mathcal{U}$-petit if and only if $\overline{\overline{x}} \in \overline{\overline{\mathcal{U}}}$.

(50) $\{x\}$ is $\mathcal{U}$-petit.

Let $\mathcal{U}$ be a universal class and G be a group. We say that G is $\mathcal{U}$-element if and only if

(Def. 14) the carrier of G is an element of $\mathcal{U}$.

Now we state the proposition:

(51) Let us consider a group G. Suppose G is $\mathcal{U}$-element. Then the multiplication of G is an element of $\mathcal{U}$.

Let $\mathcal{U}$ be a universal class and G be a group. We say that G is $\mathcal{U}$-petit if and only if

(Def. 15) there exists a group H such that H is $\mathcal{U}$-element and G and H are isomorphic.

Let $\mathcal{C}$ be an object-category. We say that $\mathcal{C}$ is $\mathcal{U}$-element if and only if

(Def. 16) the carrier of $\mathcal{C}$ is an element of $\mathcal{U}$ and the carrier' of $\mathcal{C}$ is an element of $\mathcal{U}$.

Now we state the propositions:

(52) Let us consider an object-category $\mathcal{C}$. Suppose $\mathcal{C}$ is $\mathcal{U}$-element. Then

(i) the source of $\mathcal{C}$ is an element of $\mathcal{U}$, and

(ii) the target of $\mathcal{C}$ is an element of $\mathcal{U}$, and

(iii) the composition of $\mathcal{C}$ is an element of $\mathcal{U}$.

(53) Let us consider elements o, m of $\mathcal{U}$. Then $\dot{\circlearrowleft}(o, m)$ is $\mathcal{U}$-element.

Let $\mathcal{U}$ be a universal class. Note that there exists an object-category which is $\mathcal{U}$-element.

Let $\mathcal{C}$ be an object-category. We say that $\mathcal{C}$ is $\mathcal{U}$-petit if and only if

(Def. 17) there exists a strict object-category $\mathcal{D}$ such that $\mathcal{D}$ is $\mathcal{U}$-element and $\mathcal{C} \cong \mathcal{D}$.

Now we state the propositions:

(54) Let us consider an object-category $\mathcal{C}$, and an object a of $\mathcal{C}$. Then $\langle\langle \mathrm{id}_a, \mathrm{id}_a\rangle, \mathrm{id}_a\rangle \in$ the composition of $\mathcal{C}$.

(55) Let us consider objects o, m. Then the composition of $\dot{\circlearrowleft}(o, m) = \{\langle\langle m, m\rangle, m\rangle\}$.

Proof: Set $\mathcal{C} = \dot{\circlearrowleft}(o, m)$. The composition of $\mathcal{C} \subseteq \{\langle\langle m, m\rangle, m\rangle\}$. $\square$

(56) Let us consider objects o, m, and an object c of $\dot{\circlearrowleft}(o, m)$. Then $c = o$.

(57) Let us consider objects o, m, and an element c of $\dot{\circlearrowleft}(o, m)$. Then

(i) c is an object of $\dot{\circlearrowleft}(o, m)$, and

(ii) $c = o$, and

(iii) $\mathrm{id}_c = m$.

(58) Let us consider objects o_1, o_2, m_1, m_2. Then $\dot{\circlearrowleft}(o_1, m_1) \cong \dot{\circlearrowleft}(o_2, m_2)$. The theorem is a consequence of (57).

(59) Let us consider objects o, m. Then $\dot{\circlearrowleft}(o, m)$ is $\mathcal{U}$-petit. The theorem is a consequence of (53) and (58).

Let $\mathcal{U}$ be a universal class. Let us note that there exists an object-category which is $\mathcal{U}$-petit. Now we state the propositions:

(60) There exists a $\mathcal{U}$-petit object-category $\mathcal{C}$ such that

(i) the carrier of $\mathcal{C}$ is not an element of $\mathcal{U}$, and

(ii) the carrier' of $\mathcal{C}$ is an element of $\mathcal{U}$.

The theorem is a consequence of (59) and (18).

(61) There exists a $\mathcal{U}$-petit object-category $\mathcal{C}$ such that

(i) the carrier of $\mathcal{C}$ is not an element of $\mathcal{U}$, and

(ii) the carrier' of $\mathcal{C}$ is not an element of $\mathcal{U}$.

The theorem is a consequence of (59) and (18).

(62) There exists a $\mathcal{U}$-petit object-category $\mathcal{C}$ such that

(i) the carrier of $\mathcal{C}$ is an element of $\mathcal{U}$, and

(ii) the carrier' of $\mathcal{C}$ is not an element of $\mathcal{U}$.

The theorem is a consequence of (59) and (18).

(63) There exists a $\mathcal{U}$-petit object-category $\mathcal{C}$ such that $\mathcal{C}$ is not $\mathcal{U}$-element. The theorem is a consequence of (62).

(64) Let us consider a strict object-category $\mathcal{C}$. Suppose $\mathcal{C}$ is $\mathcal{U}$-element. Then $\mathcal{C}$ is $\mathcal{U}$-petit.

Let $\mathcal{U}$ be a universal class and $\mathcal{C}$ be an object-category. We say that $\mathcal{C}$ is $\mathcal{U}$-category if and only if

(Def. 18) for every objects x, y of $\mathcal{C}$, $\hom(x, y)$ is $\mathcal{U}$-petit.

Let us note that there exists an object-category which is $\mathcal{U}$-category. Now we state the proposition:

(65) Let us consider object-categories $\mathcal{C}$, $\mathcal{D}$, and a functor F from $\mathcal{C}$ to $\mathcal{D}$. Then $F \subseteq$ (the carrier' of $\mathcal{C}$) $\times$ (the carrier' of $\mathcal{D}$).

Let us consider object-categories $\mathcal{C}$, $\mathcal{D}$. Now we state the propositions:

(66) $\mathrm{Funct}(\mathcal{C}, \mathcal{D}) \subseteq 2^{\alpha \times \beta}$, where α is the carrier' of $\mathcal{C}$ and β is the carrier' of $\mathcal{D}$.

(67) $\mathrm{NatTrans}(\mathcal{C}, \mathcal{D}) \subseteq (2^{\alpha \times \beta} \times 2^{\alpha \times \beta}) \times 2^{\gamma \times \beta}$, where α is the carrier' of $\mathcal{C}$, β is the carrier' of $\mathcal{D}$, and γ is the carrier of $\mathcal{C}$.

(68) Let us consider sets X, Y, Z. Suppose X, Y, $Z \in \mathcal{U}$. Then $2^{(2^{X \times Y} \times 2^{X \times Y}) \times 2^{Z \times Y}} \in \mathcal{U}$.

(69) Let us consider non empty sets X, Y. Suppose Y^X is an element of $\mathcal{U}$. Then X is an element of $\mathcal{U}$. The theorem is a consequence of (5).

(70) PROP 1.1.1 A) SGA 4:
Let us consider object-categories $\mathcal{C}$, $\mathcal{D}$. Suppose $\mathcal{C}$ is $\mathcal{U}$-element and $\mathcal{D}$ is $\mathcal{U}$-element. Then $\mathrm{Functors}(\mathcal{D}, \mathcal{C})$ is $\mathcal{U}$-element. The theorem is a consequence of (66) and (67).

(71) Let us consider a set c. Suppose $c \in \overline{\overline{\mathcal{U}}}$. Then $\overline{\overline{2^c}} \in \mathcal{U}$.

(72) Let us consider cardinal numbers c_1, c_2. Suppose $c_1, c_2 \in \overline{\overline{\mathcal{U}}}$. Then $\overline{\overline{2^{c_1 \times c_2}}} \in \mathcal{U}$.

6. Categories of Groups and Universes

Let x be an object. The functor $\mathrm{op}_0(x)$ yielding an element of $\{x\}$ is defined by the term

(Def. 19) x.

The functor $\mathrm{op}_1(x)$ yielding a unary operation on $\{x\}$ is defined by the term

(Def. 20) $x \dot{\longmapsto} x$.

The functor $\mathrm{op}_2(x)$ yielding a binary operation on $\{x\}$ is defined by the term

(Def. 21) $[\langle x, x\rangle \mapsto x]$.

Now we state the proposition:

(73) (i) $\mathrm{op}_0(0) = \mathrm{op}_0$, and

(ii) $\mathrm{op}_1(0) = \mathrm{op}_1$, and

(iii) $\mathrm{op}_2(0) = \mathrm{op}_2$.

Let x be an object. The functor Trivial-addLoopStr(x) yielding a non empty additive loop structure is defined by the term

(Def. 22) $\langle \{x\}, \mathrm{op}_2(x), \mathrm{op}_0(x)\rangle$.

Now we state the propositions:

(74) Trivial-addLoopStr = Trivial-addLoopStr(0).

(75) Let us consider an object x. Then Trivial-addLoopStr(x) is a strict group.

(76) (i) $\mathrm{op}_0(x)$ is an element of $\mathcal{U}$, and

(ii) $\mathrm{op}_1(x)$ is an element of $\mathcal{U}$, and

(iii) $\mathrm{op}_2(x)$ is an element of $\mathcal{U}$.

(77) comp Trivial-addLoopStr(x) is an element of $\mathcal{U}$.

(78) There exists an element y of $\mathcal{U}$ such that $\mathrm{P_{ob}}\, y$, Trivial-addLoopStr(x). The theorem is a consequence of (76) and (77).

(79) $\bigcup$ the set of all the carrier of Trivial-addLoopStr(x) where x is an element of $\mathcal{U} = \mathcal{U}$.

(80) Trivial-addLoopStr$(x) \in \mathrm{GroupObj}(\mathcal{U})$. The theorem is a consequence of (78).

(81) $\mathrm{GroupObj}(\mathcal{U}) \approx \mathcal{U}$.

Proof: Set $G_1 = \mathrm{GroupObj}(\mathcal{U})$. Reconsider $G_2 = G_1$ as a non empty set. Define $\mathcal{P}[\text{object}, \text{object}] \equiv \$_2 \in \mathcal{U}$ and $\mathrm{P_{ob}}\, \$_2, \$_1$. For every element x of G_2, there exists an element y of $\mathcal{U}$ such that $\mathcal{P}[x, y]$. Consider f being a function from G_2 into $\mathcal{U}$ such that for every element x of G_2, $\mathcal{P}[x, f(x)]$. Define $\mathcal{Q}$(object) = Trivial-addLoopStr($\$_1$). For every object x such that $x \in \mathcal{U}$ holds $\mathcal{Q}(x) \in$ GroupObj($\mathcal{U}$). Consider g being a function from $\mathcal{U}$ into GroupObj($\mathcal{U}$) such that for every object x such that $x \in \mathcal{U}$ holds $g(x) = \mathcal{Q}(x)$. □

(82) GroupObj($\mathcal{U}$) is not $\mathcal{U}$-petit. The theorem is a consequence of (81).

7. Object-category Represented by a Set

Let $\mathcal{C}$ be an object-category. The functor CatToSet($\mathcal{C}$) yielding a set is defined by the term

(Def. 23) $\langle$the carrier of $\mathcal{C}$, the carrier' of $\mathcal{C}$, the source of $\mathcal{C}$, the target of $\mathcal{C}$, the composition of $\mathcal{C}\rangle$.

Let S be a quintuple set. We say that S is StrCategory-like if and only if

(Def. 24) there exist sets y_1, y_2, y_3, y_4, y_5 such that $y_1 = (S)_{\mathbf{1}}$ and $y_2 = (S)_{\mathbf{2}}$ and $y_3 = (S)_{\mathbf{3}}$ and $y_4 = (S)_{\mathbf{4}}$ and $y_5 = (S)_{\mathbf{5}}$ and y_3 is a function from y_2 into y_1 and y_4 is a function from y_2 into y_1 and y_5 is a partial function from $y_2 \times y_2$ to y_2.

Note that there exists a quintuple set which is StrCategory-like.

Let S be a StrCategory-like, quintuple set. The functor SetToCat(S) yielding a strict category structure is defined by

(Def. 25) there exist sets y_1, y_2 and there exist functions y_3, y_4 from y_2 into y_1 and there exists a partial function y_5 from $y_2 \times y_2$ to y_2 such that $y_1 = (S)_{\mathbf{1}}$ and $y_2 = (S)_{\mathbf{2}}$ and $y_3 = (S)_{\mathbf{3}}$ and $y_4 = (S)_{\mathbf{4}}$ and $y_5 = (S)_{\mathbf{5}}$ and $it = \langle y_1, y_2, y_3, y_4, y_5 \rangle$.

We say that S is category-like if and only if

(Def. 26) there exist sets y_1, y_2 and there exist functions y_3, y_4 from y_2 into y_1 and there exists a partial function y_5 from $y_2 \times y_2$ to y_2 such that $y_1 = (S)_{\mathbf{1}}$ and $y_2 = (S)_{\mathbf{2}}$ and $y_3 = (S)_{\mathbf{3}}$ and $y_4 = (S)_{\mathbf{4}}$ and $y_5 = (S)_{\mathbf{5}}$ and $\langle y_1, y_2, y_3, y_4, y_5 \rangle$ is an object-category.

Let us note that there exists a StrCategory-like, quintuple set which is category-like and there exists a StrCategory-like, quintuple set which is non empty and category-like.

Let S be a category-like, StrCategory-like, quintuple set. The functor Obj S yielding a set is defined by the term

(Def. 27) $(S)_{\mathbf{1}}$.

The functor Mor S yielding a set is defined by the term

(Def. 28) $(S)_{\mathbf{2}}$.

We say that S is non-empty if and only if

(Def. 29) Obj S is not empty.

Note that there exists a category-like, StrCategory-like, quintuple set which is non-empty.

A CategorySet is a non-empty, category-like, StrCategory-like, quintuple set. Now we state the proposition:

(83) Every CategorySet is not empty.

Note that every CategorySet is non empty.

Let S be a CategorySet. The functor SetToCat(S) yielding a strict object-category is defined by

(Def. 30) there exist sets y_1, y_2 and there exist functions y_3, y_4 from y_2 into y_1 and there exists a partial function y_5 from $y_2 \times y_2$ to y_2 such that $y_1 = (S)_{\mathbf{1}}$ and $y_2 = (S)_{\mathbf{2}}$ and $y_3 = (S)_{\mathbf{3}}$ and $y_4 = (S)_{\mathbf{4}}$ and $y_5 = (S)_{\mathbf{5}}$ and $it = \langle y_1, y_2, y_3, y_4, y_5 \rangle$.

Let $\mathcal{C}$ be a strict object-category. Let us observe that the functor CatToSet($\mathcal{C}$) yields a CategorySet. Now we state the propositions:

(84) Let us consider a CategorySet S. Then CatToSet(SetToCat(S)) $= S$.

(85) Let us consider a strict object-category $\mathcal{C}$. Then SetToCat(CatToSet($\mathcal{C}$)) $= \mathcal{C}$.

(86) Let us consider an object-category $\mathcal{C}$. Then $\mathcal{C}$ is $\mathcal{U}$-element if and only if CatToSet($\mathcal{C}$) is $\mathcal{U}$-Set. The theorem is a consequence of (52), (13), and (14).

(87) Let us consider a CategorySet S. Then S is $\mathcal{U}$-Set if and only if SetToCat (S) is $\mathcal{U}$-element. The theorem is a consequence of (14) and (84).

Let S, T be CategorySets. We say that $S \cong T$ if and only if

(Def. 31) SetToCat(S) $\cong$ SetToCat(T).

Now we state the proposition:

(88) Let us consider strict object-categories $\mathcal{C}$, $\mathcal{D}$. Suppose $\mathcal{C} \cong \mathcal{D}$. Then CatToSet($\mathcal{C}$) $\cong$ CatToSet($\mathcal{D}$). The theorem is a consequence of (85).

Let $\mathcal{U}$ be a universal class and S be a CategorySet. We say that S is $\mathcal{U}$-petit if and only if

(Def. 32) there exists a CategorySet T such that T is $\mathcal{U}$-Set and $S \cong T$.

Now we state the proposition:

(89) Let us consider a strict object-category $\mathcal{C}$. Then $\mathcal{C}$ is $\mathcal{U}$-petit if and only if CatToSet($\mathcal{C}$) is $\mathcal{U}$-petit. The theorem is a consequence of (86), (88), (85), and (87).

Let S, T be CategorySets. The functor $\mathrm{Funct}(S,T)$ yielding a set is defined by the term

(Def. 33) $\mathrm{Funct}(\mathrm{SetToCat}(S), \mathrm{SetToCat}(T))$.

The functor $\mathrm{Functors}(T,S)$ yielding a CategorySet is defined by the term

(Def. 34) $\mathrm{CatToSet}(\mathrm{Functors}(\mathrm{SetToCat}(T), \mathrm{SetToCat}(S)))$.

Now we state the proposition:

(90) Let us consider CategorySets S, T. Then

   (i) $\mathrm{Obj}\,\mathrm{Functors}(T,S) = \mathrm{Funct}(\mathrm{SetToCat}(S), \mathrm{SetToCat}(T))$, and

   (ii) $\mathrm{Mor}\,\mathrm{Functors}(T,S) = \mathrm{NatTrans}(\mathrm{SetToCat}(S), \mathrm{SetToCat}(T))$.

(91) PROP 1.1.1 A) SGA 4:
Let us consider CategorySets S, T. Suppose S is $\mathcal{U}$-Set and T is $\mathcal{U}$-Set. Then $\mathrm{Functors}(T,S)$ is $\mathcal{U}$-Set. The theorem is a consequence of (87), (70), and (86).

8. Small and Locally-small Categories

Let $\mathcal{U}$ be a universal class and $\mathcal{C}$ be an object-category. We introduce the notation $\mathcal{C}$ is $\mathcal{U}$-small as a synonym of $\mathcal{C}$ is $\mathcal{U}$-element.

Note that there exists an object-category which is $\mathcal{U}$-small. Now we state the propositions:

(92) Let us consider sets o, m. Suppose m is not $\mathcal{U}$-Set or o is not $\mathcal{U}$-Set. Then $\dot{\circlearrowright}(o,m)$ is not $\mathcal{U}$-small. The theorem is a consequence of (18).

(93) Let us consider objects o, m. Suppose $\dot{\circlearrowright}(o,m)$ is $\mathcal{U}$-small. Then

   (i) m is $\mathcal{U}$-Set, and

   (ii) o is $\mathcal{U}$-Set.

The theorem is a consequence of (92).

Let $\mathcal{U}$ be a universal class. Observe that there exists an object-category which is non $\mathcal{U}$-small.

Let $\mathcal{C}$ be an object-category. We say that $\mathcal{C}$ is $\mathcal{U}$-locally small if and only if

(Def. 35) for every objects x, y of $\mathcal{C}$, $\hom(x,y)$ is $\mathcal{U}$-Set.

One can verify that there exists an object-category which is $\mathcal{U}$-locally small and there exists a non void, non empty object-category which is $\mathcal{U}$-locally small.

Now we state the propositions:

(94) Every $\mathcal{U}$-small object-category is $\mathcal{U}$-locally small.

(95) Let us consider an object o. Then $\dot{\circlearrowright}(o,\mathcal{U})$ is not $\mathcal{U}$-locally small.
PROOF: Set $\mathcal{C} = \dot{\circlearrowright}(o',\mathcal{U})$. $\mathcal{C}$ is not $\mathcal{U}$-locally small. $\square$

Let $\mathcal{U}$ be a universal class. Let us note that there exists an object-category which is non $\mathcal{U}$-locally small.

Let us consider a $\mathcal{U}$-locally small object-category $\mathcal{C}$. Now we state the propositions:

(96) Suppose the carrier of $\mathcal{C}$ is $\mathcal{U}$-Set. Then $\bigcup$ the set of all $\hom(a, b)$ where a, b are objects of $\mathcal{C}$ is an element of $\mathcal{U}$.

PROOF: Define $\mathcal{P}$[object of $\mathcal{C}$, element of $\mathcal{U}$] $\equiv \bigcup$ the set of all $\hom(\$_1, b)$ where b is an object of $\mathcal{C} = \$_2$. Consider f being a function from the carrier of $\mathcal{C}$ into $\mathcal{U}$ such that for every element x of the carrier of $\mathcal{C}$, $\mathcal{P}[x, f(x)]$. For every object x such that $x \in \operatorname{dom} f$ holds $f(x) \in \mathcal{U}$. □

(97) If the carrier of $\mathcal{C}$ is $\mathcal{U}$-Set, then $\mathcal{C}$ is $\mathcal{U}$-small. The theorem is a consequence of (96).

(98) Let us consider $\mathcal{U}$-small object-categories $\mathcal{C}$, $\mathcal{D}$. Then

(i) Functors$(\mathcal{D}, \mathcal{C})$ is $\mathcal{U}$-small, and

(ii) NatTrans$(\mathcal{C}, \mathcal{D})$ is $\mathcal{U}$-Set.

The theorem is a consequence of (70).

(99) Let us consider a $\mathcal{U}$-small object-category $\mathcal{C}$. Then $\mathcal{C}^{\mathrm{op}}$ is a $\mathcal{U}$-small object-category.

(100) Let us consider a $\mathcal{U}$-locally small object-category $\mathcal{C}$. Then $\mathcal{C}^{\mathrm{op}}$ is a $\mathcal{U}$-locally small object-category.

9. Examples

Let X be a set. Observe that the functor id_X yields an element of X^X. Now we state the propositions:

(101) Funcs $\mathcal{U} \subset \mathcal{U}$. The theorem is a consequence of [5, (81)].

(102) Funcs $\mathcal{U}$ is $\mathcal{U}$-Class. The theorem is a consequence of (101).

(103) $(\mathcal{U} \times \mathcal{U}) \setminus (\mathcal{U} \setminus \{\emptyset\} \times \{\emptyset\})$ is not an element of $\mathcal{U}$.

(104) (i) $\pi_1(\text{Maps}\,\mathcal{U}) \subseteq \mathcal{U} \times \mathcal{U}$, and

(ii) $(\mathcal{U} \times \mathcal{U}) \setminus (\mathcal{U} \setminus \{\emptyset\} \times \{\emptyset\}) \subseteq \pi_1(\text{Maps}\,\mathcal{U})$, and

(iii) $\pi_2(\text{Maps}\,\mathcal{U}) = \text{Funcs}\,\mathcal{U}$.

PROOF: $\pi_1(\text{Maps}\,\mathcal{U}) \subseteq \mathcal{U} \times \mathcal{U}$. $(\mathcal{U} \times \mathcal{U}) \cap \{\langle A, B \rangle$, where A, B are elements of $\mathcal{U}$: if B $= \emptyset$, then $A = \emptyset\} \subseteq \pi_1(\text{Maps}\,\mathcal{U})$. $\pi_2(\text{Maps}\,\mathcal{U}) = \text{Funcs}\,\mathcal{U}$. □

(105) Maps $\mathcal{U} \subseteq \mathcal{U}$.

(106) The carrier' of $\mathbf{Ens}_{\mathcal{U}}$ is $\mathcal{U}$-Class. The theorem is a consequence of (102), (104), and (105).

(107) (i) $\mathbf{Ens}_{\mathcal{U}}$ is a non $\mathcal{U}$-small object-category, and

(ii) the carrier of $\mathbf{Ens}_{\mathcal{U}}$ is $\mathcal{U}$-Class, and

(iii) the carrier' of $\mathbf{Ens}_{\mathcal{U}}$ is $\mathcal{U}$-Class.

The theorem is a consequence of (106).

(108) Let us consider universal classes $\mathcal{U}$, $\mathcal{V}$. Suppose $\mathcal{U} \in \mathcal{V}$. Then $\mathbf{Ens}_{\mathcal{U}}$ is a $\mathcal{V}$-small object-category. The theorem is a consequence of (107).

Let G be an Abelian group. The functor $\# G$ yielding a function from (the carrier of G) $\times$ (the carrier of G) into the carrier of G is defined by the term

(Def. 36) the addition of G.

Let F be a field, o be an object, and n be a natural number. The functor $\mathsf{Mat}_F(o,n)$ yielding a non empty, non void, strict category structure is defined by the term

(Def. 37) $\langle \{o\}$, the carrier of $F_{\rm G}^{n\times n}$, ((the carrier of $F_{\rm G}^{n\times n}) \longmapsto o)$, ((the carrier of $F_{\rm G}^{n\times n}) \longmapsto o), \# F_{\rm G}^{n\times n}\rangle$.

Observe that $\mathsf{Mat}_F(o,n)$ is category-like and $\mathsf{Mat}_F(o,n)$ is transitive and $\mathsf{Mat}_F(o,n)$ is associative and $\mathsf{Mat}_F(o,n)$ is reflexive.

Let K be a field. Let us observe that $\mathsf{Mat}_K(o,n)$ has identities. Now we state the proposition:

(109) Let us consider a field K, an element o of $\mathcal{U}$, and a non zero natural number n. Suppose the carrier of K is an element of $\mathcal{U}$. Then

(i) the carrier of $\mathsf{Mat}_K(o,n)$ is trivial, and

(ii) $\mathsf{Mat}_K(o,n)$ is $\mathcal{U}$-small object-category and $\mathcal{U}$-locally small object-category.

The theorem is a consequence of (18) and (94).

Let us consider an element o of $\mathbf{U}_0$ and a non zero natural number n. Now we state the propositions:

(110) (i) the carrier of $\mathsf{Mat}_{\mathbb{R}_{\rm F}}(o,n)$ is trivial and $\mathbf{U}_0$-Set, and

(ii) $\mathsf{Mat}_{\mathbb{R}_{\rm F}}(o,n)$ is not a $\mathbf{U}_0$-small object-category, and

(iii) $\mathsf{Mat}_{\mathbb{R}_{\rm F}}(o,n)$ is not a $\mathbf{U}_0$-locally small object-category, and

(iv) $\mathsf{Mat}_{\mathbb{R}_{\rm F}}(o,n)$ is $\mathbf{U}_1$-small object-category and $\mathbf{U}_1$-locally small object-category.

The theorem is a consequence of (18), (6), and (109).

(111) (i) the carrier of $\mathsf{Mat}_{\mathbb{C}_{\rm F}}(o,n)$ is trivial and $\mathbf{U}_0$-Set, and

(ii) $\mathsf{Mat}_{\mathbb{C}_{\rm F}}(o,n)$ is not a $\mathbf{U}_0$-small object-category, and

(iii) $\mathsf{Mat}_{\mathbb{C}_{\rm F}}(o,n)$ is not a $\mathbf{U}_0$-locally small object-category, and

(iv) $\mathsf{Mat}_{\mathbb{C}_{\mathrm{F}}}(o, n)$ is $\mathbf{U}_1$-small object-category and $\mathbf{U}_1$-locally small object-category.

The theorem is a consequence of (18), (6), and (109).

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