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Integral of Continuous Three Variable Functions[1]

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Summary. In this article we continue our proofs on integrals of continuous functions of three variables in Mizar. In fact, we use similar techniques as in the case of two variables: we deal with projections of continuous function, the continuity of three variable functions in general, aiming at pure real-valued functions (not necessarily extended real-valued functions), concluding with integrability and iterated integrals of continuous functions of three variables.

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Introduction

In this article, following the previous article [9], we continue our proofs on integrals of continuous functions of three variables in Mizar [2], [3]; for a survey of formalizations of real analysis in another proof-assistants like ACL2 [11], Isabelle/HOL [10], Coq [4], see [5].

In the first section, continuity of functions of three variables is shown. These are used in the proofs of the later sections.

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The second section summarizes the basic properties of the projection of a continuous function in three variables, a result that is almost as obvious as in two variables, but is used to transform [8] Riemann and Lebesgue integrals for real-valued functions (not extended real-valued functions).

In the last section, we prove integrability and iterated integrals of continuous functions of three variables. Throughout the paper, the basic proof steps follow [1], [16], and [12].

1. Preliminaries

Now we state the propositions:

(1) Let us consider real normed spaces X, Y, Z, a point u of $X \times Y \times Z$, a point x of X, a point y of Y, and a point z of Z. Suppose $u = \langle x, y, z \rangle$. Then

(i) $\|u\| \leqslant \|x\| + \|y\| + \|z\|$, and

(ii) $\|x\| \leqslant \|u\|$, and

(iii) $\|y\| \leqslant \|u\|$, and

(iv) $\|z\| \leqslant \|u\|$.

(2) Let us consider closed interval subsets I, J, K of $\mathbb{R}$, and a subset E of ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$). If $E = (I \times J) \times K$, then E is compact.

(3) Let us consider a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a set E.

Suppose $f = g$ and $E \subseteq \operatorname{dom} f$. Then f is uniformly continuous on E if and only if for every real number e such that $0 < e$ there exists a real number r such that $0 < r$ and for every real numbers x_1, x_2, y_1, y_2, z_1, z_2 such that $\langle x_1, y_1, z_1 \rangle$, $\langle x_2, y_2, z_2 \rangle \in E$ and $|x_2 - x_1| < r$ and $|y_2 - y_1| < r$ and $|z_2 - z_1| < r$ holds $|g(\langle x_2, y_2, z_2 \rangle) - g(\langle x_1, y_1, z_1 \rangle)| < e$.

Proof: For every real number e such that $0 < e$ there exists a real number r such that $0 < r$ and for every points p_1, p_2 of ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) such that $p_1, p_2 \in E$ and $\|p_1 - p_2\| < r$ holds $\|f_{/p_1} - f_{/p_2}\| < e$. □

(4) Let us consider intervals I, J, K. Then

(i) $(I \times J) \times K$ is a subset of ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$), and

(ii) $(I \times J) \times K \in \sigma(\text{MeasRect}(\sigma(\text{MeasRect}(\text{L-Field}, \text{L-Field})), \text{L-Field}))$.

(5) Let us consider a point u of (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$), and a real number r. Suppose $0 < r$. Then there exist real numbers s, x, y, z such that

(i) $0 < s < r$, and

(ii) $u = \langle x, y, z \rangle$, and

(iii) $]x - s, x + s[\times]y - s, y + s[\times]z - s, z + s[\subseteq \mathrm{Ball}(u, r)$.

Let us consider a subset A of (the real normed space of $\mathbb{R}$)$\times$(the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$). Now we state the propositions:

(6) Suppose for every real numbers a, b, c such that $\langle a, b, c \rangle \in A$ there exists a real-membered set R_{12} such that R_{12} is non empty and upper bounded and $R_{12} = \{r$, where r is a real number : $0 < r$ and $]a-r, a+r[\times]b - r, b + r[\times]c - r, c + r[\subseteq A\}$. Then there exists a function F from A into $\mathbb{R}$ such that for every real numbers a, b, c such that $\langle a, b, c \rangle \in A$ there exists a real-membered set R_{12} such that R_{12} is non empty and upper bounded and $R_{12} = \{r$, where r is a real number : $0 < r$ and $]a-r, a+r[\times]b - r, b + r[\times]c - r, c + r[\subseteq A\}$ and $F(\langle a, b, c \rangle) = \frac{\sup R_{12}}{2}$.

Proof: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exist real numbers a, b, c and there exists a real-membered set R_{12} such that $\$_1 = \langle a, b, c \rangle$ and R_{12} is non empty and upper bounded and $R_{12} = \{r$, where r is a real number : $0 < r$ and $]a - r, a + r[\times]b - r, b + r[\times]c - r, c + r[\subseteq A\}$ and $\$_2 = \frac{\sup R_{12}}{2}$.

For every object x such that $x \in A$ there exists an object y such that $y \in \mathbb{R}$ and $\mathcal{P}[x, y]$. Consider F being a function from A into $\mathbb{R}$ such that for every object x such that $x \in A$ holds $\mathcal{P}[x, F(x)]$. For every real numbers a, b, c such that $\langle a, b, c \rangle \in A$ there exists a real-membered set R_{12} such that R_{12} is non empty and upper bounded and $R_{12} = \{r$, where r is a real number : $0 < r$ and $]a - r, a + r[\times]b - r, b + r[\times]c - r, c + r[\subseteq A\}$ and $F(\langle a, b, c \rangle) = \frac{\sup R_{12}}{2}$. $\square$

(7) If A is open, then $A \in \sigma(\mathrm{MeasRect}(\sigma(\mathrm{MeasRect}(\text{L-Field}, \text{L-Field})), \text{L-Field}))$. The theorem is a consequence of (5), (6), and (1).

(8) Let us consider closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, and a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$. Suppose f is continuous on $(I \times J) \times K$ and $f = g$. Let us consider a real number e. Suppose $0 < e$. Then there exists a real number r such that

(i) $0 < r$, and

(ii) for every real numbers x_1, x_2, y_1, y_2, z_1, z_2 such that x_1, $x_2 \in I$ and y_1, $y_2 \in J$ and z_1, $z_2 \in K$ and $|x_2 - x_1| < r$ and $|y_2 - y_1| < r$ and

$|z_2 - z_1| < r$ holds $|g(\langle x_2, y_2, z_2\rangle) - g(\langle x_1, y_1, z_1\rangle)| < e$.

PROOF: Set $E = (I \times J) \times K$. f is uniformly continuous on E. Consider r being a real number such that $0 < r$ and for every real numbers x_1, x_2, y_1, y_2, z_1, z_2 such that $\langle x_1, y_1, z_1\rangle$, $\langle x_2, y_2, z_2\rangle \in E$ and $|x_2 - x_1| < r$ and $|y_2 - y_1| < r$ and $|z_2 - z_1| < r$ holds $|g(\langle x_2, y_2, z_2\rangle) - g(\langle x_1, y_1, z_1\rangle)| < e$. For every real numbers x_1, x_2, y_1, y_2, z_1, z_2 such that $x_1, x_2 \in I$ and y_1, $y_2 \in J$ and $z_1, z_2 \in K$ and $|x_2 - x_1| < r$ and $|y_2 - y_1| < r$ and $|z_2 - z_1| < r$ holds $|g(\langle x_2, y_2, z_2\rangle) - g(\langle x_1, y_1, z_1\rangle)| < e$. □

(9) Let us consider a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, and a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$. If $f = g$, then $\|f\| = |g|$.

(10) Let us consider closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, and a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$. Suppose f is continuous on $(I \times J) \times K$ and $f = g$. Let us consider a real number e. Suppose $0 < e$. Then there exists a real number r such that

   (i) $0 < r$, and

   (ii) for every real numbers x_1, x_2, y_1, y_2, z_1, z_2 such that $x_1, x_2 \in I$ and $y_1, y_2 \in J$ and $z_1, z_2 \in K$ and $|x_2 - x_1| < r$ and $|y_2 - y_1| < r$ and $|z_2 - z_1| < r$ holds $||g|(\langle x_2, y_2, z_2\rangle) - |g|(\langle x_1, y_1, z_1\rangle)| < e$.

The theorem is a consequence of (9) and (8).

2. Properties on the Projective Function of a Three Variable Function

Now we state the propositions:

(11) Let us consider a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and elements x, y of $\mathbb{R}$. Suppose f is continuous on $\operatorname{dom} f$ and $f = g$. Then ProjPMap1$(g, \langle x, y\rangle)$ is continuous.

PROOF: For every real number z_0 such that $z_0 \in$ dom(ProjPMap1$(g, \langle x, y\rangle)$) holds ProjPMap1$(g, \langle x, y\rangle)$ is continuous in z_0 by [13, (4)]. □

(12) Let us consider a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, a partial

function p_2 from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, and an element z of $\mathbb{R}$. Suppose f is continuous on $\operatorname{dom} f$ and $f = g$ and $p_2 = \mathrm{ProjPMap2}(g, z)$. Then p_2 is continuous on $\operatorname{dom} p_2$.

PROOF: For every point x_4 of (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) such that $x_4 \in \operatorname{dom} p_2$ holds $p_2 \restriction \operatorname{dom} p_2$ is continuous in x_4 by [15, (18)], [14, (9)]. □

(13) Let us consider a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and elements x, y of $\mathbb{R}$. Suppose f is continuous on $\operatorname{dom} f$ and $f = g$. Then $\mathrm{ProjPMap1}(|g|, \langle x, y\rangle)$ is continuous. The theorem is a consequence of (11).

(14) Let us consider a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, a partial function p_2 from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, and an element z of $\mathbb{R}$. Suppose f is continuous on $\operatorname{dom} f$ and $f = g$ and $p_2 = \mathrm{ProjPMap2}(|g|, z)$. Then p_2 is continuous on $\operatorname{dom} p_2$. The theorem is a consequence of (12).

(15) Let us consider a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and elements x, y of $\mathbb{R}$. Suppose f is uniformly continuous on $\operatorname{dom} f$ and $f = g$. Then $\mathrm{ProjPMap1}(g, \langle x, y\rangle)$ is uniformly continuous.

PROOF: For every real number r such that $0 < r$ there exists a real number s such that $0 < s$ and for every real numbers z_1, z_2 such that $z_1, z_2 \in \operatorname{dom}(\mathrm{ProjPMap1}(g, \langle x, y\rangle))$ and $|z_1 - z_2| < s$ holds $|(\mathrm{ProjPMap1}(g, \langle x, y\rangle))(z_1) - (\mathrm{ProjPMap1}(g, \langle x, y\rangle))(z_2)| < r$. □

(16) Let us consider a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, a partial function p_2 from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, and an element z of $\mathbb{R}$. Suppose f is uniformly continuous on $\operatorname{dom} f$ and $f = g$ and $p_2 = \mathrm{ProjPMap2}(g, z)$. Then p_2 is uniformly continuous on $\operatorname{dom} p_2$.

(17) Let us consider elements x, y of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from

$(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function P_8 from $\mathbb{R}$ to $\mathbb{R}$. Suppose f is continuous on $\operatorname{dom} f$ and $f = g$ and $P_8 = \text{ProjPMap1}(\overline{\mathbb{R}}(g), \langle x, y \rangle)$. Then P_8 is continuous. The theorem is a consequence of (11).

(18) Let us consider an element z of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function P_7 from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$. Suppose f is continuous on $\operatorname{dom} f$ and $f = g$ and $P_7 = \text{ProjPMap2}(\overline{\mathbb{R}}(g), z)$. Then P_7 is continuous on $\operatorname{dom} P_7$. The theorem is a consequence of (12).

(19) Let us consider elements x, y of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function P_8 from $\mathbb{R}$ to $\mathbb{R}$. Suppose f is continuous on $\operatorname{dom} f$ and $f = g$ and $P_8 = \text{ProjPMap1}(|\overline{\mathbb{R}}(g)|, \langle x, y \rangle)$. Then P_8 is continuous. The theorem is a consequence of (13).

(20) Let us consider an element z of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function P_7 from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$. Suppose f is continuous on $\operatorname{dom} f$ and $f = g$ and $P_7 = \text{ProjPMap2}(|\overline{\mathbb{R}}(g)|, z)$. Then P_7 is continuous on $\operatorname{dom} P_7$. The theorem is a consequence of (14).

3. Integral of Continuous Three Variable Function

Let us consider subsets I, J of $\mathbb{R}$, a non empty, closed interval subset K of $\mathbb{R}$, elements x, y of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function P_8 from $\mathbb{R}$ to $\mathbb{R}$. Now we state the propositions:

(21) Suppose $x \in I$ and $y \in J$ and $\operatorname{dom} f = (I \times J) \times K$ and f is continuous on $(I \times J) \times K$ and $f = g$ and $P_8 = \text{ProjPMap1}(\overline{\mathbb{R}}(g), \langle x, y \rangle)$. Then

(i) $P_8 {\restriction} K$ is bounded, and

(ii) P_8 is integrable on K.

The theorem is a consequence of (17).

(22) Suppose $x \in I$ and $y \in J$ and $\operatorname{dom} f = (I \times J) \times K$ and f is continuous on $(I \times J) \times K$ and $f = g$ and $P_8 = \text{ProjPMap1}(\overline{\mathbb{R}}(g), \langle x, y \rangle)$. Then

(i) P_8 is integrable on L-Meas, and

(ii) $\int\limits_K P_8(x)dx = \int P_8 \, \mathrm{d}\,\text{L-Meas}$, and

(iii) $\int\limits_K P_8(x)dx = \int \text{ProjPMap1}(\overline{\mathbb{R}}(g), \langle x, y \rangle) \, \mathrm{d}\,\text{L-Meas}$, and

(iv) $\int\limits_K P_8(x)dx = (\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)))(\langle x, y \rangle)$.

The theorem is a consequence of (21).

(23) Let us consider non empty, closed interval subsets I, J of $\mathbb{R}$, a subset K of $\mathbb{R}$, an element z of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function P_9 from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$. Suppose $z \in K$ and $\operatorname{dom} f = (I \times J) \times K$ and f is continuous on $(I \times J) \times K$ and $f = g$ and $P_9 = \text{ProjPMap2}(\overline{\mathbb{R}}(g), z)$. Then

(i) P_9 is integrable on ProdMeas(L-Meas, L-Meas), and

(ii) $\int P_9 \, \mathrm{d}\,\text{ProdMeas}(\text{L-Meas}, \text{L-Meas}) =$
$\int \text{ProjPMap2}(\overline{\mathbb{R}}(g), z) \, \mathrm{d}\,\text{ProdMeas}(\text{L-Meas}, \text{L-Meas})$, and

(iii) $\int P_9 \, \mathrm{d}\,\text{ProdMeas}(\text{L-Meas}, \text{L-Meas}) =$
$(\text{Integral1}(\text{ProdMeas}(\text{L-Meas}, \text{L-Meas}), \overline{\mathbb{R}}(g)))(z)$.

The theorem is a consequence of (18).

(24) Let us consider subsets I, J of $\mathbb{R}$, a non empty, closed interval subset K of $\mathbb{R}$, elements x, y of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function P_8 from $\mathbb{R}$ to $\mathbb{R}$. Suppose $x \in I$ and $y \in J$ and $\operatorname{dom} f = (I \times J) \times K$ and f is continuous on $(I \times J) \times K$ and $f = g$ and $P_8 = \text{ProjPMap1}(|\overline{\mathbb{R}}(g)|, \langle x, y \rangle)$. Then

(i) $P_8 \restriction K$ is bounded, and

(ii) P_8 is integrable on K.

The theorem is a consequence of (19).

(25) Let us consider subsets I, J of $\mathbb{R}$, a non empty, closed interval subset K of $\mathbb{R}$, elements x, y of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, a partial function P_8 from $\mathbb{R}$ to $\mathbb{R}$, and an element E of L-Field. Suppose

$x \in I$ and $y \in J$ and $\operatorname{dom} f = (I \times J) \times K$ and f is continuous on $(I \times J) \times K$ and $f = g$ and $P_8 = \mathrm{ProjPMap1}(|\overline{\mathbb{R}}(g)|, \langle x, y \rangle)$ and $E = K$. Then P_8 is E-measurable. The theorem is a consequence of (24).

(26) Let us consider subsets I, J of $\mathbb{R}$, a non empty, closed interval subset K of $\mathbb{R}$, elements x, y of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function P_8 from $\mathbb{R}$ to $\mathbb{R}$. Suppose $x \in I$ and $y \in J$ and $\operatorname{dom} f = (I \times J) \times K$ and f is continuous on $(I \times J) \times K$ and $f = g$ and $P_8 = \mathrm{ProjPMap1}(|\overline{\mathbb{R}}(g)|, \langle x, y \rangle)$. Then

(i) P_8 is integrable on L-Meas, and

(ii) $\int\limits_K P_8(x)dx = \int P_8 \, \mathrm{d}\, \text{L-Meas}$, and

(iii) $\int\limits_K P_8(x)dx = \int \mathrm{ProjPMap1}(|\overline{\mathbb{R}}(g)|, \langle x, y \rangle) \, \mathrm{d}\, \text{L-Meas}$, and

(iv) $\int\limits_K P_8(x)dx = (\mathrm{Integral2}(\text{L-Meas}, |\overline{\mathbb{R}}(g)|))(\langle x, y \rangle)$.

The theorem is a consequence of (24).

(27) Let us consider non empty, closed interval subsets I, J of $\mathbb{R}$, a subset K of $\mathbb{R}$, an element z of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, a partial function P_9 from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$, and an element E of $\sigma(\mathrm{MeasRect}(\text{L-Field}, \text{L-Field}))$. Suppose $z \in K$ and $\operatorname{dom} f = (I \times J) \times K$ and f is continuous on $(I \times J) \times K$ and $f = g$ and $P_9 = \mathrm{ProjPMap2}(|\overline{\mathbb{R}}(g)|, z)$ and $E = I \times J$. Then P_9 is E-measurable. The theorem is a consequence of (20).

(28) Let us consider non empty, closed interval subsets I, J of $\mathbb{R}$, a subset K of $\mathbb{R}$, an element z of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function P_9 from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$. Suppose $z \in K$ and $\operatorname{dom} f = (I \times J) \times K$ and f is continuous on $(I \times J) \times K$ and $f = g$ and $P_9 = \mathrm{ProjPMap2}(|\overline{\mathbb{R}}(g)|, z)$. Then

(i) P_9 is integrable on ProdMeas(L-Meas, L-Meas), and

(ii) $\int P_9 \, \mathrm{d}\, \mathrm{ProdMeas}(\text{L-Meas}, \text{L-Meas}) =$
$\int \mathrm{ProjPMap2}(|\overline{\mathbb{R}}(g)|, z) \, \mathrm{d}\, \mathrm{ProdMeas}(\text{L-Meas}, \text{L-Meas})$, and

(iii) $\int P_9 \,\mathrm{d}\,\mathrm{ProdMeas}(\text{L-Meas}, \text{L-Meas}) =$
$(\mathrm{Integral1}(\mathrm{ProdMeas}(\text{L-Meas}, \text{L-Meas}), |\overline{\mathbb{R}}(g)|))(z)$.

The theorem is a consequence of (20).

(29) Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and an element E of $\sigma(\mathrm{MeasRect}(\sigma(\mathrm{MeasRect}(\text{L-Field}, \text{L-Field})), \text{L-Field}))$. Suppose $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$ and $E = (I \times J) \times K$. Then g is E-measurable.

PROOF: For every real number r, $E \cap \text{LE-dom}(g, r) \in \sigma(\mathrm{MeasRect}(\sigma(\mathrm{MeasRect}(\text{L-Field}, \text{L-Field})), \text{L-Field}))$. $\square$

Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, elements x, y of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a real number e. Now we state the propositions:

(30) Suppose $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$. Then suppose $0 < e$. Then there exists a real number r such that

(i) $0 < r$, and

(ii) for every elements u_1, u_2 of $\mathbb{R} \times \mathbb{R}$ and for every real numbers x_1, y_1, x_2, y_2 such that $u_1 = \langle x_1, y_1 \rangle$ and $u_2 = \langle x_2, y_2 \rangle$ and $|x_2 - x_1| < r$ and $|y_2 - y_1| < r$ and $u_1, u_2 \in I \times J$ for every element z of $\mathbb{R}$ such that $z \in K$ holds $|(\mathrm{ProjPMap1}(|\overline{\mathbb{R}}(g)|, u_2))(z) - (\mathrm{ProjPMap1}(|\overline{\mathbb{R}}(g)|, u_1))(z)| < e$.

PROOF: For every element x of $\mathbb{R} \times \mathbb{R}$ and for every element y of $\mathbb{R}$ such that $x \in I \times J$ and $y \in K$ holds $(\mathrm{ProjPMap1}(|\overline{\mathbb{R}}(g)|, x))(y) = |\overline{\mathbb{R}}(g)|(x, y)$ and $|\overline{\mathbb{R}}(g)|(x, y) = |g|(\langle x, y \rangle)$. Consider r being a real number such that $0 < r$ and for every real numbers x_1, x_2, y_1, y_2, z_1, z_2 such that $x_1, x_2 \in I$ and $y_1, y_2 \in J$ and $z_1, z_2 \in K$ and $|x_2 - x_1| < r$ and $|y_2 - y_1| < r$ and $|z_2 - z_1| < r$ holds $||g|(\langle x_2, y_2, z_2 \rangle) - |g|(\langle x_1, y_1, z_1 \rangle)| < e$. $\square$

(31) Suppose $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$. Then suppose $0 < e$. Then there exists a real number r such that

(i) $0 < r$, and

(ii) for every elements u_1, u_2 of $\mathbb{R} \times \mathbb{R}$ and for every real numbers x_1, y_1, x_2, y_2 such that $u_1 = \langle x_1, y_1 \rangle$ and $u_2 = \langle x_2, y_2 \rangle$ and $|x_2 - x_1| < r$ and $|y_2 - y_1| < r$ and $u_1, u_2 \in I \times J$ for every element z of $\mathbb{R}$ such that $z \in$

K holds $|(\mathrm{ProjPMap1}(\overline{\mathbb{R}}(g), u_2))(z) - (\mathrm{ProjPMap1}(\overline{\mathbb{R}}(g), u_1))(z)| < e$.

The theorem is a consequence of (8).

(32) Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, and a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$. Suppose $(I \times J) \times K = \mathrm{dom}\, f$ and f is continuous on $(I \times J) \times K$ and $f = g$. Then

(i) Integral2(L-Meas, $|\overline{\mathbb{R}}(g)|$) is a function from $\mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$, and

(ii) Integral2(L-Meas, $|\overline{\mathbb{R}}(g)|) \restriction (I \times J)$ is a partial function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$, and

(iii) Integral2(L-Meas, $\overline{\mathbb{R}}(g)$) is a function from $\mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$, and

(iv) Integral2(L-Meas, $\overline{\mathbb{R}}(g)) \restriction (I \times J)$ is a partial function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$.

The theorem is a consequence of (26) and (22).

Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function F_4 from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$. Now we state the propositions:

(33) Suppose $(I \times J) \times K = \mathrm{dom}\, f$ and f is continuous on $(I \times J) \times K$ and $f = g$ and $F_4 = $ Integral2(L-Meas, $|\overline{\mathbb{R}}(g)|) \restriction (I \times J)$. Then F_4 is uniformly continuous on $I \times J$. The theorem is a consequence of (30), (19), and (24).

(34) Suppose $(I \times J) \times K = \mathrm{dom}\, f$ and f is continuous on $(I \times J) \times K$ and $f = g$ and $F_4 = $ Integral2(L-Meas, $\overline{\mathbb{R}}(g)) \restriction (I \times J)$. Then F_4 is uniformly continuous on $I \times J$. The theorem is a consequence of (31), (17), (21), and (22).

(35) Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, and a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$. Suppose $(I \times J) \times K = \mathrm{dom}\, f$ and f is continuous on $(I \times J) \times K$ and $f = g$. Then

(i) Integral1(ProdMeas(L-Meas, L-Meas), $|\overline{\mathbb{R}}(g)|$) is a function from $\mathbb{R}$ into $\mathbb{R}$, and

(ii) Integral1(ProdMeas(L-Meas, L-Meas), $|\overline{\mathbb{R}}(g)|) \restriction K$ is a partial function from $\mathbb{R}$ to $\mathbb{R}$, and

(iii) Integral1(ProdMeas(L-Meas, L-Meas), $\overline{\mathbb{R}}(g)$) is a function from $\mathbb{R}$ into $\mathbb{R}$, and

(iv) Integral1(ProdMeas(L-Meas, L-Meas), $\overline{\mathbb{R}}(g)$)$\restriction K$ is a partial function from $\mathbb{R}$ to $\mathbb{R}$.

The theorem is a consequence of (20), (28), (18), and (23).

Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function G_3 from $\mathbb{R}$ to $\mathbb{R}$. Now we state the propositions:

(36) Suppose $(I \times J) \times K = \mathrm{dom}\, f$ and f is continuous on $(I \times J) \times K$ and $f = g$ and $G_3 =$ Integral1(ProdMeas(L-Meas, L-Meas), $|\overline{\mathbb{R}}(g)|$)$\restriction K$. Then G_3 is continuous.

PROOF: Consider a, b being real numbers such that $I = [a, b]$. Consider c, d being real numbers such that $J = [c, d]$. For every real number e such that $0 < e$ there exists a real number r such that $0 < r$ and for every real numbers z_1, z_2 such that $|z_2 - z_1| < r$ and $z_1, z_2 \in K$ for every real numbers x, y such that $x \in I$ and $y \in J$ holds $||g|(\langle x, y, z_2\rangle) - |g|(\langle x, y, z_1\rangle)| < e$. Set $R_{11} = \overline{\mathbb{R}}(g)$. For every elements x, y, z of $\mathbb{R}$ such that $x \in I$ and $y \in J$ and $z \in K$ holds $(\mathrm{ProjPMap2}(|R_{11}|, z))(x, y) = |R_{11}|(\langle x, y\rangle, z)$ and $|R_{11}|(\langle x, y\rangle, z) = |g(\langle x, y, z\rangle)|$ and $|R_{11}|(\langle x, y\rangle, z) = |g|(\langle x, y, z\rangle)$. For every real number e such that $0 < e$ there exists a real number r such that $0 < r$ and for every elements z_1, z_2 of $\mathbb{R}$ such that $|z_2 - z_1| < r$ and $z_1, z_2 \in K$ for every elements x, y of $\mathbb{R}$ such that $x \in I$ and $y \in J$ holds $|(\mathrm{ProjPMap1}(\mathrm{ProjPMap2}(|R_{11}|, z_2), x))(y) - (\mathrm{ProjPMap1}(\mathrm{ProjPMap2}(|R_{11}|, z_1), x))(y)| < e$. For every real numbers z_0, r such that $z_0 \in K$ and $0 < r$ there exists a real number s such that $0 < s$ and for every real number z_1 such that $z_1 \in K$ and $|z_1 - z_0| < s$ holds $|G_3(z_1) - G_3(z_0)| < r$. □

(37) Suppose $(I \times J) \times K = \mathrm{dom}\, f$ and f is continuous on $(I \times J) \times K$ and $f = g$ and $G_3 =$ Integral1(ProdMeas(L-Meas, L-Meas), $\overline{\mathbb{R}}(g)$)$\restriction K$. Then G_3 is continuous.

PROOF: Consider a, b being real numbers such that $I = [a, b]$. Consider c, d being real numbers such that $J = [c, d]$. For every real number e such that $0 < e$ there exists a real number r such that $0 < r$ and for every real numbers z_1, z_2 such that $|z_2 - z_1| < r$ and $z_1, z_2 \in K$ for every real numbers x, y such that $x \in I$ and $y \in J$ holds $|g(\langle x, y, z_2\rangle) - g(\langle x, y, z_1\rangle)| < e$. Set $R_{11} = \overline{\mathbb{R}}(g)$. For every elements x, y, z of $\mathbb{R}$ such that $x \in I$ and $y \in J$ and $z \in K$ holds $(\mathrm{ProjPMap2}(R_{11}, z))(x, y) = R_{11}(\langle x, y\rangle, z)$

and $R_{11}(\langle x, y\rangle, z) = g(\langle x, y, z\rangle)$ and $R_{11}(\langle x, y\rangle, z) = g(\langle x, y, z\rangle)$.

For every real number e such that $0 < e$ there exists a real number r such that $0 < r$ and for every elements z_1, z_2 of $\mathbb{R}$ such that $|z_2 - z_1| < r$ and $z_1, z_2 \in K$ for every elements x, y of $\mathbb{R}$ such that $x \in I$ and $y \in J$ holds $|(\mathrm{ProjPMap1}(\mathrm{ProjPMap2}(R_{11}, z_2), x))(y) - (\mathrm{ProjPMap1}(\mathrm{ProjPMap2}(R_{11}, z_1), x))(y)| < e$. For every real numbers z_0, r such that $z_0 \in K$ and $0 < r$ there exists a real number s such that $0 < s$ and for every real number z_1 such that $z_1 \in K$ and $|z_1 - z_0| < s$ holds $|G_3(z_1) - G_3(z_0)| < r$. □

Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, and a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$. Now we state the propositions:

(38) Suppose $(I \times J) \times K = \mathrm{dom}\, f$ and f is continuous on $(I \times J) \times K$ and $f = g$. Then Integral2(L-Meas, $|\overline{\mathbb{R}}(g)|$) is non-negative. The theorem is a consequence of (24) and (25).

(39) Suppose $(I \times J) \times K = \mathrm{dom}\, f$ and f is continuous on $(I \times J) \times K$ and $f = g$. Then Integral1(ProdMeas(L-Meas, L-Meas), $|\overline{\mathbb{R}}(g)|$) is non-negative. The theorem is a consequence of (20) and (27).

(40) Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, an element u of $\mathbb{R} \times \mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, and a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$. Suppose $(I \times J) \times K = \mathrm{dom}\, f$ and f is continuous on $(I \times J) \times K$ and $f = g$. Then (Integral2(L-Meas, $|\overline{\mathbb{R}}(g)|$))$(u) < +\infty$. The theorem is a consequence of (32).

(41) Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, an element z of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, and a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$. Suppose $(I \times J) \times K = \mathrm{dom}\, f$ and f is continuous on $(I \times J) \times K$ and $f = g$. Then (Integral1(ProdMeas(L-Meas, L-Meas), $|\overline{\mathbb{R}}(g)|$))$(z) < +\infty$. The theorem is a consequence of (35).

(42) Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and an element E of σ(MeasRect(L-Field, L-Field)). Suppose $(I \times J) \times K = \mathrm{dom}\, f$ and f is continuous on $(I \times J) \times K$ and $f = g$. Then Integral2(L-Meas, $|\overline{\mathbb{R}}(g)|$) is E-measurable.

Proof: Set $F = \text{Integral2}(\text{L-Meas}, |\overline{\mathbb{R}}(g)|)$. Set $I_1 = I \times J$. Reconsider $G = \text{Integral2}(\text{L-Meas}, |\overline{\mathbb{R}}(g)|) \restriction I_1$ as a partial function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$. Reconsider $R_4 = \text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \restriction I_1$ as a partial function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$. Reconsider $G_1 = G$ as a partial function from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$.

Reconsider $R_6 = R_4$ as a partial function from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$. G_1 is uniformly continuous on $I \times J$. R_6 is uniformly continuous on $I \times J$. F is non-negative. Reconsider $H = \mathbb{R} \times \mathbb{R}$ as an element of $\sigma(\text{MeasRect}(\text{L-Field}, \text{L-Field}))$. For every real number r, $H \cap \text{LE-dom}(F, r) \in \sigma(\text{MeasRect}(\text{L-Field}, \text{L-Field}))$. □

Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, and a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$. Now we state the propositions:

(43) Suppose $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$. Then

- (i) g is integrable on ProdMeas(ProdMeas(L-Meas, L-Meas), L-Meas), and
- (ii) for every element u of $\mathbb{R} \times \mathbb{R}$, $\text{ProjPMap1}(\overline{\mathbb{R}}(g), u)$ is integrable on L-Meas, and
- (iii) for every element U of $\sigma(\text{MeasRect}(\text{L-Field}, \text{L-Field}))$, $\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))$ is U-measurable, and
- (iv) $\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))$ is integrable on ProdMeas(L-Meas, L-Meas), and
- (v) $\int g \, \mathrm{d}\, \text{ProdMeas}(\text{ProdMeas}(\text{L-Meas}, \text{L-Meas}), \text{L-Meas}) = \int \text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \, \mathrm{d}\, \text{ProdMeas}(\text{L-Meas}, \text{L-Meas})$.

Proof: Set $F = \text{Integral2}(\text{L-Meas}, |\overline{\mathbb{R}}(g)|)$. Set $I_1 = I \times J$. Reconsider $G = \text{Integral2}(\text{L-Meas}, |\overline{\mathbb{R}}(g)|) \restriction I_1$ as a partial function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$. Reconsider $R_4 = \text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \restriction I_1$ as a partial function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$. Reconsider $A_1 = I \times J$ as an element of $\sigma(\text{MeasRect}(\text{L-Field}, \text{L-Field}))$. Reconsider $G_1 = G$ as a partial function from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$. Reconsider $R_6 = R_4$ as a partial function from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$. G_1 is uniformly continuous on $I \times J$. R_6 is uniformly continuous on $I \times J$. Reconsider $N_1 = (\mathbb{R} \times \mathbb{R}) \setminus A_1$ as an element of $\sigma(\text{MeasRect}(\text{L-Field}, \text{L-Field}))$. F is non-negative. Reconsider $H = \mathbb{R} \times \mathbb{R}$ as an element of $\sigma(\text{MeasRect}(\text{L-Field}, \text{L-Field}))$.

F is H-measurable. Set $F_1 = F \restriction N_1$. For every object x such that $x \in \operatorname{dom} F_1$ holds $F_1(x) = 0$. Reconsider $K_1 = (I \times J) \times K$ as an element of $\sigma(\text{MeasRect}(\sigma(\text{MeasRect}(\text{L-Field}, \text{L-Field})), \text{L-Field}))$. g is K_1-measurable. For every element x of $\mathbb{R} \times \mathbb{R}$, $(\text{Integral2}(\text{L-Meas}, |\overline{\mathbb{R}}(g)|))(x) < +\infty$. □

(44) Suppose $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$. Then

- (i) for every element z of $\mathbb{R}$, $\text{ProjPMap2}(\overline{\mathbb{R}}(g), z)$ is integrable on $\text{ProdMeas}(\text{L-Meas}, \text{L-Meas})$, and
- (ii) for every element V of L-Field, $\text{Integral1}(\text{ProdMeas}(\text{L-Meas}, \text{L-Meas}), \overline{\mathbb{R}}(g))$ is V-measurable, and
- (iii) $\text{Integral1}(\text{ProdMeas}(\text{L-Meas}, \text{L-Meas}), \overline{\mathbb{R}}(g))$ is integrable on L-Meas, and
- (iv) $\int g \, \mathrm{d}\, \text{ProdMeas}(\text{ProdMeas}(\text{L-Meas}, \text{L-Meas}), \text{L-Meas}) = \int \text{Integral1}(\text{ProdMeas}(\text{L-Meas}, \text{L-Meas}), \overline{\mathbb{R}}(g)) \, \mathrm{d}\, \text{L-Meas}$.

The theorem is a consequence of (43) and (41).

(45) Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, an element x of $\mathbb{R}$, and an element E of L-Field. Suppose $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$ and $x \in I$. Then $\text{ProjPMap1}(|\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))|, x)$ is E-measurable.

Proof: Set $F_4 = \text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))$. Reconsider $G_4 = \text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))$ as a function from $\mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$. Reconsider $G = G_4 \restriction (I \times J)$ as a partial function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$. Reconsider $F = G$ as a partial function from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$. F is uniformly continuous on $I \times J$. Set $F_5 = \text{ProjPMap1}(|F_4|, x)$. Set $L_0 = F_5 \restriction J$. For every element t of $\mathbb{R}$ such that $t \in J$ holds $0 \leqslant L_0(t)$. Reconsider $H = \mathbb{R}$ as an element of L-Field. For every real number r, $H \cap \text{LE-dom}(F_5, r) \in \text{L-Field}$. □

(46) Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, and a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$. Suppose $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$. Then

- (i) for every element x of $\mathbb{R}$, $(\text{Integral2}(\text{L-Meas}, |\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))|))(x) < +\infty$, and

(ii) for every element x of $\mathbb{R}$, ProjPMap1(Integral2(L-Meas, $\overline{\mathbb{R}}(g)$), x) is integrable on L-Meas.

PROOF: Reconsider G_4 = Integral2(L-Meas, $\overline{\mathbb{R}}(g)$) as a function from $\mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$. Reconsider $G = G_4 \restriction (I \times J)$ as a partial function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$. Reconsider $F = G$ as a partial function from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$. F is uniformly continuous on $I \times J$. For every element x of $\mathbb{R}$, (Integral2(L-Meas, | Integral2(L-Meas, $\overline{\mathbb{R}}(g)$)|))(x) $< +\infty$ by [6, (5)], [7, (75)]. Integral2(L-Meas, $\overline{\mathbb{R}}(g)$) is integrable on ProdMeas(L-Meas, L-Meas). □

(47) Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, an element y of $\mathbb{R}$, and an element E of L-Field. Suppose $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$ and $y \in J$. Then ProjPMap2(| Integral2(L-Meas, $\overline{\mathbb{R}}(g)$)|, y) is E-measurable.

PROOF: Set F_4 = Integral2(L-Meas, $\overline{\mathbb{R}}(g)$). Reconsider G_4 = Integral2(L-Meas, $\overline{\mathbb{R}}(g)$) as a function from $\mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$. Reconsider $G = G_4 \restriction (I \times J)$ as a partial function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$. Reconsider $F = G$ as a partial function from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$. F is uniformly continuous on $I \times J$. Set F_6 = ProjPMap2($|F_4|$, y). Set $L_0 = F_6 \restriction I$. For every element t of $\mathbb{R}$ such that $t \in I$ holds $0 \leqslant L_0(t)$. For every element r of $\mathbb{R}$, $0_{\overline{\mathbb{R}}} \leqslant F_6(r)$. Reconsider $H = \mathbb{R}$ as an element of L-Field. For every real number r, $H \cap$ LE-dom(F_6, r) $\in$ L-Field. □

(48) Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, and a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$. Suppose $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$. Then

(i) for every element y of $\mathbb{R}$, (Integral1(L-Meas, | Integral2(L-Meas, $\overline{\mathbb{R}}(g)$)|))(y) $< +\infty$, and

(ii) for every element y of $\mathbb{R}$, ProjPMap2(Integral2(L-Meas, $\overline{\mathbb{R}}(g)$), y) is integrable on L-Meas.

PROOF: Reconsider G_4 = Integral2(L-Meas, $\overline{\mathbb{R}}(g)$) as a function from $\mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$. Reconsider $G = G_4 \restriction (I \times J)$ as a partial function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$. Reconsider $F = G$ as a partial function from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$. F is uniformly continuous on $I \times J$. For every element y of $\mathbb{R}$, (Integral1(L-Meas, | Integral2(L-Meas, $\overline{\mathbb{R}}(g)$)|))(y) $< +\infty$. Integral2(L-Meas, $\overline{\mathbb{R}}(g)$) is integrable on ProdMeas(L-Meas, L-Meas). □

(49) Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) × (the real normed space of $\mathbb{R}$)) × (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and an element E of σ(MeasRect(L-Field, L-Field)). Suppose $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$. Then Integral2(L-Meas, $|\overline{\mathbb{R}}(g)|$) is E-measurable.

PROOF: Set F = Integral2(L-Meas, $|\overline{\mathbb{R}}(g)|$). Set $F_0 = F{\restriction}(I \times J)$. Reconsider $G = F_0$ as a partial function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$. Reconsider $G_1 = G$ as a partial function from (the real normed space of $\mathbb{R}$) × (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$. G_1 is uniformly continuous on $I \times J$. Reconsider $R_2 = \mathbb{R} \times \mathbb{R}$ as an element of σ(MeasRect(L-Field, L-Field)). F is non-negative. For every real number r, $R_2 \cap$LE-dom$(F, r) \in \sigma$(MeasRect(L-Field, L-Field)). □

(50) Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) × (the real normed space of $\mathbb{R}$)) × (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and an element E of L-Field. Suppose $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$. Then Integral1(ProdMeas(L-Meas, L-Meas), $|\overline{\mathbb{R}}(g)|$) is E-measurable.

PROOF: Set F = Integral1(ProdMeas(L-Meas, L-Meas), $|\overline{\mathbb{R}}(g)|$). Set $F_0 = F{\restriction}K$. Reconsider $G = F_0$ as a partial function from $\mathbb{R}$ to $\mathbb{R}$. $G{\restriction}K$ is bounded and G is integrable on K. Reconsider $R = \mathbb{R}$ as an element of L-Field. F is non-negative. For every real number r, $R \cap$ LE-dom$(F, r) \in$ L-Field. □

(51) Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) × (the real normed space of $\mathbb{R}$)) × (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and an element x of $\mathbb{R}$. Suppose $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$. Then

(i) ProjPMap1(Integral2(L-Meas, $\overline{\mathbb{R}}(g)$), x) is a function from $\mathbb{R}$ into $\mathbb{R}$, and

(ii) ProjPMap1(| Integral2(L-Meas, $\overline{\mathbb{R}}(g)$)|, x) is a function from $\mathbb{R}$ into $\mathbb{R}$.

The theorem is a consequence of (32).

(52) Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and an element y of $\mathbb{R}$. Suppose $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$. Then

(i) $\mathrm{ProjPMap2}(\mathrm{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)), y)$ is a function from $\mathbb{R}$ into $\mathbb{R}$, and

(ii) $\mathrm{ProjPMap2}(|\,\mathrm{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))|, y)$ is a function from $\mathbb{R}$ into $\mathbb{R}$.

The theorem is a consequence of (32).

(53) Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, and a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$. Suppose $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$. Then $|\,\mathrm{Integral1}(\mathrm{ProdMeas}(\text{L-Meas}, \text{L-Meas}), \overline{\mathbb{R}}(g))|$ is a function from $\mathbb{R}$ into $\mathbb{R}$. The theorem is a consequence of (35).

(54) Let us consider an element x of $\mathbb{R}$, non empty, closed interval subsets I, J, K of $\mathbb{R}$, and a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$. Suppose $(I \times J) \times K = \operatorname{dom} g$. Then $\int \mathrm{ProjPMap1}(\mathrm{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)), x) \restriction \mathbb{R} \setminus J \, \mathrm{d}\,\text{L-Meas} = 0$.

(55) Let us consider an element y of $\mathbb{R}$, non empty, closed interval subsets I, J, K of $\mathbb{R}$, and a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$. Suppose $(I \times J) \times K = \operatorname{dom} g$. Then $\int \mathrm{ProjPMap2}(\mathrm{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)), y) \restriction \mathbb{R} \setminus I \, \mathrm{d}\,\text{L-Meas} = 0$.

(56) Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, and a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$. Suppose $(I \times J) \times K = \operatorname{dom} g$. Then $\int \mathrm{Integral1}(\mathrm{ProdMeas}(\text{L-Meas}, \text{L-Meas}), \overline{\mathbb{R}}(g)) \restriction \mathbb{R} \setminus K \, \mathrm{d}\,\text{L-Meas} = 0$.

(57) Let us consider an element x of $\mathbb{R}$, non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function P_1 from $\mathbb{R}$ to $\mathbb{R}$. Suppose $x \in I$ and $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$ and $P_1 = \mathrm{ProjPMap1}(\mathrm{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)), x) \restriction J$. Then P_1 is continuous. The theorem is a consequence of (32) and (34).

(58) Let us consider an element y of $\mathbb{R}$, non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function P_2

from $\mathbb{R}$ to $\mathbb{R}$. Suppose $y \in J$ and $(I\times J)\times K = \operatorname{dom} f$ and f is continuous on $(I\times J)\times K$ and $f = g$ and $P_2 = \text{ProjPMap2}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)), y)\restriction I$. Then P_2 is continuous. The theorem is a consequence of (32) and (34).

(59) Let us consider an element x of $\mathbb{R}$, non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R}\times\mathbb{R})\times\mathbb{R}$ to $\mathbb{R}$, and a partial function P_1 from $\mathbb{R}$ to $\mathbb{R}$. Suppose $x \in I$ and $(I\times J)\times K = \operatorname{dom} f$ and f is continuous on $(I\times J)\times K$ and $f = g$ and $P_1 = \text{ProjPMap1}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)), x)\restriction J$. Then

(i) $P_1 \restriction J$ is bounded, and

(ii) P_1 is integrable on J.

The theorem is a consequence of (32) and (34).

(60) Let us consider an element y of $\mathbb{R}$, non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R}\times\mathbb{R})\times\mathbb{R}$ to $\mathbb{R}$, and a partial function P_2 from $\mathbb{R}$ to $\mathbb{R}$. Suppose $y \in J$ and $(I\times J)\times K = \operatorname{dom} f$ and f is continuous on $(I\times J)\times K$ and $f = g$ and $P_2 = \text{ProjPMap2}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)), y)\restriction I$. Then

(i) $P_2 \restriction I$ is bounded, and

(ii) P_2 is integrable on I.

The theorem is a consequence of (32) and (34).

(61) Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function G_3 from $\mathbb{R}$ to $\mathbb{R}$. Suppose $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$ and $G_3 = \text{Integral1}(\text{ProdMeas}(\text{L-Meas}, \text{L-Meas}), \overline{\mathbb{R}}(g))\restriction K$. Then

(i) $G_3 \restriction K$ is bounded, and

(ii) G_3 is integrable on K.

The theorem is a consequence of (37).

(62) Let us consider an element x of $\mathbb{R}$, non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R}\times\mathbb{R})\times\mathbb{R}$ to $\mathbb{R}$, and a partial function P_1 from $\mathbb{R}$ to $\mathbb{R}$. Suppose $x \in I$ and $(I\times J)\times K = \operatorname{dom} f$ and f is continuous on

$(I\times J)\times K$ and $f = g$ and $P_1 = \text{ProjPMap1}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)), x)\restriction J$. Then

(i) $\text{ProjPMap1}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)), x)\restriction J$ is integrable on L-Meas, and

(ii) $\int\limits_J P_1(x)dx = \int \text{ProjPMap1}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)), x)\restriction J \,\text{d}\,\text{L-Meas}$, and

(iii) $\int\limits_J P_1(x)dx = \int \text{ProjPMap1}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)), x) \,\text{d}\,\text{L-Meas}$, and

(iv) $\int\limits_J P_1(x)dx = (\text{Integral2}(\text{L-Meas}, \text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))))(x)$.

The theorem is a consequence of (46), (59), and (54).

(63) Let us consider an element y of $\mathbb{R}$, non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R}\times\mathbb{R})\times\mathbb{R}$ to $\mathbb{R}$, and a partial function P_2 from $\mathbb{R}$ to $\mathbb{R}$. Suppose $y \in J$ and $(I\times J)\times K = \text{dom}\, f$ and f is continuous on $(I\times J)\times K$ and $f = g$ and $P_2 = \text{ProjPMap2}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)), y)\restriction I$. Then

(i) $\text{ProjPMap2}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)), y)\restriction I$ is integrable on L-Meas, and

(ii) $\int\limits_I P_2(x)dx = \int \text{ProjPMap2}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)), y)\restriction I \,\text{d}\,\text{L-Meas}$, and

(iii) $\int\limits_I P_2(x)dx = \int \text{ProjPMap2}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)), y) \,\text{d}\,\text{L-Meas}$, and

(iv) $\int\limits_I P_2(x)dx = (\text{Integral1}(\text{L-Meas}, \text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))))(y)$.

The theorem is a consequence of (48), (60), and (55).

Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, and a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$. Now we state the propositions:

(64) Suppose $(I \times J) \times K = \text{dom}\, f$ and f is continuous on $(I \times J) \times K$ and $f = g$. Then

(i) for every element U of L-Field, $\text{Integral2}(\text{L-Meas}, \text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)))$ is U-measurable, and

(ii) Integral2(L-Meas, Integral2(L-Meas, $\overline{\mathbb{R}}(g)$)) is integrable on L-Meas, and

(iii) $\int$ Integral2(L-Meas, $\overline{\mathbb{R}}(g)$) d ProdMeas(L-Meas, L-Meas) = $\int$ Integral2(L-Meas, Integral2(L-Meas, $\overline{\mathbb{R}}(g)$)) d L-Meas, and

(iv) $\int g$ d ProdMeas(ProdMeas(L-Meas, L-Meas), L-Meas) = $\int$ Integral2(L-Meas, Integral2(L-Meas, $\overline{\mathbb{R}}(g)$)) d L-Meas, and

(v) Integral2(L-Meas, $\overline{\mathbb{R}}(g)$)$\restriction(I \times J)$ is integrable on ProdMeas(L-Meas, L-Meas), and

(vi) $\int$ Integral2(L-Meas, $\overline{\mathbb{R}}(g)$)$\restriction(I \times J)$ d ProdMeas(L-Meas, L-Meas) = $\int$ Integral2(L-Meas, Integral2(L-Meas, $\overline{\mathbb{R}}(g)$)$\restriction(I \times J)$) d L-Meas.

The theorem is a consequence of (32), (43), (46), (40), and (34).

(65) Suppose $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$. Then

(i) for every element V of L-Field, Integral1(L-Meas, Integral2(L-Meas, $\overline{\mathbb{R}}(g)$)) is V-measurable, and

(ii) Integral1(L-Meas, Integral2(L-Meas, $\overline{\mathbb{R}}(g)$)) is integrable on L-Meas, and

(iii) $\int$ Integral2(L-Meas, $\overline{\mathbb{R}}(g)$) d ProdMeas(L-Meas, L-Meas) = $\int$ Integral1(L-Meas, Integral2(L-Meas, $\overline{\mathbb{R}}(g)$)) d L-Meas, and

(iv) $\int g$ d ProdMeas(ProdMeas(L-Meas, L-Meas), L-Meas) = $\int$ Integral1(L-Meas, Integral2(L-Meas, $\overline{\mathbb{R}}(g)$)) d L-Meas, and

(v) $\int$ Integral2(L-Meas, $\overline{\mathbb{R}}(g)$)$\restriction(I \times J)$ d ProdMeas(L-Meas, L-Meas) = $\int$ Integral1(L-Meas, Integral2(L-Meas, $\overline{\mathbb{R}}(g)$)$\restriction(I \times J)$) d L-Meas.

The theorem is a consequence of (32), (43), (48), (40), and (34).

(66) Let us consider an element x of $\mathbb{R}$, non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function P_1 from $\mathbb{R}$ to $\mathbb{R}$. Suppose $x \in I$ and $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$ and P_1 = ProjPMap1(Integral2(L-Meas, $\overline{\mathbb{R}}(g)$)$\restriction(I \times J)$, x). Then

(i) P_1 is continuous, and

(ii) dom(ProjPMap1(Integral2(L-Meas, $\overline{\mathbb{R}}(g)$)$\restriction(I \times J)$, x)) = J, and

(iii) $P_1 \restriction J$ is bounded, and

(iv) P_1 is integrable on J, and

(v) $\int\limits_J P_1(x)dx = \int \text{ProjPMap1}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))\restriction(I \times J), x)\, \text{d}\,\text{L-}$ Meas, and

(vi) $\int\limits_J P_1(x)dx = (\text{Integral2}(\text{L-Meas}, \text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))\restriction(I \times J)))(x)$, and

(vii) $\text{ProjPMap1}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))\restriction(I \times J), x)$ is integrable on L-Meas.

The theorem is a consequence of (32) and (34).

(67) Let us consider an element y of $\mathbb{R}$, non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function P_2 from $\mathbb{R}$ to $\mathbb{R}$. Suppose $y \in J$ and $(I \times J) \times K = \text{dom}\, f$ and f is continuous on $(I \times J) \times K$ and $f = g$ and $P_2 = \text{ProjPMap2}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))\restriction(I \times J), y)$. Then

(i) P_2 is continuous, and

(ii) $\text{dom}(\text{ProjPMap2}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))\restriction(I \times J), y)) = I$, and

(iii) $P_2\restriction I$ is bounded, and

(iv) P_2 is integrable on I, and

(v) $\int\limits_I P_2(x)dx = \int \text{ProjPMap2}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))\restriction(I \times J), y)\, \text{d}\,\text{L-}$ Meas, and

(vi) $\int\limits_I P_2(x)dx = (\text{Integral1}(\text{L-Meas}, \text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))\restriction(I \times J)))(y)$, and

(vii) $\text{ProjPMap2}(\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))\restriction(I \times J), y)$ is integrable on L-Meas.

The theorem is a consequence of (32) and (34).

(68) Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function G_8 from $\mathbb{R}$ to $\mathbb{R}$. Suppose $(I \times J) \times K = \text{dom}\, f$ and f is continuous on $(I \times J) \times K$ and $f = g$ and $G_8 = \text{Integral2}(\text{L-Meas}, \text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))\restriction(I \times J))\restriction I$. Then

(i) $\text{dom}\, G_8 = I$, and

(ii) G_8 is continuous, and

(iii) $G_8 \restriction I$ is bounded, and

(iv) G_8 is integrable on I, and

(v) Integral2(L-Meas, Integral2(L-Meas, $\overline{\mathbb{R}}(g)) \restriction (I \times J)) \restriction I$ is integrable on L-Meas, and

(vi) $\int$ Integral2(L-Meas, Integral2(L-Meas, $\overline{\mathbb{R}}(g)) \restriction (I \times J)) \restriction I \, \mathrm{d}$ L-Meas $= \int\limits_I G_8(x)dx$, and

(vii) $\int$ Integral2(L-Meas, $\overline{\mathbb{R}}(g)) \restriction (I \times J) \, \mathrm{d}$ ProdMeas(L-Meas, L-Meas) $= \int\limits_I G_8(x)dx$.

The theorem is a consequence of (32) and (34).

(69) Let us consider non empty, closed interval subsets I, J, K of $\mathbb{R}$, a partial function f from ((the real normed space of $\mathbb{R}) \times$ (the real normed space of $\mathbb{R})) \times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function G_7 from $\mathbb{R}$ to $\mathbb{R}$. Suppose $(I \times J) \times K = \operatorname{dom} f$ and f is continuous on $(I \times J) \times K$ and $f = g$ and $G_7 =$ Integral1(L-Meas, Integral2(L-Meas, $\overline{\mathbb{R}}(g)) \restriction (I \times J)) \restriction J$. Then

(i) $\operatorname{dom} G_7 = J$, and

(ii) G_7 is continuous, and

(iii) $G_7 \restriction J$ is bounded, and

(iv) G_7 is integrable on J, and

(v) Integral1(L-Meas, Integral2(L-Meas, $\overline{\mathbb{R}}(g)) \restriction (I \times J)) \restriction J$ is integrable on L-Meas, and

(vi) $\int$ Integral1(L-Meas, Integral2(L-Meas, $\overline{\mathbb{R}}(g)) \restriction (I \times J)) \restriction J \, \mathrm{d}$ L-Meas $= \int\limits_J G_7(x)dx$, and

(vii) $\int$ Integral2(L-Meas, $\overline{\mathbb{R}}(g)) \restriction (I \times J) \, \mathrm{d}$ ProdMeas(L-Meas, L-Meas) $= \int\limits_J G_7(x)dx$.

The theorem is a consequence of (32) and (34).

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