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Elementary Number Theory Problems. Part XIII

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Summary. This paper formalizes problems 41, 92, 121–123, 172, 182, 183, 191, 192 and 192a from "250 Problems in Elementary Number Theory" by Wacław Sierpiński [8].

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Introduction

In this paper, Problems 41 from Section I, 92, 121, 122, 123 from Section IV, 172, 182, 183, 191, 192, and 192a from Section V of [8] are formalized, using the Mizar formalism [2], [1]. The paper is a part of the project *Formalization of Elementary Number Theory in Mizar* [7], [4], [5], [6], [3].

In the preliminary section, we proved some trivial but useful facts about numbers.

In problem 92 the inequality $p_{k+1} + p_{k+2} \leqslant p_1 \cdot p_2 \cdot \ldots \cdot p_k$ should be justified for any integer $k \geqslant 3$, where p_k denotes the k-th prime. Because we count primes starting from the index 0, we formulated the fact as:

```
3 <= k implies
   primenumber(k) + primenumber(k+1) <= Product primesFinS(k);
```

where `primesFinS(k)` denotes the finite sequence of primes of the length `k`, and elements of finite sequences are indexed from 1.

Problem 121 about finding the least positive integer n for which $k \cdot 2^{2^n} + 1$ is composite is represented as separated theorems for every positive $k \leqslant 10$.

Problem 122 requires finding all positive integers $k \leqslant 10$ such that every number $k \cdot 2^{2^n} + 1$ $(n = 1, 2, \dots)$ is composite. The proof lies in the fact that numbers $(3 \cdot t + 2) \cdot 2^{2^n} + 1$ are all divisible by 3 and greater than 3, for every natural t, and every positive natural n. In the book, there are minor misprints in the proof, where $2 \cdot 2^{2^2} + 1$ should be $2 \cdot 2^{2^n} + 1$ and $5 \cdot 2^{2^2} + 1$ should be $5 \cdot 2^{2^n} + 1$.

Problems 191 and 192 are generalized from positive integers to non-zero integers.

Problem 192a is formulated incorrectly in the book. It asks to prove that the system of two equations $x^2 + 7y^2 = z^2$ and $7x^2 + y^2 = t^2$ has no solutions in positive integers x, y, z, and t. However, it has solutions, for instance, $x = 3$, $y = 1$, $z = 4$, and $t = 8$. The example is provided in the book.

Proofs of other problems are straightforward formalizations of solutions given in the book.

1. Preliminaries

From now on a, b, c, k, m, n denote natural numbers, i, j denote integers, and p denotes a prime number.

Now we state the propositions:

(1) If $n < 3$, then $n = 0$ or $n = 1$ or $n = 2$.

(2) If $n < 4$, then $n = 0$ or $n = 1$ or $n = 2$ or $n = 3$.

(3) If $n < 5$, then $n = 0$ or $n = 1$ or $n = 2$ or $n = 3$ or $n = 4$.

Let us note that $\frac{1}{2}$ is non integer and there exists a rational number which is non natural and there exists a rational number which is non integer.

Now we state the proposition:

(4) If $j \neq 0$ and $\frac{i}{j}$ is integer, then $j \mid i$.

Let q be a non integer rational number. One can verify that $q^{\mathbf{2}}$ is non integer. Now we state the proposition:

(5) If $\frac{a}{b} \cdot c$ is natural and $b \neq 0$ and a and b are relatively prime, then there exists a natural number d such that $c = b \cdot d$.

2. PROBLEM 41

Let us consider an integer k. Now we state the propositions:

(6) $2 \cdot k + 1$ and $9 \cdot k + 4$ are relatively prime.

(7) $\gcd(2 \cdot k - 1, 9 \cdot k + 4) = \gcd(k + 8, 17)$.

3. PROBLEM 92

Now we state the proposition:

(8) If $m > 1$ and $n > 1$ and m and n are relatively prime, then there exist prime numbers p, q such that $p \mid m$ and $p \nmid n$ and $q \mid n$ and $q \nmid m$ and $p \neq q$.

Let us consider k. The functor primesFinS(k) yielding a finite sequence of elements of $\mathbb{N}$ is defined by

(Def. 1) $\operatorname{len} it = k$ and for every natural number i such that $i < k$ holds $it(i+1) = \operatorname{pr}(i)$.

Let us observe that primesFinS(0) is empty.

Now we state the propositions:

(9) primesFinS(1) $= \langle 2 \rangle$.

(10) primesFinS(2) $= \langle 2, 3 \rangle$.

(11) primesFinS(3) $= \langle 2, 3, 5 \rangle$.

(12) $p < \operatorname{pr}(k)$ if and only if primeindex(p) $< k$.

(13) If primeindex(p) $< k$, then $1 +$ primeindex(p) $\in$ dom(primesFinS(k)).

(14) If primeindex(p) $< k$, then (primesFinS(k))($1 +$ primeindex(p)) $= p$.

(15) If $p < \operatorname{pr}(k)$, then $p \in \operatorname{rng}$ primesFinS(k). The theorem is a consequence of (13), (12), and (14).

(16) If p and $\prod$ primesFinS(k) are relatively prime, then $\operatorname{pr}(k) \leqslant p$. The theorem is a consequence of (15).

Let us consider k. Let us note that primesFinS(k) is positive yielding and primesFinS(k) is increasing.

Let R be an extended real-valued binary relation. We say that R has values greater or equal one if and only if

(Def. 2) for every extended real r such that $r \in \operatorname{rng} R$ holds $r \geqslant 1$.

Observe that $\langle 1 \rangle$ has values greater or equal one and there exists a natural-valued finite sequence which has values greater or equal one.

Let f be an extended real-valued function. Let us observe that f has values greater or equal one if and only if the condition (Def. 3) is satisfied.

(Def. 3) for every object x such that $x \in \operatorname{dom} f$ holds $f(x) \geqslant 1$.

Let f be an extended real-valued finite sequence. One can verify that f has values greater or equal one if and only if the condition (Def. 4) is satisfied.

(Def. 4) for every natural number n such that $1 \leqslant n \leqslant \operatorname{len} f$ holds $f(n) \geqslant 1$.

One can verify that every extended real-valued binary relation which is empty has also values greater or equal one and every extended real-valued binary relation which has values greater or equal one is also positive yielding.

Now we state the propositions:

(17) If $m \leqslant n$, then $\operatorname{primesFinS}(n) \restriction m = \operatorname{primesFinS}(m)$.

(18) Let us consider extended real-valued binary relations P, R. Suppose $\operatorname{rng} P \subseteq \operatorname{rng} R$ and R has values greater or equal one. Then P has values greater or equal one.

(19) Let us consider extended real-valued finite sequences f, g. Suppose $f \frown g$ has values greater or equal one. Then

(i) f has values greater or equal one, and

(ii) g has values greater or equal one.

(20) Let us consider an extended real r. If $\langle r \rangle$ has values greater or equal one, then $r \geqslant 1$.

Let us consider a real-valued finite sequence f with values greater or equal one. Now we state the propositions:

(21) $\prod f \geqslant 1$.

PROOF: Define $\mathcal{P}$[finite sequence of elements of $\mathbb{R}$] $\equiv$ for every real-valued finite sequence g with values greater or equal one such that $g = \$_1$ holds $\prod \$_1 \geqslant 1$. For every finite sequence p of elements of $\mathbb{R}$ and for every element x of $\mathbb{R}$ such that $\mathcal{P}[p]$ holds $\mathcal{P}[p \frown \langle x \rangle]$. For every finite sequence p of elements of $\mathbb{R}$, $\mathcal{P}[p]$. □

(22) $\prod (f \restriction n) \leqslant \prod f$. The theorem is a consequence of (19) and (20).

Let us consider k. One can verify that $\operatorname{primesFinS}(k)$ has values greater or equal one.

Now we state the proposition:

(23) If $3 \leqslant k$, then $\operatorname{pr}(k) + \operatorname{pr}(k+1) \leqslant \prod \operatorname{primesFinS}(k)$. The theorem is a consequence of (8) and (16).

4. Problem 121

Let k, n be natural numbers. We say that n satisfies Sierpiński Problem 121 for k if and only if

(Def. 5) $k \cdot 2^{2^n} + 1$ is composite and for every positive natural number m such that $m < n$ holds $k \cdot 2^{2^m} + 1$ is not composite.

Now we state the propositions:

(24) 5 satisfies Sierpiński Problem 121 for 1. The theorem is a consequence of (3).

(25) 1 satisfies Sierpiński Problem 121 for 2.

(26) 2 satisfies Sierpiński Problem 121 for 3.

(27) 2 satisfies Sierpiński Problem 121 for 4.

(28) 1 satisfies Sierpiński Problem 121 for 5.

(29) 1 satisfies Sierpiński Problem 121 for 6.

(30) 3 satisfies Sierpiński Problem 121 for 7. The theorem is a consequence of (1).

(31) 1 satisfies Sierpiński Problem 121 for 8.

(32) 2 satisfies Sierpiński Problem 121 for 9.

(33) 2 satisfies Sierpiński Problem 121 for 10.

5. Problem 122

Let us consider a positive natural number n.

Now we state the propositions:

(34) $3 \mid (3 \cdot a + 2) \cdot 2^{2^n} + 1$.

(35) $2 \cdot 2^{2^n} + 1$ is composite.

(36) $5 \cdot 2^{2^n} + 1$ is composite. The theorem is a consequence of (34).

(37) $8 \cdot 2^{2^n} + 1$ is composite. The theorem is a consequence of (34).

(38) Let us consider a positive natural number k. Then $k \leqslant 10$ and for every positive natural number n, $k \cdot 2^{2^n} + 1$ is composite if and only if $k \in \{2, 5, 8\}$. The theorem is a consequence of (24), (26), (27), (30), (32), (33), (35), (36), and (37).

6. Problem 123

Now we state the propositions:

(39) $2^{2^{n+1}} + 2^{2^n} + 1 \geqslant 7$.

(40) If $n > 0$, then $2^{2^{n+1}} + 2^{2^n} + 1 \geqslant 21$.

(41) If $n > 1$, then $2^{2^{n+1}} + 2^{2^n} + 1 \geqslant 273$.

(42) If m is even or $m = 2 \cdot n$, then $2^m \bmod 3 = 1$.
PROOF: Define $\mathcal{P}$[natural number] $\equiv 2^{2\cdot\$_1} \bmod 3 = 1$. For every k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every k, $\mathcal{P}[k]$. □

(43) If m is odd or $m = 2 \cdot n + 1$, then $2^m \bmod 3 = 2$.
PROOF: Define $\mathcal{P}$[natural number] $\equiv 2^{2\cdot\$_1+1} \bmod 3 = 2$. For every k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every k, $\mathcal{P}[k]$. □

(44) Let us consider a non zero natural number n. Then $3 \mid 2^{2^{n+1}} + 2^{2^n} + 1$. The theorem is a consequence of (42).

(45) $7 \mid 2^{2^{n+1}} + 2^{2^n} + 1$. The theorem is a consequence of (42) and (43).

Let n be a non zero natural number. Note that $\frac{1}{3} \cdot (2^{2^{n+1}} + 2^{2^n} + 1)$ is natural. Now we state the proposition:

(46) Let us consider a non zero natural number n. If $n > 1$, then $\frac{1}{3} \cdot (2^{2^{n+1}} + 2^{2^n} + 1)$ is composite. The theorem is a consequence of (39), (45), (44), and (41).

7. Problem 172

Now we state the proposition:

(47) Let us consider positive natural numbers n, x, y, z. Then $n^x + n^y = n^z$ if and only if $n = 2$ and $y = x$ and $z = x + 1$.

8. Problem 182

Now we state the proposition:

(48) Let us consider real numbers a, b, c. If $c > 1$ and $c^a = c^b$, then $a = b$.

Let us consider positive natural numbers n, x, y, z, t. Now we state the propositions:

(49) If $x \leqslant y \leqslant z$, then $n^x + n^y + n^z = n^t$ iff $n = 2$ and $y = x$ and $z = x + 1$ and $t = x + 2$ or $n = 3$ and $y = x$ and $z = x$ and $t = x + 1$.

(50) $n^x + n^y + n^z = n^t$ if and only if $n = 2$ and $y = x$ and $z = x + 1$ and $t = x + 2$ or $n = 2$ and $y = x + 1$ and $z = x$ and $t = x + 2$ or $n = 2$ and $z = y$ and $x = y + 1$ and $t = y + 2$ or $n = 3$ and $y = x$ and $z = x$ and $t = x + 1$. The theorem is a consequence of (49).

9. Problem 183

Now we state the proposition:

(51) Let us consider positive natural numbers x, y, z, t. Then $4^x+4^y+4^z \neq 4^t$.

10. Problem 191

Now we state the proposition:

(52) Let us consider non zero integers x, y, z, t. Then

(i) $x^{\mathbf{2}} + 5 \cdot y^{\mathbf{2}} \neq z^{\mathbf{2}}$, or

(ii) $5 \cdot x^{\mathbf{2}} + y^{\mathbf{2}} \neq t^{\mathbf{2}}$.

11. Problem 192

Now we state the propositions:

(53) Let us consider non zero integers x, y, z, t. Then

(i) $x^{\mathbf{2}} + 6 \cdot y^{\mathbf{2}} \neq z^{\mathbf{2}}$, or

(ii) $6 \cdot x^{\mathbf{2}} + y^{\mathbf{2}} \neq t^{\mathbf{2}}$.

(54) (i) $3^{\mathbf{2}} + 7 \cdot 1^{\mathbf{2}} = 4^{\mathbf{2}}$, and

(ii) $7 \cdot 3^{\mathbf{2}} + 1^{\mathbf{2}} = 8^{\mathbf{2}}$.

References

[1] Grzegorz Bancerek, Czesław Byliński, Adam Grabowski, Artur Korniłowicz, Roman Matuszewski, Adam Naumowicz, Karol Pąk, and Josef Urban. Mizar: State-of-the-art and beyond. In Manfred Kerber, Jacques Carette, Cezary Kaliszyk, Florian Rabe, and Volker Sorge, editors, *Intelligent Computer Mathematics*, volume 9150 of *Lecture Notes in Computer Science*, pages 261–279. Springer International Publishing, 2015. ISBN 978-3-319-20614-1. doi:10.1007/978-3-319-20615-8_17.

[2] Grzegorz Bancerek, Czesław Byliński, Adam Grabowski, Artur Korniłowicz, Roman Matuszewski, Adam Naumowicz, and Karol Pąk. The role of the Mizar Mathematical Library for interactive proof development in Mizar. *Journal of Automated Reasoning*, 61(1):9–32, 2018. doi:10.1007/s10817-017-9440-6.

[3] Adam Grabowski. Elementary number theory problems. Part XII – primes in arithmetic progression. *Formalized Mathematics*, 31(1):277–286, 2023. doi:10.2478/forma-2023-0022.

[4] Artur Korniłowicz. Elementary number theory problems. Part IX. *Formalized Mathematics*, 31(1):161–169, 2023. doi:10.2478/forma-2023-0015.

[5] Artur Korniłowicz. Elementary number theory problems. Part X – Diophantine equations. *Formalized Mathematics*, 31(1):171–180, 2023. doi:10.2478/forma-2023-0016.

[6] Adam Naumowicz. Elementary number theory problems. Part XI. *Formalized Mathematics*, 31(1):229–275, 2023. doi:10.2478/forma-2023-0021.

[7] Adam Naumowicz. Dataset description: Formalization of elementary number theory in Mizar. In Christoph Benzmüller and Bruce R. Miller, editors, *Intelligent Computer Mathematics – 13th International Conference, CICM 2020, Bertinoro, Italy, July 26–31, 2020, Proceedings*, volume 12236 of *Lecture Notes in Computer Science*, pages 303–308. Springer, 2020. doi:10.1007/978-3-030-53518-6_22.

[8] Wacław Sierpiński. *250 Problems in Elementary Number Theory*. Elsevier, 1970.

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