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GOODNESS-OF-FIT TESTS FOR DOUBLY TRUNCATED DATA[1]

Summary

Purpose – The paper regards the goodness-of-fit (GOF) tests for doubly truncated continuous data with known truncation points. The first goal of the paper is to derive computing formulas of several test statistics for doubly truncated data, when the number of truncated data is unknown. The second goal is to develop statistical inference procedure based on the derived formulas, which includes information regarding the number of truncated data, when it is available.

Research method – The formulas and the inference procedure are developed with the use of the methods proposed by Chernobai, Rachev and Fabozzi [2015], who already developed GOF tests for the left truncated data, when the number of truncated data is unknown.

Results – Several tests are developed in case of double truncation. Depending on the chosen truncation points, the tests for left, right or doubly truncated samples might be obtained. When no truncation occurs, the tests are reduced to the complete sample tests. The quality of the tests is assessed on the basis of the FTSE100 return distributions.

Originality/value/implications/recommendations – To the best knowledge of the author, computing formulas of the GOF test statistics for doubly truncated distributions with known truncation points, when the number of truncated data is unknown, have not been presented in the literature yet.

Keywords: goodness-of-fit, return distribution, truncated distribution

JEL classification: C12, C14, C15, C24, G12

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1. Introduction

The possible range of applications of doubly truncated data is wide, including economics, medicine, astronomy and engineering. This article is about the goodness-of-fit (GOF) tests for doubly truncated continuous data with known truncation points. The tests considered in this article are in particular the one-sample tests and they are based on an empirical cumulative distribution function (CDF). The tests presented in this paper were developed with the use of the method proposed by Chernobai, Rachev and Fabozzi [2015] and should be perceived as the generalization of the left truncated GOF tests proposed there.

Referring to the literature, it is worth to distinguish two types of the GOF tests for truncated data: when the number of truncated data is known and when it is unknown. The GOF tests for truncated data, when the number of truncated data is known, might be used to determine the goodness-of-fit in the certain proportion of the random sample. Such modifications were already considered in the literature for the Kolmogorov-Smirnov statistic [Barr, Davidson, 1973; Dufour, Maag, 1978] and the statistics of Cramér-von Mises, Anderson-Darling and Watson [Pettitt, Stephens, 1976]. The GOF tests for truncated data, when the number of truncated data is unknown, were considered only for the left truncated case [Chernobai, Rachev, Fabozzi, 2015]. In the aforementioned paper, one can find exact formulas for the statistics of Kolmogorov-Smirnov, Kuiper, Cramér-von Mises and Anderson-Darling. The Anderson-Darling statistic is presented there in two versions (supremum and quadratic) and with two weighing functions (one giving more importance to both tails and the other one giving more importance to the upper tail of a distribution). Both of the above mentioned concepts have certain limitations. In the first concept, when the number of truncated data is known, the range of available statistics is confined and there are no statistics dedicated to test the goodness-of-fit in the chosen tail of a distribution. The main disadvantage of the second approach, when the number of truncated data is unknown, is the possibility to test the goodness-of-fit only in the left truncated distributions. The mentioned constraints and the research carried out on the GOF tests for the left truncated distributions [Echaust, Lach, 2017; Lach, Smaga, 2018] became the inspiration for the doctoral dissertation, parts of which are presented in this article. The intention of the research was to create a flexible tool to test the goodness-of-fit of any selected part of an unconditional distribution and to examine its application values. The most important part of this research regarded the development of the GOF test statistics for the doubly truncated data, when the number of truncated data is unknown.

The paper is organized as follows. Firstly, the complete sample GOF test statistics, which are the subject of the modifications, are presented. Next, the modified test statistics and the inference procedure for the doubly truncated samples are introduced. The third section contains the data overview as well as the distributions used later in the study. In the next section, the simulations are conducted to examine the finite sample performance of the tests. The last section contains conclusions. The derivation of the chosen test statistic for the doubly truncated case might be found in the Appendix 1.

2. Goodness-of-fit tests for complete samples

The test statistics for the complete distributions (Table 1), the modifications of which are presented in this paper, are both Kolmogorov–Smirnov type and Cramér–von Mises type. Following Chernobai, Rachev and Fabozzi [2015], the first ones are also described as supremum, and the second ones as quadratic.

The Kolmogorov-Smirnov statistic is more sensitive in the central part of the distribution. The Anderson–Darling statistic, depending on the weighing function, provides more weight to both tails of the distribution, to the upper tail [Sinclair, Spurr, Ahmad, 1987] or to the lower tail [Ahmad, Sinclair, Spurr, 1988].

TABLE 1

General formulas of the test statistics for the complete samples

Type	Statistic	General formula	
Supremum	Kolmogorov–Smirnov	$KS = \sqrt{n}\sup_x \left\| F_n(x) - F_0(x) \right\|$	(1)
	Kuiper	$V = \sqrt{n}\left(\sup_x \left\{ F_n(x) - F_0(x) \right\} + \sup_x \left\{ F_0(x) - F_n(x) \right\} \right)$	(2)
	Anderson–Darling	$AD = \sqrt{n}\sup_x \left\| \frac{F_n(x) - F_0(x)}{\sqrt{F_0(x)\left(1 - F_0(x)\right)}} \right\|$	(3)

Type	Statistic	General formula	
Supremum [cont.]	Anderson–Darling for the upper tail	$AD_{up} = \sqrt{n} \sup_x \left\| \frac{F_n(x) - F_0(x)}{1 - F_0(x)} \right\|$	(4)
	Anderson–Darling for the lower tail	$AD_{down} = \sqrt{n} \sup_x \left\| \frac{F_n(x) - F_0(x)}{F_0(x)} \right\|$	(5)
Quadratic	Cramér–von Mises	$W^2 = n \int \left(F_n(x) - F_0(x)\right)^2 dF_0(x)$	(6)
	Anderson–Darling	$AD^2 = n \int \frac{\left(F_n(x) - F_0(x)\right)^2}{F_0(x)\left(1 - F_0(x)\right)} dF_0(x)$	(7)
	Anderson–Darling for the upper tail	$AD_{up}^2 = n \int \frac{\left(F_n(x) - F_0(x)\right)^2}{\left(1 - F_0(x)\right)^2} dF_0(x)$	(8)
	Anderson–Darling for the lower tail	$AD_{down}^2 = n \int \frac{\left(F_n(x) - F_0(x)\right)^2}{\left(F_0(x)\right)^2} dF_0(x)$	(9)

Symbols: F_n (*x*) – the empirical cumulative distribution function (ECDF); F_0 (*x*) – the theoretical cumulative distribution function; *n* – the sample size.

Source: the author's own elaboration on the basis of: [Chernobai, Rachev, Fabozzi, 2015: 586].

In the GOF testing, the null and alternative hypotheses are usually formulated as: $H_0 : F \in F_0$ and $H_1 : F \notin F_0$. The theoretical distribution function F_0 might depend on the parameters that are estimated from the sample, which is particularly common when verifying hypotheses concerning financial markets. In such cases, when hypotheses are composite, the distributions of the test statistics are usually unknown and must be approximated. Let's assume, that $\mathbf{X} = \left(X_1, \ldots, X_n\right)'$ denote an *n*-element sample of i.i.d. random variables with an unknown cumulative distribution function *F*. A bootstrap algorithm for testing aforementioned hypotheses might be as follows (Algorithm 1):

1. For the n-element sample $\boldsymbol{X} = \left(X_1, \ldots, X_n\right)'$:
   1.1. Estimate the parameters $\hat{\theta}$ of $\hat{F}_0$.
   1.2. Compute the test statistic T_0.
2. Generate bootstrap samples from $\hat{F}_0$: $\boldsymbol{X}^{(i)} = \left(X_1^{(i)}, \ldots, X_n^{(i)}\right)$, where $i = 1, \ldots, B$.
3. For each bootstrap sample: $\boldsymbol{X}^{(i)} = \left(X_1^{(i)}, \ldots, X_n^{(i)}\right)$:
   3.1. Estimate the parameters of $\hat{\theta}^{(i)}$ of F_0.
   3.2. Compute the test statistic T_i.
4. Compute the p-value according to the formula: $$\hat{p} = \frac{\#\{1 \le i \le B : T_i > T_0\} + 1}{B + 1}. \quad (10)$$
5. Reject the null hypothesis if the $\hat{p}$-value is less than or equal to the nominal significance level α.

Goodness-of-fit tests for doubly truncated samples

This section presents modifications of the test statistics (1)–(9). Chernobai, Rachev and Fabozzi [2015] had already derived the computational formulas for the test statistics (1)–(4) and (6)–(8), when the sample is left truncated and the number of truncated data is unknown. In this article, the computational formulas for all the mentioned test statistics (1)–(9) are derived, when the sample is doubly truncated and the number of truncated data is unknown. The formulas were derived using the method adopted by Chernobai, Rachev and Fabozzi [2015], therefore they should be considered as the generalization of the left truncated data tests presented therein. The statistics were derived under the assumptions that the truncation points and the sample size are not random variables.

Firstly, from the doubly truncated perspective, the ECDF and the theoretical CDF are defined. Next, the computing formulas for the test statistics (1)–(9), when the sample is doubly truncated, are presented. Finally, the statistical inference procedure to test the GOF in case of double truncation is introduced. This procedure includes the information regarding the number of truncated data, because it is available in case of further analyzed financial data, i.e. the FTSE100 percentage log-returns. The statistical inference procedure, when the number of truncated data is unknown, has already been presented in the literature for the left-truncated case. This procedure, with slight changes, can also be applied with the statistics for double-truncated distributions presented in this paper.

In the remainder of the article, the observations are denoted as $\mathbf{x} = \left(x_1, \ldots, x_n\right)'$ and the ordered sample of observations as $x_{(1)} \le x_{(2)} \le \ldots \le x_{(n)}$. The truncation points are indicated as $H1$ and $H2$. The test statistics considered in this paper are based on the ECDF. The definition of the ECDF for a doubly truncated sample is the same, as the ECDF for a complete sample and is defined as follows:

$$F_n\left(x\right) = \begin{cases} 0 & x < x_{(1)}, \\ \dfrac{j}{n} & x_{(j)} \le x < x_{(j+1)}, j = 1, 2, \ldots, n-1 \\ 1 & x \ge x_{(n)}. \end{cases} \tag{11}$$

To construct the test statistics, the definition of the theoretical CDF for the doubly truncated sample is also needed. Let F_0 denote the theoretical CDF for the complete sample. Furthermore, let z_{H1} denote the value of the theoretical CDF at $H1$ and z_{H2} the value of the theoretical CDF at $H2$, i.e. $z_{H1} = F_0(H1)$ and $z_{H2} = F_0(H2)$, where $H1 < H2$. The theoretical CDF for the doubly truncated sample is as follows:

$$F_0^{**}\left(x\right) = \begin{cases} 0 & x \le H1, \\ \dfrac{F_0\left(x\right) - F_0\left(H1\right)}{F_0\left(H2\right) - F_0\left(H1\right)} & H1 < x \le H2, \\ 1 & x > H2. \end{cases} \tag{12}$$

Under the null hypothesis, we have
$F_0\left(X\right) \sim U\left[F_0\left(H1\right), F_0\left(H2\right)\right]$ and $F_0^{**}\left(x\right) \sim U\left[0,1\right]$.

The test statistics for the doubly truncated samples were derived with the use of equations (11) and (12), their general and computing formulas are presented in Table 2. Appendix 1 contains derivation of the chosen statistic (AD^{**}), derivation of the remaining statistics is available upon request. The statistics are reduced to the complete sample statistics when no truncation occurs (i.e. when $z_{H1} = 0$ and $z_{H2} = 1$). For $z_{H1} > 0$ and $z_{H2} = 1$, the statistics for the left truncated samples are obtained. For $z_{H1} = 0$ and $z_{H2} < 1$, the statistics for the right truncated samples are obtained. Finally, for $z_{H1} > 0$ and $z_{H2} < 1$, the statistics for the doubly truncated samples occurs.

TABLE 2

General and computing formulas of the test statistics for the doubly truncated samples

Symbol	General and computing formulas
KS^{**}	$KS^{**} = \sqrt{n}\sup_{x}\left\lvert F_n(x) - \hat{F}_0^{**}(x)\right\rvert$
	$KS^{**} = \frac{\sqrt{n}}{z_{H2}-z_{H1}}\max\left\{\sup_{j}\left\{z_{H1}+\frac{j}{n}(z_{H2}-z_{H1})-z_j\right\},\sup_{j}\left\{z_j-\left(z_{H1}+\frac{j-1}{n}(z_{H2}-z_{H1})\right)\right\}\right\}$
V^{**}	$V^{**} = \sqrt{n}\left(\sup_{x}\left\{F_n(x)-\hat{F}_0^{**}(x)\right\}+\sup_{x}\left\{\hat{F}_0^{**}(x)-F_n(x)\right\}\right)$
	$V^{**} = \frac{\sqrt{n}}{z_{H2}-z_{H1}}\left(\sup_{j}\left\{z_{H1}+\frac{j}{n}(z_{H2}-z_{H1})-z_j\right\}+\sup_{j}\left\{z_j-\left(z_{H1}+\frac{j-1}{n}(z_{H2}-z_{H1})\right)\right\}\right)$
AD^{**}	$AD^{**} = \sqrt{n}\sup_{x}\left\lvert\frac{F_n(x)-\hat{F}_0^{**}(x)}{\sqrt{\hat{F}_0^{**}(x)\left(1-\hat{F}_0^{**}(x)\right)}}\right\rvert$
	$AD^{**} = \sqrt{n}\max\left\{\sup_{j}\left\{\frac{z_{H1}+\frac{j}{n}(z_{H2}-z_{H1})-z_j}{\sqrt{(z_j-z_{H1})(z_{H2}-z_j)}}\right\},\sup_{j}\left\{\frac{z_j-\left(z_{H1}+\frac{j-1}{n}(z_{H2}-z_{H1})\right)}{\sqrt{(z_j-z_{H1})(z_{H2}-z_j)}}\right\}\right\}$
AD_{up}^{**}	$AD_{up}^{**} = \sqrt{n}\sup_{x}\left\lvert\frac{F_n(x)-\hat{F}_0^{**}(x)}{1-\hat{F}_0^{**}(x)}\right\rvert$
	$AD_{up}^{**} = \sqrt{n}\max\left\{\sup_{j}\left\{\frac{z_{H1}+\frac{j}{n}(z_{H2}-z_{H1})-z_j}{z_{H2}-z_j}\right\},\sup_{j}\left\{\frac{z_j-\left(z_{H1}+\frac{j-1}{n}(z_{H2}-z_{H1})\right)}{z_{H2}-z_j}\right\}\right\}$

AD^{**}_{down}	$AD^{**}_{down} = \sqrt{n}\sup_x \left\lvert \frac{F_n(x) - \hat{F}_0^{**}(x)}{\hat{F}_0^{**}(x)} \right\rvert$
	$AD^{**}_{down} = \sqrt{n}\ \max\left\{\sup_j\left\{\frac{z_{H1}+\frac{j}{n}(z_{H2}-z_{H1})-z_j}{z_j-z_{H1}}\right\},\sup_j\left\{\frac{z_j-\left(z_{H1}+\frac{j-1}{n}(z_{H2}-z_{H1})\right)}{z_j-z_{H1}}\right\}\right\}$
W^{2**}	$W^{2**} = n\int_{H1}^{H2}\left(F_n(x)-\hat{F}_0^{**}(x)\right)^2 d\hat{F}_0^{**}(x)$
	$W^{2**} = \frac{n}{3}+\frac{nz_{H1}}{(z_{H2}-z_{H1})}+\frac{1}{n(z_{H2}-z_{H1})}\sum_{j=1}^{n}(1-2j)z_j+\frac{1}{(z_{H2}-z_{H1})^2}\left(\sum_{j=1}^{n}(z_j-z_{H1})^2\right)$
AD^{2**}	$AD^{2**} = n\int_{H1}^{H2}\frac{\left(F_n(x)-\hat{F}_0^{**}(x)\right)^2}{\hat{F}_0^{**}(x)\left(1-\hat{F}_0^{**}(x)\right)}d\hat{F}_0^{**}(x)$
	$AD^{2**} = -n+2n\ln(z_{H2}-z_{H1})+\frac{1}{n}\sum_{j=1}^{n}(1-2j)\ln(z_j-z_{H1})-\frac{1}{n}\sum_{j=1}^{n}(1+2(n-j))\ln(z_{H2}-z_j)$
AD^{2**}_{up}	$AD^{**}_{up} = n\int_{H1}^{H2}\frac{\left(F_n(x)-\hat{F}_0^{**}(x)\right)^2}{\left(1-\hat{F}_0^{**}(x)\right)^2}d\hat{F}_0^{**}(x)$
	$AD^{**}_{up} = -2n\ln(z_{H2}-z_{H1})+2\sum_{j=1}^{n}\ln(z_{H2}-z_j)+\frac{(z_{H2}-z_{H1})}{n}\sum_{j=1}^{n}(1+2(n-j))\frac{1}{z_{H2}-z_j}$
AD^{2**}_{down}	$AD^{2**}_{down} = n\int_{H1}^{H2}\frac{\left(F_n(x)-\hat{F}_0^{**}(x)\right)^2}{\left(\hat{F}_0^{**}(x)\right)^2}d\hat{F}_0^{**}(x)$
	$AD^{2**}_{down} = -2n\ln(z_{H2}-z_{H1})+2\sum_{j=1}^{n}\ln(z_j-z_{H1})+\frac{(z_{H2}-z_{H1})}{n}\sum_{j=1}^{n}(1-2j)\frac{1}{z_{H1}-z_j}$

Symbols: $F_n(x)$ – the empirical cumulative distribution function (ECDF) for the doubly truncated sample; $F_0^{**}(x)$ – the theoretical cumulative distribution function for the doubly truncated sample; n – the sample size; $H1$, $H2$ – the truncation points; z_{H1}, z_{H2} – values of the theoretical CDF at $H1$ and $H2$ respectively.

Source: the author's own work.

The null distributions of the test statistics presented above are unknown and must be estimated. The null and alternative hypotheses of the GOF test, in case of double truncation, are formulated as: $H_0 : F^{**} \in F_0^{**}$ and $H_1 : F^{**} \notin F_0^{**}$, where "**" denotes double truncation. To determine the goodness-of-fit in the certain proportion of the random sample, following bootstrap algorithm is proposed (Algorithm 2):

1. For the *n*-element sample $\boldsymbol{X} = \left(X_1, \ldots, X_n\right)'$:
   1.1. Estimate the parameters $\hat{\theta}$ of F_0, where F_0 denotes the CDF of the complete distribution.
   1.2. Determine truncation points $z_{H1} = \hat{F}_0\left(H1\right)$ and $z_{H2} = \hat{F}_0\left(H2\right)$. Count the number of observations in the following intervals: $(-\infty, H1]$, $(H1, H2]$, $(H2, \infty)$ and denote them as n_1, n_2, n_3. Since now z_{H1}, z_{H2}, n_1, n_2 and n_3 are fixed.
   1.3. Compute the chosen test statistic T_0^{**} for the doubly truncated sample.
2. Generate *B* bootstrap samples from $\hat{F}_0$: $\boldsymbol{X}^{(i)} = \left(X_1^{(i)}, \ldots, X_n^{(i)}\right)$, where $i = 1, \ldots, B$. In every bootstrap sample, the number of observations in the intervals $(\hat{F}_0^{-1}\left(0\right), \hat{F}_0^{-1}\left(z_{H1}\right)], (\hat{F}_0^{-1}\left(z_{H1}\right), \hat{F}_0^{-1}\left(z_{H2}\right)], (\hat{F}_0^{-1}\left(z_{H2}\right), \hat{F}_0^{-1}\left(1\right)]$ is respectively n_1, n_2 and n_3.
3. For each bootstrap sample $\boldsymbol{X}^{(i)} = \left(X_1^{(i)}, \ldots, X_n^{(i)}\right)$,
   3.1. Estimate the parameters $\hat{\theta}^{(i)}$ of F_0 (for the complete sample).
   3.2. Compute the test statistic T_i^{**} (for the doubly truncated sample).
4. Compute the p-value according to the formula:

$$\hat{p} = \frac{\#\{1 \leq i \leq B : T_i^{**} > T_0\} + 1}{B + 1}. \tag{13}$$

5. Reject the null hypothesis, if the $\hat{p}$-value is less than or equal to the nominal significance level α.

4. Data and return distributions

The next section is based on the daily percentage log-returns of the FTSE100 closing prices. The time scope embraces two following years, 2019 and 2020. The FTSE100 closing prices and percentage log-returns based on them differ strongly between the 2019 and 2020 year, as the first one indicates the pre-pandemic and the second one pandemic period (Table 3 and Chart 1).

TABLE 3

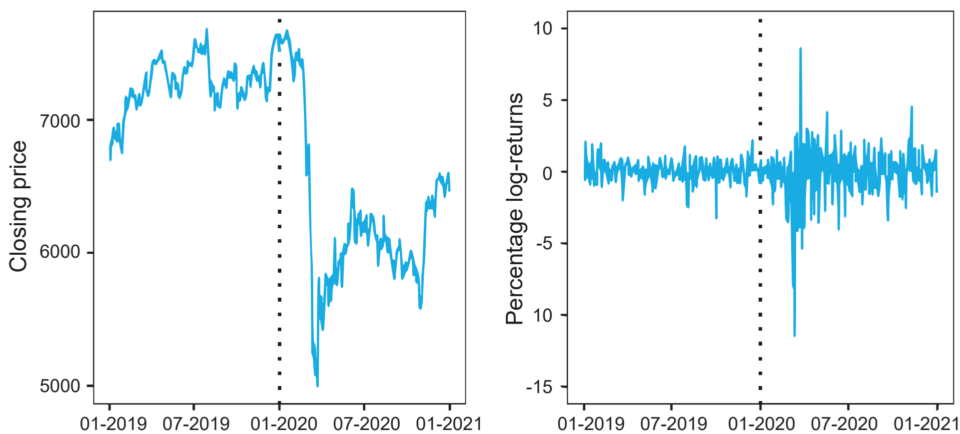
Descriptive statistics of the daily percentage log-returns of the FTSE100

Year	n	Min	Max	Arithmetic mean	Standard deviation	Skewness	Excess kurtosis
2019	253	-3,2839	2,2272	0,0452	0,7397	-0,4402	2,1702
2020	254	-11,5124	8,6668	-0,0610	1,8584	-0,9965	7,8597

Source: the author's own work.

CHART 1

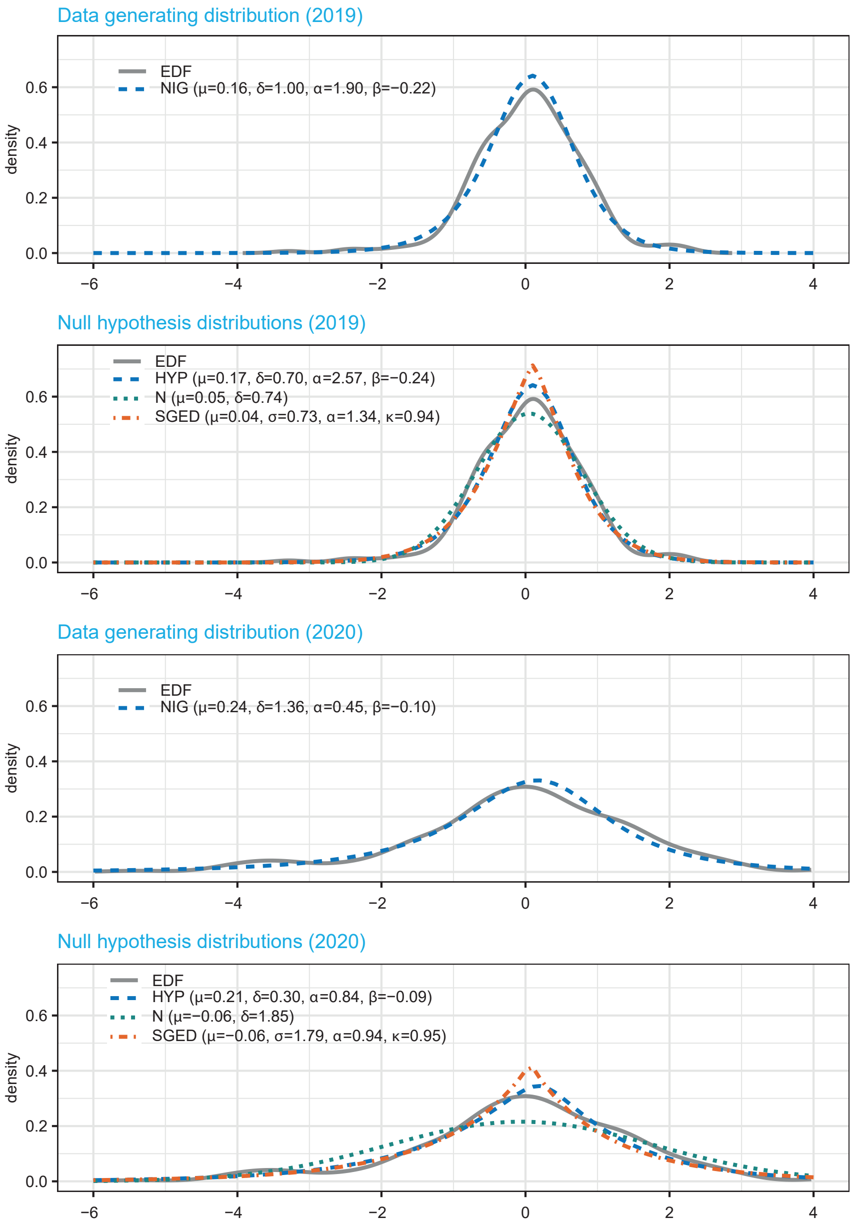
Closing prices and percentage log-returns of the FTSE100



Source: the author's own work.

CHART 2

The data generating and null hypothesis distributions fitted to the FTSE100 daily percentage log-returns



Source: the author's own work.

The distributions of the daily returns have fat tails, they are leptokurtic and asymmetric, therefore they should have at least four parameters [Cont, 2001]. Thus the chosen distributions are: the hyperbolic distribution (HYP), the normal inverse Gaussian distribution (NIG) and the skewed generalized error distribution (SGED). The two-parameter normal distribution (N) has also been included in the research for comparative purposes. The HYP and the NIG distributions, both being the subclasses of the broader generalized hyperbolic family, have the same four parameters μ, δ, α and β, describing, respectively, the location, the scale, the kurtosis and the skewness [Prause, 1999]. The SGED distribution is presented here in the parametrization of Ayebo and Kozubowski [2003], with the parameters μ, σ β and $\varkappa$ depicting the location, the scale, the shape and the skewness. The normal distribution has the location parameter μ and the scale parameter δ.

The empirical probability density function of the log-returns and the fitted distributions, broken down by years, are presented in Chart 2. In each analyzed year the data generating distribution and the null hypothesis distributions are distinguished. The data are generated from the NIG distribution. Its' parameters are estimated upon daily percentage log-returns of the FTSE100, separately calculated for the 2019 and 2020 year. In the null hypothesis, four aforementioned distributions were considered: HYP, N, NIG and SGED. As the NIG and HYP distributions are very close, the NIG distribution is not included on the null hypothesis distributions' figures.

5. Simulation study

In order to analyze the finite sample behavior of the developed tests, broad Monte Carlo simulations were carried out. Simulations were conducted in accordance with the following algorithm (Algorithm 3):

1. Determine the data generating distribution F and its parameters.
2. Generate Monte Carlo trials from $F : \boldsymbol{X}^{(j)} = \left(X_1^{(j)}, \ldots, X_n^{(j)}\right)$, where $j = 1, \ldots, M$.
3. For each Monte Carlo trail $\boldsymbol{X}^{(j)} = \left(X_1^{(j)}, \ldots, X_n^{(j)}\right)$ verify the null hypothesis $H_0 : F^{**} \in F_0^{**}$ using Algorithm 2.
4. Count the number of rejections and divide it by M.

When estimating the empirical levels of the tests, the data generating distribution F and the theoretical distribution included in the null hypothesis F_0^{**} are the same. Here the empirical size is presented only for the NIG distribution. When estimating the empirical powers, F and F_0^{**} are different. As was already mentioned in the previous section, the article presents the results when the data generating distribution is NIG and the null hypothesis distributions are HYP, N or SGED.

There are no clear indications what the right number of simulations is [Martinez, Martinez, 2008]. Stute, Manteiga and Quindimil [1993], for the samples of size n = 20, 50, 100 and two-parameter data generating distributions, set the number of Monte Carlo trials M to 1000 and the number of bootstrap replications B to 500. Years later, despite the significant growth of the computing power, Pawsey [2018] assumed that M = 1000 and B = 199 for the samples of size n = 20, 50, 100 and four-parameter data generating distributions. In the conducted research, due to the available computational resources, it was assumed that M = 1000 and B = 100. It should be emphasized that the simulations were conducted for the samples of size n = 100, 200, 300, 400, 500 and 1000. All the tests were carried out at the 5% level of significance. The results are shown in Table 4.

The empirical levels of the tests are emphasized in italic in Table 4. The empirical levels are close to the 5% significance level in most of the cases, but it can also be noticed that the tests for the central part of the distribution are conservative. However, according to Tollenaar and Mooijaart [2003], conservative tests with reasonable power are still useful. The empirical powers of the tests are presented in regular font in Table 4. As was expected, the rejection rates increase with the rise of the sample size. It can also be noticed that the empirical powers of the tests are low, when the parameters of the data generating distribution are estimated upon the prepandemic year 2019. During the dynamic year 2020 the distributions became more recognizable and the empirical powers of the tests increased. Generally, the tests for the complete distributions are more powerful than the tests for the selected parts of the distribution, but they do not indicate where the biggest differences appear. The introduced modified tests make it easy to notice it. The difference between the NIG and the HYP distributions appears mainly in the lower tail. The discrepancy between the NIG and the N distribution originates from the lower tail and from the central part of the distribution. Finally, the difference between the NIG and the SGED distributions become visible in the central part of the distributions. One might also observe that different tests are suitable for different parts of the distribution. Regarding the central part of the distribution, the tests based on the V^{**} statistic are generally more powerful than other considered tests. Concerning the tails of the distribution, the tests based on the $AD_{up/down}^{**}$ and $AD_{up/down}^{2**}$ statistics usually have more power than other tests. Finally, for the complete distribution, the quadratic versions of the statistics seem to be more powerful than their supremum versions.

TABLE 4

Empirical sizes and powers of the tests for the complete and doubly truncated samples (in %)

Theoretical distribution	Truncation	n	True distribution													
			NIG (0.16; 1.00; 1.90; -0.22)							NIG (0.24; 1.36; 0.45; -0.10)						
			Test statistic													
			[1]	[2]	[3]	[4]	[5]	[6]	[7]	[1]	[2]	[3]	[4]	[5]	[6]	[7]
HYP	[0, 1]	100	3,8	3,1	6,8	–	3,2	2,8	–	7,0	6,6	13,9	–	7,0	8,7	–
		200	4,8	5,5	6,7	–	4,0	4,0	–	7,5	7,2	15,6	–	8,4	8,9	–
		300	3,8	5,6	6,7	–	4,2	4,9	–	9,0	8,2	20,0	–	10,4	12,5	–
		400	5,0	5,3	6,1	–	4,3	3,9	–	11,1	10,7	20,7	–	12,0	16,9	–
		500	4,3	4,8	9,2	–	5,3	5,2	–	12,3	14,1	25,6	–	16,3	20,4	–
		1000	6,3	6,1	7,4	–	6,0	6,0	–	24,8	24,1	29,2	–	30,4	39,6	–
	[0, 0.05]	5	4,3	5,2	5,2	5,8	3,7	4,0	5,1	6,0	5,3	11,3	16,7	7,4	8,1	17,8
		10	4,2	3,6	5,6	6,4	4,4	3,9	6,4	8,5	6,3	13,4	17,2	9,7	11,6	20,4
		15	4,7	4,2	4,9	5,0	4,7	3,9	5,6	9,0	7,0	19,4	24,2	9,8	12,4	27,4
		20	4,6	4,0	6,5	6,1	4,9	4,9	6,4	9,8	5,5	18,9	22,6	11,0	14,7	26,3
		25	5,1	5,2	5,7	6,9	4,1	4,4	6,7	9,5	7,6	19,7	25,7	10,6	14,8	31,7
		50	5,3	6,0	7,4	9,0	5,2	5,8	8,8	17,8	10,3	27,7	31,5	20,5	28,4	45,2
	[0.05, 0.95]	90	2,5	3,6	3,4	–	2,7	3,4	–	4,1	4,4	5,3	–	5,3	4,5	–
		180	4,3	4,5	3,8	–	3,7	4,2	–	6,0	5,7	4,7	–	5,8	5,2	–
		270	4,4	4,6	4,8	–	3,2	4,2	–	5,3	7,3	5,2	–	5,2	6,5	–
		360	3,2	4,1	5,1	–	3,8	3,6	–	7,6	8,1	3,5	–	7,4	8,9	–
		450	3,5	4,4	4,6	–	3,8	4,2	–	7,5	8,5	4,0	–	7,0	9,5	–
		900	5,4	5,4	4,6	–	4,1	4,1	–	16,4	18,6	4,0	–	16,3	18,8	–
	[0.95, 1]	5	4,0	4,6	3,7	2,8	3,0	3,4	2,9	6,2	6,8	6,6	8,0	6,3	5,6	7,8
		10	4,4	4,3	5,3	4,3	4,1	4,4	3,8	5,0	5,0	7,9	9,5	4,2	4,9	9,7
		15	4,1	3,9	5,8	5,4	3,8	3,9	5,2	5,2	5,4	9,7	10,4	5,6	6,1	11,2
		20	3,5	4,3	4,4	4,6	3,6	3,6	4,3	5,1	5,7	11,2	13,0	5,2	6,3	14,5
		25	4,0	6,2	6,4	5,8	4,5	4,3	6,2	4,6	5,0	10,7	13,6	4,5	4,9	13,9
		50	4,2	3,9	5,7	6,1	4,0	3,9	5,9	6,5	4,4	17,0	17,7	7,0	9,1	20,9

Theoretical distribution	Truncation	n	True distribution													
			NIG (0.16; 1.00; 1.90; -0.22)							NIG (0.24; 1.36; 0.45; -0.10)						
			Test statistic													
			[1]	[2]	[3]	[4]	[5]	[6]	[7]	[1]	[2]	[3]	[4]	[5]	[6]	[7]
N	[0, 1]	100	22,3	27,0	35,2	–	30,5	33,5	–	74,6	81,6	68,4	–	85,7	87,4	–
		200	40,5	49,2	47,6	–	53,5	57,8	–	94,9	97,0	82,3	–	98,2	98,8	–
		300	55,6	64,1	58,9	–	72,7	76,9	–	99,7	99,9	92,5	–	100,0	100,0	–
		400	66,3	76,4	67,9	–	82,3	87,3	–	99,9	100,0	97,2	–	100,0	100,0	–
		500	78,2	85,4	72,1	–	90,0	93,3	–	100,0	100,0	98,8	–	100,0	100,0	–
		1000	97,1	99,0	86,4	–	99,6	99,7	–	100,0	100,0	99,9	–	100,0	100,0	–
	[0, 0.05]	5	11,1	5,8	29,6	34,6	12,4	19,3	36,5	24,1	11,9	59,2	64,9	28,0	50,3	69,0
		10	16,9	11,2	43,4	50,3	19,6	32,6	54,9	45,4	25,9	80,9	85,1	52,4	77,7	89,9
		15	26,0	14,4	52,3	58,6	30,3	47,3	67,8	60,9	40,1	92,2	92,7	66,1	88,3	96,6
		20	31,5	18,5	58,3	64,3	36,5	55,4	74,7	75,8	54,4	95,9	96,5	78,5	94,7	99,3
		25	38,9	22,9	66,0	70,5	45,4	63,9	81,9	83,4	64,6	97,2	97,4	85,1	97,2	99,2
		50	66,2	44,0	82,0	83,8	71,0	88,4	94,3	98,3	93,9	99,9	100,0	99,1	100,0	100,0
	[0.05, 0.95]	90	13,7	21,7	3,5	–	17,1	18,7	–	64,8	75,7	2,2	–	77,0	78,3	–
		180	27,7	37,7	3,3	–	35,1	39,2	–	94,6	96,7	2,9	–	97,6	98,2	–
		270	45,2	56,6	3,6	–	57,3	61,1	–	99,6	99,7	3,4	–	99,9	99,9	–
		360	57,5	68,9	3,6	–	69,2	73,8	–	100,0	100,0	5,5	–	100,0	100,0	–
		450	68,5	78,2	3,5	–	81,4	85,7	–	100,0	100,0	10,4	–	100,0	100,0	–
		900	95,4	98,0	3,8	–	99,0	99,5	–	100,0	100,0	49,9	–	100,0	100,0	–
	[0.95, 1]	5	6,2	5,7	17,1	20,0	6,5	11,1	21,3	7,1	4,5	26,7	32,2	8,0	16,2	32,8
		10	7,1	5,6	21,0	25,5	7,8	12,0	28,1	9,5	6,7	34,8	38,5	11,7	18,9	41,4
		15	9,6	6,0	23,7	27,8	10,8	15,3	32,4	12,3	8,7	42,5	46,3	14,3	25,7	50,0
		20	9,5	8,4	30,0	34,3	10,2	16,5	39,1	13,5	10,1	50,5	53,7	13,5	27,0	59,1
		25	10,4	7,6	30,7	35,3	12,0	18,8	42,0	15,6	12,1	54,2	55,8	18,1	33,9	64,0
		50	15,9	12,7	45,9	51,2	18,0	30,7	61,4	23,0	20,0	72,1	72,4	26,5	49,5	81,4

Theoretical distribution	Truncation	n	True distribution													
			NIG (0.16; 1.00; 1.90; -0.22)							NIG (0.24; 1.36; 0.45; -0.10)						
			Test statistic													
			[1]	[2]	[3]	[4]	[5]	[6]	[7]	[1]	[2]	[3]	[4]	[5]	[6]	[7]
NIG	[0, 1]	100	*3,9*	*4,2*	*5,8*	–	*3,6*	*2,9*	–	*3,6*	*3,8*	*5,2*	–	*3,4*	*3,4*	–
		200	*3,9*	*4,0*	*4,7*	–	*3,8*	*3,9*	–	*4,7*	*6,1*	*5,4*	–	*4,4*	*3,7*	–
		300	*3,8*	*3,9*	*5,0*	–	*3,1*	*3,3*	–	*4,4*	*3,8*	*6,2*	–	*3,9*	*3,6*	–
		400	*4,3*	*5,3*	*5,4*	–	*5,5*	*5,5*	–	*5,1*	*5,5*	*6,2*	–	*5,5*	*4,8*	–
		500	*4,0*	*4,4*	*5,7*	–	*5,0*	*4,4*	–	*5,5*	*5,5*	*5,5*	–	*5,4*	*4,7*	–
		1000	*4,4*	*3,9*	*5,6*	–	*4,4*	*4,3*	–	*5,9*	*5,4*	*5,1*	–	*6,0*	*4,6*	–
	[0, 0.05]	5	*4,0*	*3,9*	*5,1*	*4,3*	*4,1*	*3,5*	*3,6*	*4,0*	*5,4*	*5,4*	*3,6*	*4,7*	*5,0*	*3,0*
		10	*6,2*	*5,6*	*5,5*	*4,7*	*5,0*	*5,9*	*4,1*	*4,9*	*4,2*	*5,1*	*3,2*	*5,0*	*4,3*	*3,4*
		15	*3,3*	*3,9*	*5,2*	*5,1*	*3,4*	*4,0*	*3,8*	*4,6*	*5,4*	*4,4*	*2,9*	*4,7*	*4,7*	*2,6*
		20	*4,7*	*4,8*	*3,6*	*3,3*	*5,3*	*5,0*	*3,1*	*5,1*	*5,4*	*4,3*	*3,3*	*4,5*	*4,7*	*2,9*
		25	*4,7*	*5,1*	*4,5*	*3,5*	*4,4*	*4,2*	*4,0*	*4,5*	*5,2*	*4,0*	*4,5*	*4,4*	*4,0*	*4,3*
		50	*4,1*	*5,4*	*5,6*	*4,6*	*5,5*	*5,3*	*4,3*	*4,7*	*4,7*	*4,1*	*4,9*	*4,9*	*4,8*	*3,8*
	[0.05, 0.95]	90	*3,6*	*4,2*	*4,7*	–	*4,6*	*3,2*	–	*4,8*	*5,4*	*5,4*	–	*3,4*	*3,2*	–
		180	*3,3*	*2,5*	*4,2*	–	*3,0*	*2,4*	–	*4,5*	*4,5*	*4,9*	–	*3,5*	*3,4*	–
		270	*3,9*	*5,0*	*5,4*	–	*4,2*	*3,6*	–	*3,6*	*3,7*	*5,9*	–	*4,3*	*5,1*	–
		360	*3,8*	*4,0*	*4,6*	–	*2,9*	*3,4*	–	*4,8*	*4,9*	*6,8*	–	*4,2*	*3,7*	–
		450	*3,6*	*4,2*	*5,7*	–	*4,0*	*3,6*	–	*4,1*	*5,2*	*4,9*	–	*4,5*	*4,0*	–
		900	*3,8*	*4,8*	*5,3*	–	*4,5*	*4,2*	–	*3,5*	*3,6*	*5,2*	–	*3,7*	*3,1*	–
	[0.95, 1]	5	*4,1*	*3,7*	*5,1*	*3,6*	*3,9*	*3,5*	*3,4*	*4,3*	*4,4*	*4,4*	*4,3*	*4,6*	*4,5*	*3,6*
		10	*5,0*	*6,6*	*5,0*	*3,9*	*4,2*	*4,5*	*4,1*	*5,6*	*4,4*	*5,7*	*4,3*	*4,8*	*4,2*	*4,1*
		15	*4,9*	*4,6*	*4,2*	*2,6*	*4,5*	*4,1*	*2,4*	*4,7*	*5,1*	*5,0*	*4,4*	*4,0*	*4,1*	*4,1*
		20	*4,7*	*5,4*	*5,7*	*4,5*	*4,5*	*3,9*	*4,4*	*4,0*	*5,7*	*4,8*	*4,3*	*4,3*	*4,8*	*4,1*
		25	*5,1*	*6,3*	*4,3*	*4,4*	*4,1*	*4,1*	*3,9*	*4,9*	*4,7*	*5,2*	*3,9*	*4,7*	*4,5*	*4,0*
		50	*5,2*	*5,2*	*4,8*	*4,7*	*5,2*	*5,3*	*4,7*	*4,5*	*5,6*	*5,3*	*6,0*	*3,8*	*4,6*	*5,4*

Theoretical distribution	Truncation	n	True distribution													
			NIG (0.16; 1.00; 1.90; -0.22)							NIG (0.24; 1.36; 0.45; -0.10)						
			Test statistic													
			[1]	[2]	[3]	[4]	[5]	[6]	[7]	[1]	[2]	[3]	[4]	[5]	[6]	[7]
SGED	[0, 1]	100	5,3	5,4	7,7	–	4,1	5,3	–	6,2	6,8	12,3	–	5,6	6,7	–
		200	5,5	6,3	12,9	–	6,9	6,3	–	11,0	12,9	18,4	–	13,3	14,6	–
		300	7,7	7,5	13,4	–	7,9	8,8	–	13,1	14,3	19,0	–	14,8	16,6	–
		400	7,8	9,0	17,2	–	9,6	11,7	–	18,2	21,8	21,2	–	19,8	23,8	–
		500	8,6	10,5	17,8	–	10,5	11,6	–	19,5	26,8	22,9	–	24,7	28,0	–
		1000	16,0	19,3	21,8	–	19,4	20,1	–	34,0	46,3	28,5	–	46,9	52,6	–
	[0, 0.05]	5	5,5	5,1	7,0	8,5	5,1	6,0	8,5	6,7	5,3	9,0	11,4	6,6	8,3	12,5
		10	6,3	5,6	8,8	10,8	6,1	6,8	12,7	6,5	5,5	10,6	16,9	6,1	7,7	17,7
		15	4,3	3,7	9,2	13,2	5,0	6,1	14,4	8,7	7,4	13,8	22,2	8,6	10,0	26,1
		20	6,8	7,2	10,4	15,0	6,0	7,0	16,9	8,7	7,2	14,5	20,8	8,9	10,8	24,1
		25	5,9	6,4	10,0	13,2	4,5	5,2	14,4	9,6	7,6	17,3	25,2	11,1	15,0	32,8
		50	6,1	5,8	13,4	20,0	6,8	8,2	22,0	15,2	10,5	23,4	28,8	16,5	23,9	42,5
	[0.05, 0.95]	90	4,8	4,2	4,8	–	4,1	3,6	–	8,2	6,7	5,5	–	6,9	5,5	–
		180	6,3	7,4	4,4	–	5,4	4,5	–	8,5	10,3	4,1	–	7,5	5,7	–
		270	8,8	8,9	4,8	–	7,1	5,4	–	13,1	15,7	4,0	–	10,4	9,4	–
		360	7,1	9,4	3,8	–	6,3	5,4	–	12,5	17,8	4,2	–	9,8	9,7	–
		450	8,6	12,4	3,1	–	7,3	4,6	–	17,6	24,5	5,5	–	13,9	11,9	–
		900	13,6	19,2	3,7	–	15,7	9,8	–	27,6	45,4	5,3	–	26,4	25,1	–
	[0.95, 1]	5	4,0	4,4	4,4	6,6	4,5	3,9	5,3	5,2	5,1	5,3	6,3	4,7	5,3	5,8
		10	5,1	4,8	7,4	8,6	4,3	5,0	9,1	4,5	4,6	7,4	7,8	5,0	4,7	8,1
		15	3,7	4,2	9,3	9,8	4,0	5,4	10,3	5,5	5,0	6,7	7,4	5,0	5,1	8,0
		20	4,3	4,8	9,5	10,3	4,2	4,8	9,9	4,6	5,0	7,4	9,9	3,9	5,5	10,5
		25	5,5	5,8	10,2	11,0	6,0	5,5	11,9	4,1	4,9	8,9	9,8	3,9	4,2	9,9
		50	5,2	4,6	10,9	12,8	5,1	6,0	14,8	5,4	6,1	9,8	11,2	5,0	5,0	12,3

Test statistics: [1]: KS^{**}; [2]: V^{**}; [3]: AD^{**}; [4]: $AD^{**}_{up}, AD^{**}_{down}$; [5]: W^{2**}; [6]: AD^{2**} [7]: $AD^{2**}_{up}, AD^{2**}_{down}$.

Source: the author's own work.

6. Conclusions

Although the tests for the complete distributions are more powerful than the tests for the certain proportions of the distribution, they do not indicate where the biggest differences between the empirical and the theoretical distributions appear. The statistics developed in this paper for doubly truncated distributions and the statistical inference procedure taking into account the number of truncated data allow to evaluate the goodness-of-fit of any part of unconditional distribution, when the number of truncated data is known. The simulation study confirmed that the tests have acceptable empirical size. Their empirical powers depend mainly on the choice of the test statistic and less on the truncation points and the sample sizes.

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Appendix 1

This appendix contains derivation of the computational formula for the AD^{**} test statistic, used to determine the goodness-of-fit when the sample is doubly truncated. The derivation is based on the scheme developed by Chernobai, Rachev and Fabozzi [2015] for the left truncated case. The derivations of the computational formulas for the remaining test statistics presented in this paper are available upon request.

General formula for the supremum version of the Anderson-Darling test statistic in the doubly truncated case is as follows:

$$AD^{**} = \sqrt{n} \sup_x \left| \frac{F_n(x) - \hat{F}_0^{**}(x)}{\sqrt{\hat{F}_0^{**}(x)\left(1 - \hat{F}_0^{**}(x)\right)}} \right|,$$

where $F_n(x)$ denotes the ECDF and $\hat{F}_0^{**}(x)$ denotes the theoretical distribution function of a doubly truncated sample (formulas 1 and 2 respectively). The computing formula in the doubly truncated case was derived as follows:

$$AD^{**} = \sqrt{n}\ \max\left\{ \sup_j \left\{ \frac{F_n\left(x_{(j)}\right) - \hat{F}_0^{**}\left(x_{(j)}\right)}{\sqrt{\hat{F}_0^{**}\left(x_{(j)}\right)\left(1 - \hat{F}_0^{**}\left(x_{(j)}\right)\right)}} \right\}, \sup_j \left\{ \frac{\hat{F}_0^{**}\left(x_{(j)}\right) - F_n\left(x_{(j)}\right)}{\sqrt{\hat{F}_0^{**}\left(x_{(j)}\right)\left(1 - \hat{F}_0^{**}\left(x_{(j)}\right)\right)}} \right\} \right\}$$

$$= \sqrt{n}\ \max\left\{ \sup_j \left\{ \frac{\frac{j}{n} - \frac{z_j - z_{H1}}{z_{H2} - z_{H1}}}{\sqrt{\frac{z_j - z_{H1}}{z_{H2} - z_{H1}}\left(1 - \frac{z_j - z_{H1}}{z_{H2} - z_{H1}}\right)}} \right\}, \sup_j \left\{ \frac{\frac{z_j - z_{H1}}{z_{H2} - z_{H1}} - \frac{j-1}{n}}{\sqrt{\frac{z_j - z_{H1}}{z_{H2} - z_{H1}}\left(1 - \frac{z_j - z_{H1}}{z_{H2} - z_{H1}}\right)}} \right\} \right\}$$

$$= \sqrt{n}\ \max\left\{ \sup_j \left\{ \frac{\frac{1}{z_{H2} - z_{H1}}\left(z_{H1} + \frac{j}{n}\left(z_{H2} - z_{H1}\right) - z_j\right)}{\sqrt{\frac{z_j - z_{H1}}{z_{H2} - z_{H1}}\left(\frac{z_{H2} - z_j}{z_{H2} - z_{H1}}\right)}} \right\}, \sup_j \left\{ \frac{\frac{1}{z_{H2} - z_{H1}}\left(z_j - \left(z_{H1} + \frac{j-1}{n}\left(z_{H2} - z_{H1}\right)\right)\right)}{\sqrt{\frac{z_j - z_{H1}}{z_{H2} - z_{H1}}\left(\frac{z_{H2} - z_j}{z_{H2} - z_{H1}}\right)}} \right\} \right\}$$

$$= \sqrt{n}\ \max\left\{ \sup_j \left\{ \frac{z_{H1} + \frac{j}{n}\left(z_{H2} - z_{H1}\right) - z_j}{\sqrt{\left(z_j - z_{H1}\right)\left(z_{H2} - z_j\right)}} \right\}, \sup_j \left\{ \frac{z_j - \left(z_{H1} + \frac{j-1}{n}\left(z_{H2} - z_{H1}\right)\right)}{\sqrt{\left(z_j - z_{H1}\right)\left(z_{H2} - z_j\right)}} \right\} \right\}.$$