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Symmetrical Piecewise Linear Functions Composed by Absolute Value Function

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Summary. We continue the formal development of the application of piecewise linear functions and centroids in the area of fuzzy set theory. The corresponding piecewise linear functions are symmetrical and composed by absolute function. In this paper we prove that the membership functions of isosceles triangle type and isosceles trapezoid type can be constructed by functions of this type.

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Introduction

In this paper, some mathematical properties of piecewise linear functions are formalized in Mizar [11], [10] in order to use them in fuzzy set theory [2], [22]. The focused piecewise linear functions are symmetrical and composed by absolute function. L-R fuzzy number is applied for various fields [1], [3], [20], [12]. Since isosceles triangle type and isosceles trapezoid type membership functions are simple [4], they are applied for the membership functions of L-R fuzzy number in most cases [17]. It is formalized that the membership functions of isosceles triangle type [16] and isosceles trapezoid type (introduced formally in Mizar in [5]) can be constructed by absolute value functions. We wanted to avoid duplication [9] of some basic functional notions, so we use extensively Mizar functor "AffineMap" denoting just linear function with two parameters.

We prove that the centroids of the composite function of two continuous functions are the weighted averages of the areas and centroids of the functions that compose them [21]. Moreover, some calculation and operation between membership functions for fuzzy approximate reasoning [19], e.g. Mamdani method [13] and the product-sum-gravity method [18] are formalized, extending also the development of both fuzzy numbers within the Mizar Mathematical Library [7] and fuzzy sets in general [14], [15], [8] (for another recent formal development in this area, see [6]).

1. Preliminaries

From now on A denotes a non empty, closed interval subset of $\mathbb{R}$. Now we state the proposition:

(1) Let us consider real numbers b, c, d. If $b > 0$ and $c > 0$ and $d > 0$, then $\frac{b-d}{\frac{b}{c}} < c$.

Let us consider real numbers a, x. Now we state the propositions:

(2) $a - |a \cdot x| \leqslant a$.

(3) $a - |x| \leqslant a$.

(4) Let us consider real numbers a, b, c, x. Then $|\frac{b \cdot (a-x-a)}{c}| = |\frac{b \cdot (a+x-a)}{c}|$.

Let us consider real numbers a, b, c. Now we state the propositions:

(5) $|\max(c, a) - \max(c, b)| \leqslant |a - b|$.

(6) $|\min(c, a) - \min(c, b)| \leqslant |a - b|$.

(7) Let us consider real numbers a, b, c, d. Then $|\min(c, \max(d, a)) - \min(c, \max(d, b))| \leqslant |a - b|$. The theorem is a consequence of (6) and (5).

2. Continuous Functions

Let us consider a real number c and partial functions f, g from $\mathbb{R}$ to $\mathbb{R}$. Now we state the propositions:

(8) Suppose $]-\infty, c] \subseteq \operatorname{dom} f$ and $[c, +\infty[\subseteq \operatorname{dom} g$.
Then $f{\upharpoonright}]-\infty, c[+\cdot g{\upharpoonright}[c, +\infty[= f{\upharpoonright}]-\infty, c] +\cdot g{\upharpoonright}[c, +\infty[$.
Proof: Set $f_1 = f{\upharpoonright}]-\infty, c[+\cdot g{\upharpoonright}[c, +\infty[$. Set $f_2 = f{\upharpoonright}]-\infty, c] +\cdot g{\upharpoonright}[c, +\infty[$. For every object x such that $x \in \operatorname{dom} f_1$ holds $f_1(x) = f_2(x)$. □

(9) Suppose f is continuous and g is continuous and $f(c) = g(c)$ and $]-\infty, c] \subseteq \operatorname{dom} f$ and $[c, +\infty[\subseteq \operatorname{dom} g$. Then $f{\upharpoonright}]-\infty, c] +\cdot g{\upharpoonright}[c, +\infty[$ is continuous.
Proof: Set $F = f{\upharpoonright}]-\infty, c] +\cdot g{\upharpoonright}[c, +\infty[$. For every real number x_0 such that $x_0 \in \operatorname{dom} F$ holds F is continuous in x_0. □

(10) Let us consider a real number c, and functions f, g from $\mathbb{R}$ into $\mathbb{R}$. Suppose f is continuous and g is continuous and $f(c) = g(c)$. Then $f{\restriction}]-\infty, c]{+\cdot}g{\restriction}[c, +\infty[$ is a continuous function from $\mathbb{R}$ into $\mathbb{R}$. The theorem is a consequence of (9).

(11) Let us consider real numbers a, b, c, and functions f, g, h from $\mathbb{R}$ into $\mathbb{R}$. Suppose $a \leqslant b \leqslant c$ and f is continuous and g is continuous and $h{\restriction}[a, c] = f{\restriction}[a, b]{+\cdot}g{\restriction}[b, c]$ and $f(b) = g(b)$. Then $\int\limits_{[a,c]} h(x)dx = \int\limits_{[a,b]} f(x)dx + \int\limits_{[b,c]} g(x)dx$.

(12) Let us consider a function f from $\mathbb{R}$ into $\mathbb{R}$, and real numbers a, b, c. Suppose $a \leqslant b \leqslant c$ and $[a, c] \subseteq \operatorname{dom} f$ and $f{\restriction}[a, b]$ is bounded and $f{\restriction}[b, c]$ is bounded and f is integrable on $[a, b]$ and f is integrable on $[b, c]$. Then

(i) f is integrable on $[a, c]$, and

(ii) $\int\limits_a^c f(x)dx = \int\limits_a^b f(x)dx + \int\limits_b^c f(x)dx$.

(13) Let us consider real numbers a, b, c, and a function f from $\mathbb{R}$ into $\mathbb{R}$. Suppose $a \leqslant c$ and f is integrable on $[a, c]$ and $f{\restriction}[a, c]$ is bounded and $[a, c] \subseteq \operatorname{dom} f$ and $b \in [a, c]$. Then

(i) f is integrable on $[a, b]$, and

(ii) f is integrable on $[b, c]$, and

(iii) $\int\limits_a^c f(x)dx = \int\limits_a^b f(x)dx + \int\limits_b^c f(x)dx$.

(14) Let us consider a real number a, and functions f, g, h from $\mathbb{R}$ into $\mathbb{R}$. Suppose $f{\restriction}A$ is bounded and f is integrable on A and $g{\restriction}A$ is bounded and g is integrable on A and $a \in A$ and $h = f{\restriction}]-\infty, a]{+\cdot}g{\restriction}[a, +\infty[$ and $f(a) = g(a)$. Then h is integrable on A.
Proof: For every object x such that $x \in \operatorname{dom}(f{\restriction}[\inf A, a])$ holds $(f{\restriction}[\inf A, a])(x) = (h{\restriction}[\inf A, a])(x)$. For every object x such that $x \in \operatorname{dom}(g{\restriction}[a, \sup A])$ holds $(g{\restriction}[a, \sup A])(x) = (h{\restriction}[a, \sup A])(x)$. f is integrable on $[\inf A, a]$. g is integrable on $[a, \sup A]$. $\square$

(15) Let us consider real numbers a, b, c, and functions f, g from $\mathbb{R}$ into $\mathbb{R}$. Suppose $a \leqslant b \leqslant c$. Then $(f{\restriction}]-\infty, b]{+\cdot}g{\restriction}[b, +\infty[){\restriction}[a, c] = f{\restriction}[a, b]{+\cdot}g{\restriction}[b, c]$.
Proof: For every object x such that $x \in \operatorname{dom}((f{\restriction}]-\infty, b]{+\cdot}g{\restriction}[b, +\infty[){\restriction}[a, c])$ holds $((f{\restriction}]-\infty, b]{+\cdot}g{\restriction}[b, +\infty[){\restriction}[a, c])(x) = (f{\restriction}[a, b]{+\cdot}g{\restriction}[b, c])(x)$. $\square$

(16) Let us consider real numbers a, b, c, and functions f, g, h from $\mathbb{R}$ into $\mathbb{R}$. Suppose $a \leqslant b \leqslant c$ and f is integrable on $[a, c]$ and $f\restriction[a, c]$ is bounded and g is integrable on $[a, c]$ and $g\restriction[a, c]$ is bounded and $h = f\restriction]-\infty, b]+\cdot g\restriction[b, +\infty[$ and $f(b) = g(b)$. Then $\int\limits_{[a,c]} h(x)dx = \int\limits_{[a,b]} f(x)dx + \int\limits_{[b,c]} g(x)dx$. The theorem is a consequence of (15) and (14).

3. Area and Centroid of Continuous Functions

Now we state the propositions:

(17) Let us consider functions f, g, h from $\mathbb{R}$ into $\mathbb{R}$, and real numbers a, b, c. Suppose $a \leqslant b \leqslant c$ and f is continuous and g is continuous and $h\restriction[a, c] = f\restriction[a, b]+\cdot g\restriction[b, c]$ and $\int\limits_{[a,b]} f(x)dx \neq 0$ and $\int\limits_{[b,c]} g(x)dx \neq 0$ and $f(b) = g(b)$. Then $\mathrm{centroid}(h, [a, c]) = \frac{1}{\int\limits_{[a,c]} h(x)dx} \cdot ((\mathrm{centroid}(f, [a, b])) \cdot (\int\limits_{[a,b]} f(x)dx) + (\mathrm{centroid}(g, [b, c])) \cdot (\int\limits_{[b,c]} g(x)dx))$.

(18) Let us consider a function f from $\mathbb{R}$ into $\mathbb{R}$, and real numbers a, b, c. Suppose for every real number x, $f(x) = b - |\frac{b\cdot(x-a)}{c}|$. Let us consider a real number y. Then $f(a - y) = f(a + y)$.

(19) Let us consider a function f from $\mathbb{R}$ into $\mathbb{R}$, and real numbers a, b, c, d, e. Suppose for every real number x, $f(x) = \min(d, \max(e, b - |\frac{b\cdot(x-a)}{c}|))$. Let us consider a real number y. Then $f(a - y) = f(a + y)$.

(20) Let us consider real numbers a, b, c, d. Suppose $b > 0$ and $c > 0$ and $d > 0$ and $d < b$. Let us consider a real number x. Then $(d \cdot \mathrm{TrapezoidalFS}((a - c), (a + \frac{d-b}{\frac{b}{c}}), (a + \frac{b-d}{\frac{b}{c}}), (a + c)))(x) = \min(d, \max(0, b - |\frac{b\cdot(x-a)}{c}|))$.

Proof: For every real number x, $(d\cdot \mathrm{TrapezoidalFS}((a-c), (a+ \frac{d-b}{\frac{b}{c}}), (a+ \frac{b-d}{\frac{b}{c}}), (a + c)))(x) = \min(d, \max(0, b - |\frac{b\cdot(x-a)}{c}|))$. $\square$

(21) Let us consider real numbers a, b, c, d. Suppose $b > 0$ and $c > 0$ and $d > 0$ and $d < b$. Then $\mathrm{centroid}(d \cdot \mathrm{TrapezoidalFS}((a - c), (a + \frac{d-b}{\frac{b}{c}}), (a + \frac{b-d}{\frac{b}{c}}), (a + c)), [a - c, a + c]) = a$.

Let us consider real numbers a, b, c, d and a function f from $\mathbb{R}$ into $\mathbb{R}$. Now we state the propositions:

(22) Suppose $b > 0$ and $c > 0$ and $d > 0$ and $d < b$ and for every real number x, $f(x) = \min(d, \max(0, b - |\frac{b\cdot(x-a)}{c}|))$. Then $f = d \cdot \mathrm{TrapezoidalFS}((a - c), (a + \frac{d-b}{\frac{b}{c}}), (a + \frac{b-d}{\frac{b}{c}}), (a + c))$. The theorem is a consequence of (20).

(23) Suppose $b > 0$ and $c > 0$ and $d > 0$ and $d < b$ and for every real number x, $f(x) = \min(d, \max(0, b - |\frac{b\cdot(x-a)}{c}|))$. Then $\mathrm{centroid}(f, [a - c, a + c]) = a$. The theorem is a consequence of (22) and (21).

Let us consider real numbers a, b, c, d, e and a function f from $\mathbb{R}$ into $\mathbb{R}$. Now we state the propositions:

(24) If $b \neq 0$ and $c \neq 0$ and for every real number x, $f(x) = \min(d, \max(e, b - |\frac{b\cdot(x-a)}{c}|))$, then f is Lipschitzian.
Proof: There exists a real number r such that $0 < r$ and for every real numbers x_1, x_2 such that $x_1, x_2 \in \mathrm{dom}\, f$ holds $|f(x_1) - f(x_2)| \leqslant r \cdot |x_1 - x_2|$. □

(25) If $c \neq 0$ and for every real number x, $f(x) = \min(d, \max(e, b - |\frac{b\cdot(x-a)}{c}|))$, then f is Lipschitzian. The theorem is a consequence of (24).

Let us consider real numbers a, b, c, d and a function f from $\mathbb{R}$ into $\mathbb{R}$. Now we state the propositions:

(26) Suppose $c > 0$ and for every real number x, $f(x) = \min(d, \max(0, b - |\frac{b\cdot(x-a)}{c}|))$. Then

 (i) f is integrable on A, and

 (ii) $f {\restriction} A$ is bounded.

The theorem is a consequence of (25).

(27) Suppose $b > 0$ and $c > 0$ and $d > 0$ and for every real number x, $f(x) = \min(d, \max(0, b - |\frac{b\cdot(x-a)}{c}|))$. Then

 (i) $f(\inf[a - c, a + c]) = 0$, and

 (ii) $f(\sup[a - c, a + c]) = 0$.

(28) Let us consider real numbers a, b, c. Suppose $b > 0$ and $c > 0$. Let us consider a real number x. If $x \notin [a - c, a + c]$, then $\max(0, b - |\frac{b\cdot(x-a)}{c}|) = 0$.
Proof: Define $\mathcal{H}(\text{element of } \mathbb{R}) = (\max(0, b - |\frac{b\cdot(\$_1-a)}{c}|))(\in \mathbb{R})$. Consider h being a function from $\mathbb{R}$ into $\mathbb{R}$ such that for every element x of $\mathbb{R}$, $h(x) = \mathcal{H}(x)$. For every real number x, $h(x) = \max(0, b - |\frac{b\cdot(x-a)}{c}|)$. □

(29) Let us consider real numbers a, b, c, d. Suppose $b > 0$ and $c > 0$ and $d > 0$. Let us consider a real number x. Suppose $x \notin [a - c, a + c]$. Then $\min(d, \max(0, b - |\frac{b\cdot(x-a)}{c}|)) = 0$. The theorem is a consequence of (28).

Let us consider real numbers a, b, c, d, a function f from $\mathbb{R}$ into $\mathbb{R}$, and a real number x. Now we state the propositions:

(30) Suppose $b > 0$ and $c > 0$ and $d > 0$ and for every real number x, $f(x) = \min(d, \max(0, b - |\frac{b\cdot(x-a)}{c}|))$. Then if $x \notin [a-c, a+c]$, then $f(x) = 0$. The theorem is a consequence of (29).

(31) Suppose $b > 0$ and $c > 0$ and $d > 0$ and for every real number x, $f(x) = \min(d, \max(0, b - |\frac{b\cdot(x-a)}{c}|))$. Then if $x \in A \setminus [a-c, a+c]$, then $f(x) = 0$. The theorem is a consequence of (30).

Let us consider real numbers a, b, c, d and a function f from $\mathbb{R}$ into $\mathbb{R}$. Now we state the propositions:

(32) Suppose $b > 0$ and $c > 0$ and $d > 0$ and $[a-c, a+c] \subseteq A$ and for every real number x, $f(x) = \min(d, \max(0, b - |\frac{b\cdot(x-a)}{c}|))$. Then $\text{centroid}(f, A) = a$. The theorem is a consequence of (26), (31), (27), and (23).

(33) Suppose $b > 0$ and $c > 0$ and $d > 0$ and $[a-c, a+c] \subseteq A$ and $d < b$ and for every real number x, $f(x) = \min(d, \max(0, b - |\frac{b\cdot(x-a)}{c}|))$. Then $\text{centroid}(f, A) = \text{centroid}(f, [a-c, a+c])$. The theorem is a consequence of (32) and (23).

(34) Let us consider real numbers a, b, c, d, and functions f, F from $\mathbb{R}$ into $\mathbb{R}$. Suppose $b > 0$ and $c > 0$ and $d > 0$ and for every real number x, $f(x) = \max(0, b - |\frac{b\cdot(x-a)}{c}|)$ and for every real number x, $F(x) = \min(d, \max(0, b - |\frac{b\cdot(x-a)}{c}|))$. Then $\text{centroid}(f, [a-c, a+c]) = \text{centroid}(F, [a-c, a+c])$. The theorem is a consequence of (23) and (3).

(35) Let us consider real numbers a, b, c, d, and a function f from $\mathbb{R}$ into $\mathbb{R}$. Suppose $b > 0$ and $c > 0$ and $d > 0$ and $d < b$ and for every real number x, $f(x) = \min(d, \max(0, b - |\frac{b\cdot(x-a)}{c}|))$. Then $f\restriction[a-c, a+c] = ((\text{AffineMap}(\frac{b}{c}, b - \frac{a\cdot b}{c}))\restriction[a-c, a + \frac{d-b}{\frac{b}{c}}] +\cdot (\text{AffineMap}(0, d))\restriction[a + \frac{d-b}{\frac{b}{c}}, a + \frac{b-d}{\frac{b}{c}}]) +\cdot (\text{AffineMap}(-\frac{b}{c}, b + \frac{a\cdot b}{c}))\restriction[a + \frac{b-d}{\frac{b}{c}}, a + c]$.

Proof: $-\frac{b-d}{\frac{b}{c}} > -c$. $\frac{b-d}{\frac{b}{c}} < c$. For every object x such that $x \in \text{dom}(f\restriction[a-c, a+c])$ holds $(f\restriction[a-c, a+c])(x) = (((\text{AffineMap}(\frac{b}{c}, b - \frac{a\cdot b}{c}))\restriction[a-c, a + \frac{d-b}{\frac{b}{c}}] +\cdot (\text{AffineMap}(0, d))\restriction[a + \frac{d-b}{\frac{b}{c}}, a + \frac{b-d}{\frac{b}{c}}]) +\cdot (\text{AffineMap}(-\frac{b}{c}, b + \frac{a\cdot b}{c}))\restriction[a + \frac{b-d}{\frac{b}{c}}, a + c])(x)$. □

4. Some Special Examples

Now we state the proposition:

(36) Let us consider real numbers a, b, c, d, r, s. Suppose $a < b < c < d$. Then

(i) $(\text{AffineMap}(\frac{r}{b-a}, -\frac{a\cdot r}{b-a}))(a) = 0$, and

(ii) $(\text{AffineMap}(\frac{r}{b-a}, -\frac{a\cdot r}{b-a}))(b) = r$, and

(iii) $(\mathrm{AffineMap}(\frac{s-r}{c-b}, s - \frac{c\cdot(s-r)}{c-b}))(b) = r$, and

(iv) $(\mathrm{AffineMap}(\frac{s-r}{c-b}, s - \frac{c\cdot(s-r)}{c-b}))(c) = s$, and

(v) $(\mathrm{AffineMap}(\frac{-s}{d-c}, -\frac{d\cdot(-s)}{d-c}))(c) = s$, and

(vi) $(\mathrm{AffineMap}(\frac{-s}{d-c}, -\frac{d\cdot(-s)}{d-c}))(d) = 0$.

Let us consider real numbers a, b, c, d, r, s and a function f from $\mathbb{R}$ into $\mathbb{R}$. Now we state the propositions:

(37) Suppose $a < b < c < d$ and $f\restriction[a,d] = ((\mathrm{AffineMap}(\frac{r}{b-a}, -\frac{a\cdot r}{b-a}))\restriction[a,b]+\cdot(\mathrm{AffineMap}(\frac{s-r}{c-b}, s-\frac{c\cdot(s-r)}{c-b}))\restriction[b,c])+\cdot(\mathrm{AffineMap}(\frac{-s}{d-c}, -\frac{d\cdot(-s)}{d-c}))\restriction[c,d]$. Then $\int\limits_{[a,d]}(\mathrm{id}_{\mathbb{R}}\cdot f)(x)dx = \int\limits_{[a,b]}(\mathrm{id}_{\mathbb{R}}\cdot(\mathrm{AffineMap}(\frac{r}{b-a}, -\frac{a\cdot r}{b-a})))(x)dx + \int\limits_{[b,c]}(\mathrm{id}_{\mathbb{R}}\cdot(\mathrm{AffineMap}(\frac{s-r}{c-b}, s-\frac{c\cdot(s-r)}{c-b})))(x)dx + \int\limits_{[c,d]}(\mathrm{id}_{\mathbb{R}}\cdot(\mathrm{AffineMap}(\frac{-s}{d-c}, -\frac{d\cdot(-s)}{d-c})))(x)dx$.

Proof: Set $f_3 = \mathrm{AffineMap}(\frac{r}{b-a}, -\frac{a\cdot r}{b-a})$. Set $f_4 = \mathrm{AffineMap}(\frac{s-r}{c-b}, s - \frac{c\cdot(s-r)}{c-b})$. Reconsider $h = f_3\restriction]-\infty,b[+\cdot f_4\restriction[b,+\infty[$ as a function from $\mathbb{R}$ into $\mathbb{R}$. $f_3(b) = r$. For every object x such that $x \in \mathrm{dom}(h\restriction[a,c])$ holds $(h\restriction[a,c])(x) = (f_3\restriction[a,b]+\cdot f_4\restriction[b,c])(x)$. □

(38) Suppose $a < b < c < d$ and $f\restriction[a,d] = ((\mathrm{AffineMap}(\frac{r}{b-a}, -\frac{a\cdot r}{b-a}))\restriction[a,b]+\cdot(\mathrm{AffineMap}(\frac{s-r}{c-b}, s-\frac{c\cdot(s-r)}{c-b}))\restriction[b,c])+\cdot(\mathrm{AffineMap}(\frac{-s}{d-c}, -\frac{d\cdot(-s)}{d-c}))\restriction[c,d]$. Then $\int\limits_{[a,d]}f(x)dx = \int\limits_{[a,b]}(\mathrm{AffineMap}(\frac{r}{b-a}, -\frac{a\cdot r}{b-a}))(x)dx + \int\limits_{[b,c]}(\mathrm{AffineMap}(\frac{s-r}{c-b}, s - \frac{c\cdot(s-r)}{c-b}))(x)dx + \int\limits_{[c,d]}(\mathrm{AffineMap}(\frac{-s}{d-c}, -\frac{d\cdot(-s)}{d-c}))(x)dx$.

Proof: Set $f_3 = \mathrm{AffineMap}(\frac{r}{b-a}, -\frac{a\cdot r}{b-a})$. Set $f_4 = \mathrm{AffineMap}(\frac{s-r}{c-b}, s - \frac{c\cdot(s-r)}{c-b})$. Reconsider $h = f_3\restriction]-\infty,b[+\cdot f_4\restriction[b,+\infty[$ as a function from $\mathbb{R}$ into $\mathbb{R}$. $f_3(b) = r$. For every object x such that $x \in \mathrm{dom}(h\restriction[a,c])$ holds $(h\restriction[a,c])(x) = (f_3\restriction[a,b]+\cdot f_4\restriction[b,c])(x)$. $\int\limits_{[a,c]}h(x)dx = \int\limits_{[a,b]}f_3(x)dx + \int\limits_{[b,c]}f_4(x)dx$. □

Let us consider real numbers a, b, c, d, r, s, x. Now we state the propositions:

(39) Suppose $a < b < c < d$ and $r \geqslant 0$ and $s \geqslant 0$ and ($x < a$ or $d < x$). Then $(((\mathrm{AffineMap}(\frac{r}{b-a}, -\frac{a\cdot r}{b-a}))\restriction]-\infty,b]+\cdot(\mathrm{AffineMap}(\frac{s-r}{c-b}, s - \frac{c\cdot(s-r)}{c-b}))\restriction[b,c])+\cdot(\mathrm{AffineMap}(\frac{-s}{d-c}, -\frac{d\cdot(-s)}{d-c}))\restriction[c,+\infty[)(x) \leqslant 0$.

(40) Suppose $a < b < c < d$ and $r \geqslant 0$ and $s \geqslant 0$ and $x \in [a, d]$. Then $(((\mathrm{AffineMap}(\frac{r}{b-a}, -\frac{a \cdot r}{b-a})) \restriction]-\infty, b] +\cdot (\mathrm{AffineMap}(\frac{s-r}{c-b}, s - \frac{c \cdot (s-r)}{c-b})) \restriction [b, c]) +\cdot (\mathrm{AffineMap}(\frac{-s}{d-c}, -\frac{d \cdot (-s)}{d-c})) \restriction [c, +\infty[)(x) \geqslant 0$.

(41) Let us consider real numbers a, b, c, d, r, s. Suppose $a < b < c < d$ and $r \geqslant 0$ and $s \geqslant 0$ and $r = s$. Let us consider a real number x. Then $(r \cdot \mathrm{TrapezoidalFS}(a, b, c, d))(x) = \max_{+}((((\mathrm{AffineMap}(\frac{r}{b-a}, -\frac{a \cdot r}{b-a})) \restriction]-\infty, b] +\cdot (\mathrm{AffineMap}(\frac{s-r}{c-b}, s - \frac{c \cdot (s-r)}{c-b})) \restriction [b, c]) +\cdot (\mathrm{AffineMap}(\frac{-s}{d-c}, -\frac{d \cdot (-s)}{d-c})) \restriction [c, +\infty[)(x))$.

PROOF: Set $T = \mathrm{TrapezoidalFS}(a, b, c, d)$. For every real number x, $(r \cdot T)(x) = \max_{+}((((\mathrm{AffineMap}(\frac{r}{b-a}, -\frac{a \cdot r}{b-a})) \restriction]-\infty, b] +\cdot (\mathrm{AffineMap}(\frac{s-r}{c-b}, s - \frac{c \cdot (s-r)}{c-b})) \restriction [b, c]) +\cdot (\mathrm{AffineMap}(\frac{-s}{d-c}, -\frac{d \cdot (-s)}{d-c})) \restriction [c, +\infty[)(x))$. □

(42) Let us consider real numbers a, b, c, d. Suppose $c \leqslant d$. Then

(i) $\int\limits_{[c,d]} (\mathrm{id}_{\mathbb{R}} \cdot (\mathrm{AffineMap}(a, b)))(x) dx = (d - c) \cdot (\frac{a \cdot (d \cdot d + d \cdot c + c \cdot c)}{3} + \frac{b \cdot (d + c)}{2})$, and

(ii) $\int\limits_{[c,d]} (\mathrm{AffineMap}(a, b))(x) dx = (d - c) \cdot (\frac{a \cdot (d + c)}{2} + b)$.

(43) Let us consider real numbers a, b, c, d, r, s, and a function f from $\mathbb{R}$ into $\mathbb{R}$. Suppose $a < b < c < d$ and $f \restriction [a, d] = ((\mathrm{AffineMap}(\frac{r}{b-a}, -\frac{a \cdot r}{b-a})) \restriction [a, b] +\cdot (\mathrm{AffineMap}(\frac{s-r}{c-b}, s - \frac{c \cdot (s-r)}{c-b})) \restriction [b, c]) +\cdot (\mathrm{AffineMap}(\frac{-s}{d-c}, -\frac{d \cdot (-s)}{d-c})) \restriction [c, d]$. Then $\mathrm{centroid}(f, [a, d]) = \Big((b - a) \cdot (\frac{\frac{r}{b-a} \cdot (b \cdot b + b \cdot a + a \cdot a)}{3} + \frac{(-\frac{a \cdot r}{b-a}) \cdot (b + a)}{2}) + (c - b) \cdot (\frac{\frac{s-r}{c-b} \cdot (c \cdot c + c \cdot b + b \cdot b)}{3} + \frac{(s - \frac{c \cdot (s-r)}{c-b}) \cdot (c + b)}{2}) + (d - c) \cdot (\frac{\frac{-s}{d-c} \cdot (d \cdot d + d \cdot c + c \cdot c)}{3} + \frac{(-\frac{d \cdot (-s)}{d-c}) \cdot (d + c)}{2})\Big) / \Big((b - a) \cdot (\frac{\frac{r}{b-a} \cdot (b + a)}{2} + -\frac{a \cdot r}{b-a}) + (c - b) \cdot (\frac{\frac{s-r}{c-b} \cdot (c + b)}{2} + (s - \frac{c \cdot (s-r)}{c-b})) + (d - c) \cdot (\frac{\frac{-s}{d-c} \cdot (d + c)}{2} + -\frac{d \cdot (-s)}{d-c})\Big)$. The theorem is a consequence of (37), (38), and (42).

(44) Let us consider real numbers b, c, d. Suppose $b < c$. Then $(\mathrm{AffineMap}(d \cdot \frac{1}{c-b}, d \cdot (-\frac{b}{c-b}))) + (\mathrm{AffineMap}(d \cdot (-\frac{1}{c-b}), d \cdot \frac{c}{c-b})) = \mathrm{AffineMap}(0, d)$.

(45) Let us consider real numbers a, b, c, p, q. Suppose $a < b < c$. Then $(\mathrm{AffineMap}(p, q)) \restriction [a, b] +\cdot (\mathrm{AffineMap}(p, q)) \restriction [b, c] = (\mathrm{AffineMap}(p, q)) \restriction [a, c]$.

PROOF: Set $f = \mathrm{AffineMap}(p, q)$. For every object x such that $x \in \mathrm{dom}(f \restriction [a, c])$ holds $(f \restriction [a, c])(x) = (f \restriction [a, b] +\cdot f \restriction [b, c])(x)$. □

Let us consider real numbers a, b, c and a real number x. Now we state the propositions:

(46) If $a < b < c$, then if $x \in [a, b]$, then $(\text{TriangularFS}(a, b, c))(x) = (\text{AffineMap}(\frac{1}{b-a}, -\frac{a}{b-a}))(x)$.

PROOF: For every real number x such that $x \in [a, b]$ holds $(\text{TriangularFS}(a, b, c))(x) = (\text{AffineMap}(\frac{1}{b-a}, -\frac{a}{b-a}))(x)$. □

(47) If $a < b < c$, then if $x \in [b, c]$, then $(\text{TriangularFS}(a, b, c))(x) = (\text{AffineMap}(-\frac{1}{c-b}, \frac{c}{c-b}))(x)$.

(48) If $a < b < c$, then if $x \notin]a, c[$, then $(\text{TriangularFS}(a, b, c))(x) = (\text{AffineMap}(0, 0))(x)$.

PROOF: For every real number x such that $x \notin]a, c[$ holds $(\text{TriangularFS}(a, b, c))(x) = (\text{AffineMap}(0, 0))(x)$. □

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