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The Ring of Conway Numbers in Mizar

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Summary. Conway's introduction to algebraic operations on surreal numbers with a rather simple definition. However, he combines recursion with Conway's induction on surreal numbers, more formally he combines transfinite induction-recursion with the properties of proper classes, which is difficult to introduce formally.

This article represents a further step in our ongoing efforts to investigate the possibilities offered by Mizar with Tarski-Grothendieck set theory [4] to introduce the algebraic structure of Conway numbers and to prove their ring character.

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Introduction

We present a formal analysis of the contents of Chapter 1, *The Class* **No** *is a Field* of John Conway's seminal book [5]. We formalised four sections, namely *Properties of Addition*, *Properties of Negation*, *Properties of Addition and Order* and *Properties of Multiplication*. We begin our exploration by formulating and proving two schemes (i.e., second-order theorems) for defining arithmetic operations on surreal numbers using a technique that mimics induction-infinite recursion. Then, we examine the applicability of this solution by defining the opposite surreal number but also the sum and product of surreal numbers. We prove for each such operator simultaneously its correctness and crucial properties, in particular the preservation of pre-order under the operator. For this purpose, we use transfinite induction with respect to successive generations of surreal numbers. Notice that we express the Conway induction using the transfinite induction with the Heisenberg sum of two ordinals [3, 6], formalised in [7].

The most important result is the formalisation of the following properties of the surreal numbers

$$
\begin{array}{llllll}
x+0_{\mathbf{No}}=x & (38), & -(x+y)=-x+-y & (40), & x\cdot 0_{\mathbf{No}}\approx 0_{\mathbf{No}} & (56),\\
x+y=y+x & (29), & --x=x & (9), & x\cdot 1_{\mathbf{No}}\approx x & (57),\\
(x+y)+z=x+(y+z) & (37), & x+-x\approx 0_{\mathbf{No}} & (39), & x\cdot y\approx y\cdot x & (51),
\end{array}
$$

$$
\begin{array}{llll}
(-x)\cdot y=-x\cdot y=x\cdot(-y) & (58) & (-x)\cdot(-y)=x\cdot y & (58),\\
x\cdot(y+z)\approx x\cdot y+x\cdot z & (67), & (x\cdot y)\cdot z\approx x\cdot(y\cdot z) & (69),\\
0_{\mathbf{No}}<x\wedge 0_{\mathbf{No}}<y\Rightarrow 0_{\mathbf{No}}<x\cdot y & (72), & y\leqslant z\Leftrightarrow x+y\leqslant x+z & (32).
\end{array}
$$

The formalisation is mainly based on [1, 2, 5, 10].

1. Preliminaries

From now on α, β, γ denote ordinal numbers, o denotes an object, x, y, z, t, r, l denote surreal numbers, and X, Y denote sets.

Let f be a function. One can check that f is function yielding if and only if the condition (Def. 1) is satisfied.

(Def. 1) $\operatorname{rng} f$ is functional.

One can check that there exists a transfinite sequence which is $\subseteq$-monotone and function yielding. Let f be a $\subseteq$-monotone function and X be a set. Let us observe that $f\restriction X$ is $\subseteq$-monotone. Let f be a $\subseteq$-monotone, function yielding transfinite sequence. Let us note that $\bigcup\operatorname{rng} f$ is function-like and relation-like. Now we state the propositions:

(1) Let us consider a $\subseteq$-monotone, function yielding transfinite sequence f, and an object o. Suppose $o\in\operatorname{dom}(\bigcup\operatorname{rng} f)$. Then there exists α such that

- (i) $\alpha\in\operatorname{dom} f$, and
- (ii) $o\in\operatorname{dom}(f(\alpha))$.

(2) Let us consider a $\subseteq$-monotone, function yielding transfinite sequence f, and α. Suppose $\alpha\in\operatorname{dom} f$. Then

- (i) $\operatorname{dom}(f(\alpha))\subseteq\operatorname{dom}(\bigcup\operatorname{rng} f)$, and
- (ii) for every o such that $o\in\operatorname{dom}(f(\alpha))$ holds $f(\alpha)(o)=(\bigcup\operatorname{rng} f)(o)$.

Proof: Set $U=\bigcup\operatorname{rng} f$. $\operatorname{dom}(f(\alpha))\subseteq\operatorname{dom} U$. $\square$

(3) Let us consider a $\subseteq$-monotone, function yielding transfinite sequence f, an ordinal number α, and a set X. Suppose for every o such that $o \in X$ there exists an ordinal number β such that $o \in \mathrm{dom}(f(\beta))$ and $\beta \in \alpha$. Then $(\bigcup \mathrm{rng}(f \restriction \alpha))^{\circ} X = (\bigcup \mathrm{rng}\, f)^{\circ} X$. The theorem is a consequence of (2).

2. Surreal Number Operators – Schemes

The scheme *MonoFvSExists* deals with an ordinal number θ and a unary functor δ yielding a set and a binary functor $\mathcal{H}$ yielding an object and states that

(Sch. 1) There exists a $\subseteq$-monotone, function yielding transfinite sequence S such that $\mathrm{dom}\, S = \mathrm{succ}\, \theta$ and for every ordinal number α such that $\alpha \in \mathrm{succ}\, \theta$ there exists a many sorted set S_3 indexed by $\delta(\alpha)$ such that $S(\alpha) = S_3$ and for every o such that $o \in \delta(\alpha)$ holds $S_3(o) = \mathcal{H}(o, S \restriction \alpha)$

provided

- for every $\subseteq$-monotone, function yielding transfinite sequence S such that for every ordinal number α such that $\alpha \in \mathrm{dom}\, S$ holds $\mathrm{dom}(S(\alpha)) = \delta(\alpha)$ for every ordinal number α for every o such that $o \in \mathrm{dom}(S(\alpha))$ holds $\mathcal{H}(o, S \restriction \alpha) = \mathcal{H}(o, S)$ and
- for every ordinal numbers α, β such that $\alpha \subseteq \beta$ holds $\delta(\alpha) \subseteq \delta(\beta)$.

The scheme *MonoFvSUniq* deals with an ordinal number θ and a unary functor δ yielding a set and $\subseteq$-monotone, function yielding transfinite sequences S_1, S_2 and a binary functor $\mathcal{H}$ yielding an object and states that

(Sch. 2) $S_1 \restriction \theta = S_2 \restriction \theta$

provided

- $\theta \subseteq \mathrm{dom}\, S_1$ and $\theta \subseteq \mathrm{dom}\, S_2$ and
- for every ordinal number α such that $\alpha \in \theta$ there exists a many sorted set S_3 indexed by $\delta(\alpha)$ such that $S_1(\alpha) = S_3$ and for every o such that $o \in \delta(\alpha)$ holds $S_3(o) = \mathcal{H}(o, S_1 \restriction \alpha)$ and
- for every ordinal number α such that $\alpha \in \theta$ there exists a many sorted set S_3 indexed by $\delta(\alpha)$ such that $S_2(\alpha) = S_3$ and for every o such that $o \in \delta(\alpha)$ holds $S_3(o) = \mathcal{H}(o, S_2 \restriction \alpha)$.

3. The Opposite Surreal Number

Let us consider α. The functor $\mathrm{opposite}_{\mathbf{No}}(\alpha)$ yielding a many sorted set indexed by Dayα is defined by

(Def. 2) there exists a $\subseteq$-monotone, function yielding transfinite sequence S such that $\mathrm{dom}\, S = \mathrm{succ}\, \alpha$ and $it = S(\alpha)$ and for every β such that $\beta \in \mathrm{succ}\, \alpha$ there exists a many sorted set S_5 indexed by Dayβ such that $S(\beta) = S_5$ and for every o such that $o \in \mathrm{Day}\beta$ holds $S_5(o) = \langle\!\langle (\bigcup \mathrm{rng}(S \restriction \beta))^\circ(\mathrm{R}_o), (\bigcup \mathrm{rng}(S \restriction \beta))^\circ(\mathrm{L}_o) \rangle\!\rangle$.

Now we state the propositions:

(4) Let us consider a $\subseteq$-monotone, function yielding transfinite sequence S. Suppose for every β such that $\beta \in \mathrm{dom}\, S$ there exists a many sorted set S_5 indexed by Dayβ such that $S(\beta) = S_5$ and for every o such that $o \in \mathrm{Day}\beta$ holds $S_5(o) = \langle\!\langle (\bigcup \mathrm{rng}(S \restriction \beta))^\circ(\mathrm{R}_o), (\bigcup \mathrm{rng}(S \restriction \beta))^\circ(\mathrm{L}_o) \rangle\!\rangle$. If $\alpha \in \mathrm{dom}\, S$, then $\mathrm{opposite}_{\mathbf{No}}(\alpha) = S(\alpha)$.

Proof: Define $\delta(\text{ordinal number}) = \mathrm{Day}\$_1$. Define $\mathcal{H}(\text{object}, \subseteq\text{-monotone, function yielding transfinite sequence}) = \langle\!\langle (\bigcup \mathrm{rng}\, \$_2)^\circ(\mathrm{R}_{\$_1}), (\bigcup \mathrm{rng}\, \$_2)^\circ (\mathrm{L}_{\$_1}) \rangle\!\rangle$. Consider S_2 being a $\subseteq$-monotone, function yielding transfinite sequence such that $\mathrm{dom}\, S_2 = \mathrm{succ}\, \alpha$ and $S_2(\alpha) = \mathrm{opposite}_{\mathbf{No}}(\alpha)$ and for every ordinal number β such that $\beta \in \mathrm{succ}\, \alpha$ there exists a many sorted set S_5 indexed by $\delta(\beta)$ such that $S_2(\beta) = S_5$ and for every object x such that $x \in \delta(\beta)$ holds $S_5(x) = \mathcal{H}(x, S_2 \restriction \beta)$. $S_1 \restriction \mathrm{succ}\, \alpha = S_2 \restriction \mathrm{succ}\, \alpha$. $\square$

(5) Let us consider a $\subseteq$-monotone, function yielding transfinite sequence f. Suppose $o \in \mathrm{dom}(f(\beta))$ and $\beta \in \alpha$. Then

   (i) $o \in \mathrm{dom}(\bigcup \mathrm{rng}(f \restriction \alpha))$, and

   (ii) $(\bigcup \mathrm{rng}(f \restriction \alpha))(o) = (\bigcup \mathrm{rng}\, f)(o)$.

The theorem is a consequence of (2).

(6) Let us consider a $\subseteq$-monotone, function yielding transfinite sequence f, and ordinal numbers α, β. Suppose $o \in \mathrm{dom}(f(\beta))$ and $\beta \in \alpha$. Then $(\bigcup \mathrm{rng}(f \restriction \alpha))(o) = (\bigcup \mathrm{rng}\, f)(o)$. The theorem is a consequence of (2).

Let us consider x. The functor $-x$ yielding a set is defined by the term

(Def. 3) $(\mathrm{opposite}_{\mathbf{No}}(\mathfrak{born}\, x))(x)$.

Let X be a set. The functor $\ominus X$ yielding a set is defined by

(Def. 4) $o \in it$ iff there exists a surreal number x such that $x \in X$ and $o = -x$.

Now we state the proposition:

(7) $-x = \langle\, \ominus \mathrm{R}_x, \ominus \mathrm{L}_x \,\rangle$.

Proof: Set $\alpha = \mathfrak{born}\, x$. Consider S being a $\subseteq$-monotone, function yielding transfinite sequence such that $\mathrm{dom}\, S = \mathrm{succ}\, \alpha$ and $\mathrm{opposite}_{\mathbf{No}}(\alpha) = S(\alpha)$

and for every ordinal number β such that $\beta \in \operatorname{succ}\alpha$ there exists a many sorted set S_5 indexed by Dayβ such that $S(\beta) = S_5$ and for every object x such that $x \in \text{Day}\beta$ holds $S_5(x) = \langle\!\langle (\bigcup \text{rng}(S{\restriction}\beta))^\circ(\text{L}_x), (\bigcup \text{rng}(S{\restriction}\beta))^\circ(\text{R}_x) \rangle\!\rangle$. Consider S_3 being a many sorted set indexed by Dayα such that $S(\alpha) = S_3$ and for every object x such that $x \in \text{Day}\alpha$ holds $S_3(x) = \langle\!\langle (\bigcup \text{rng}(S{\restriction}\alpha))^\circ(\text{R}_x), (\bigcup \text{rng}(S{\restriction}\alpha))^\circ(\text{L}_x) \rangle\!\rangle$. Set $U = \bigcup \text{rng}(S{\restriction}\alpha)$. $\ominus\,\text{R}_x \subseteq U^\circ(\text{R}_x)$. $U^\circ(\text{R}_x) \subseteq \ominus\,\text{R}_x$. $\ominus\,\text{L}_x \subseteq U^\circ(\text{L}_x)$. $U^\circ(\text{L}_x) \subseteq \ominus\,\text{L}_x$. □

Let us consider x. One can check that $-x$ is surreal. Let X be a set. Let us note that $\ominus X$ is surreal-membered. Now we state the propositions:

(8) (i) $\text{L}_{(-x)} = \ominus\,\text{R}_x$, and

(ii) $\text{R}_{(-x)} = \ominus\,\text{L}_x$.

The theorem is a consequence of (7).

(9) CONWAY CH. 1 TH. 4(II):
$--x = x$.

Let us consider x. Let us observe that $--x$ reduces to x. Now we state the propositions:

(10) $x \leqslant y$ if and only if $-y \leqslant -x$.

(11) Let us consider a surreal number x, and an ordinal number δ. If $x \in \text{Day}\delta$, then $-x \in \text{Day}\delta$.

(12) $\mathfrak{born}\,x = \mathfrak{born}\,(-x)$.

(13) $\mathfrak{born}_{\approx} x = \mathfrak{born}_{\approx}(-x)$. The theorem is a consequence of (10) and (12).

(14) If $x \in \mathfrak{Born}_{\approx} y$, then $-x \in \mathfrak{Born}_{\approx}(-y)$. The theorem is a consequence of (10), (13), and (12).

(15) Let us consider a surreal-membered set X. Then $\ominus \ominus X = X$.

(16) $\overline{\overline{\ominus X}} \subseteq \overline{\overline{X}}$.

PROOF: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ for every x such that $x = \$_1$ holds $\$_2 = -x$. If $o \in \ominus X$, then there exists an object u such that $\mathcal{P}[o, u]$. Consider f being a function such that $\text{dom}\, f = \ominus X$ and for every object o such that $o \in \ominus X$ holds $\mathcal{P}[o, f(o)]$. $\text{rng}\, f \subseteq X$. f is one-to-one. □

(17) Let us consider a surreal-membered set X. Then $\overline{\overline{X}} = \overline{\overline{\ominus X}}$. The theorem is a consequence of (15) and (16).

Let us consider surreal-membered sets X, Y. Now we state the propositions:

(18) $X \preceq Y$ if and only if $\ominus Y \preceq \ominus X$. The theorem is a consequence of (15).

(19) $X \ll Y$ if and only if $\ominus Y \ll \ominus X$. The theorem is a consequence of (15).

Now we state the propositions:

(20) Let us consider sets X_1, X_2. Then $\ominus(X_1 \cup X_2) = \ominus X_1 \cup \ominus X_2$.

(21) $\{-x\} = \ominus\{x\}$.

(22) $\ominus\emptyset = \emptyset$.

(23) $-\mathbf{0}_{\mathbf{No}} = \mathbf{0}_{\mathbf{No}}$. The theorem is a consequence of (7) and (22).

One can verify that $-\mathbf{0}_{\mathbf{No}}$ reduces to $\mathbf{0}_{\mathbf{No}}$. Now we state the proposition:

(24) $x \approx \mathbf{0}_{\mathbf{No}}$ if and only if $-x \approx \mathbf{0}_{\mathbf{No}}$.

Let α be an ordinal number. The functor $\mathrm{Triangle}\,\alpha$ yielding a subset of $\mathrm{Day}\alpha \times \mathrm{Day}\alpha$ is defined by

(Def. 5) for every surreal numbers x, y, $\langle x, y\rangle \in it$ iff $\mathfrak{born}\,x \oplus \mathfrak{born}\,y \subseteq \alpha$.

Observe that $\mathrm{Triangle}\,\alpha$ is non empty. Now we state the proposition:

(25) Let us consider ordinal numbers α, β. Suppose $\alpha \subseteq \beta$. Then $\mathrm{Triangle}\,\alpha \subseteq \mathrm{Triangle}\,\beta$.

4. The Sum of Surreal Numbers

Let α be an ordinal number. The functor $\mathrm{sum}_{\mathbf{No}}(\alpha)$ yielding a many sorted set indexed by $\mathrm{Triangle}\,\alpha$ is defined by

(Def. 6) there exists a $\subseteq$-monotone, function yielding transfinite sequence S such that $\mathrm{dom}\,S = \mathrm{succ}\,\alpha$ and $it = S(\alpha)$ and for every ordinal number β such that $\beta \in \mathrm{succ}\,\alpha$ there exists a many sorted set S_5 indexed by $\mathrm{Triangle}\,\beta$ such that $S(\beta) = S_5$ and for every object x such that $x \in \mathrm{Triangle}\,\beta$ holds $S_5(x) = \langle(\bigcup \mathrm{rng}(S{\restriction}\beta))^\circ(\mathrm{L}_{\mathrm{L}_x} \times \{\mathrm{R}_x\} \cup \{\mathrm{L}_x\} \times \mathrm{L}_{\mathrm{R}_x}), (\bigcup \mathrm{rng}(S{\restriction}\beta))^\circ(\mathrm{R}_{\mathrm{L}_x} \times \{\mathrm{R}_x\} \cup \{\mathrm{L}_x\} \times \mathrm{R}_{\mathrm{R}_x})\rangle$.

Now we state the proposition:

(26) Let us consider a $\subseteq$-monotone, function yielding transfinite sequence S. Suppose for every ordinal number β such that $\beta \in \mathrm{dom}\,S$ there exists a many sorted set S_5 indexed by $\mathrm{Triangle}\,\beta$ such that $S(\beta) = S_5$ and for every object x such that $x \in \mathrm{Triangle}\,\beta$ holds $S_5(x) = \langle(\bigcup \mathrm{rng}(S{\restriction}\beta))^\circ(\mathrm{L}_{\mathrm{L}_x} \times \{\mathrm{R}_x\} \cup \{\mathrm{L}_x\} \times \mathrm{L}_{\mathrm{R}_x}), (\bigcup \mathrm{rng}(S{\restriction}\beta))^\circ(\mathrm{R}_{\mathrm{L}_x} \times \{\mathrm{R}_x\} \cup \{\mathrm{L}_x\} \times \mathrm{R}_{\mathrm{R}_x})\rangle$. Let us consider an ordinal number α. If $\alpha \in \mathrm{dom}\,S$, then $\mathrm{sum}_{\mathbf{No}}(\alpha) = S(\alpha)$.

Proof: Define $\delta(\text{ordinal number}) = \mathrm{Triangle}\,\$_1$. Define $\mathcal{H}(\text{object}, \subseteq\text{-monotone, function yielding transfinite sequence}) = \langle(\bigcup \mathrm{rng}\,\$_2)^\circ(\mathrm{L}_{\mathrm{L}_{\$_1}} \times \{\mathrm{R}_{\$_1}\} \cup \{\mathrm{L}_{\$_1}\} \times \mathrm{L}_{\mathrm{R}_{\$_1}}), (\bigcup \mathrm{rng}\,\$_2)^\circ(\mathrm{R}_{\mathrm{L}_{\$_1}} \times \{\mathrm{R}_{\$_1}\} \cup \{\mathrm{L}_{\$_1}\} \times \mathrm{R}_{\mathrm{R}_{\$_1}})\rangle$. Consider S_1 being a $\subseteq$-monotone, function yielding transfinite sequence such that $\mathrm{dom}\,S_1 = \mathrm{succ}\,\alpha$ and $\mathrm{sum}_{\mathbf{No}}(\alpha) = S_1(\alpha)$ and for every ordinal number β such that $\beta \in \mathrm{succ}\,\alpha$ there exists a many sorted set S_5 indexed by $\delta(\beta)$ such that $S_1(\beta) = S_5$ and for every object x such that $x \in \delta(\beta)$ holds $S_5(x) = \mathcal{H}(x, S_1{\restriction}\beta)$. $S{\restriction}\mathrm{succ}\,\alpha = S_1{\restriction}\mathrm{succ}\,\alpha$. $\square$

Let x, y be surreal numbers. The functor $x + y$ yielding a set is defined by the term

(Def. 7) $(\mathrm{sum}_{\mathbf{No}}(\mathfrak{born}\, x \oplus \mathfrak{born}\, y))(\langle x, y \rangle)$.

Let X, Y be sets. The functor $X \oplus Y$ yielding a set is defined by

(Def. 8) $o \in it$ iff there exist surreal numbers x, y such that $x \in X$ and $y \in Y$ and $o = x + y$.

Now we state the propositions:

(27) Let us consider a set X. Then $X \oplus \emptyset = \emptyset$.

(28) Let us consider surreal numbers x, y. Then $x + y = \langle (\mathrm{L}_x \oplus \{y\}) \cup (\{x\} \oplus \mathrm{L}_y), (\mathrm{R}_x \oplus \{y\}) \cup (\{x\} \oplus \mathrm{R}_y) \rangle$.

PROOF: Set $B_3 = \mathfrak{born}\, x$. Set $B_5 = \mathfrak{born}\, y$. Set $\alpha = B_3 \oplus B_5$. Consider S being a $\subseteq$-monotone, function yielding transfinite sequence such that $\mathrm{dom}\, S = \mathrm{succ}\, \alpha$ and $\mathrm{sum}_{\mathbf{No}}(\alpha) = S(\alpha)$ and for every ordinal number β such that $\beta \in \mathrm{succ}\, \alpha$ there exists a many sorted set S_5 indexed by $\mathrm{Triangle}\, \beta$ such that $S(\beta) = S_5$ and for every object x such that $x \in \mathrm{Triangle}\, \beta$ holds $S_5(x) = \langle (\bigcup \mathrm{rng}(S \restriction \beta))^\circ(\mathrm{L}_{\mathrm{L}_x} \times \{\mathrm{R}_x\} \cup \{\mathrm{L}_x\} \times \mathrm{L}_{\mathrm{R}_x}), (\bigcup \mathrm{rng}(S \restriction \beta))^\circ(\mathrm{R}_{\mathrm{L}_x} \times \{\mathrm{R}_x\} \cup \{\mathrm{L}_x\} \times \mathrm{R}_{\mathrm{R}_x}) \rangle$. Consider S_3 being a many sorted set indexed by $\mathrm{Triangle}\, \alpha$ such that $S(\alpha) = S_3$ and for every object x such that $x \in \mathrm{Triangle}\, \alpha$ holds $S_3(x) = \langle (\bigcup \mathrm{rng}(S \restriction \alpha))^\circ(\mathrm{L}_{(x)_{\mathbf{1}}} \times \{\mathrm{R}_x\} \cup \{\mathrm{L}_x\} \times \mathrm{L}_{\mathrm{R}_x}), (\bigcup \mathrm{rng}(S \restriction \alpha))^\circ(\mathrm{R}_{\mathrm{L}_x} \times \{\mathrm{R}_x\} \cup \{\mathrm{L}_x\} \times \mathrm{R}_{\mathrm{R}_x}) \rangle$. Set $U = \bigcup \mathrm{rng}(S \restriction \alpha)$. $U^\circ(\mathrm{L}_x \times \{y\}) \subseteq \mathrm{L}_x \oplus \{y\}$. $\mathrm{L}_x \oplus \{y\} \subseteq U^\circ(\mathrm{L}_x \times \{y\})$. $U^\circ(\mathrm{R}_x \times \{y\}) \subseteq \mathrm{R}_x \oplus \{y\}$. $\mathrm{R}_x \oplus \{y\} \subseteq U^\circ(\mathrm{R}_x \times \{y\})$. $U^\circ(\{x\} \times \mathrm{L}_y) \subseteq \{x\} \oplus \mathrm{L}_y$. $\{x\} \oplus \mathrm{L}_y \subseteq U^\circ(\{x\} \times \mathrm{L}_y)$. $U^\circ(\{x\} \times \mathrm{R}_y) \subseteq \{x\} \oplus \mathrm{R}_y$. $\{x\} \oplus \mathrm{R}_y \subseteq U^\circ(\{x\} \times \mathrm{R}_y)$. $\square$

(29) COMMUTATIVITY OF ADDITION FOR SURREAL NUMBER, CONWAY CH. 1 TH. 3(II):
$x + y = y + x$.

PROOF: Define $\mathcal{P}[\text{ordinal number}] \equiv$ for every surreal numbers x, y such that $\mathfrak{born}\, x \oplus \mathfrak{born}\, y \subseteq \$_1$ holds $x + y = y + x$. For every ordinal number δ such that for every ordinal number γ such that $\gamma \in \delta$ holds $\mathcal{P}[\gamma]$ holds $\mathcal{P}[\delta]$. For every ordinal number δ, $\mathcal{P}[\delta]$. $\square$

Let x, y be surreal numbers. Let us note that the functor $x + y$ is commutative. Now we state the proposition:

(30) Let us consider sets X, Y. Then $X \oplus Y = Y \oplus X$.

Let X, Y be sets. One can verify that the functor $X \oplus Y$ is commutative.

Let us consider x and y. Let us note that $x + y$ is surreal. Let x, y be surreal numbers. The functor $x - y$ yielding a surreal number is defined by the term

(Def. 9) $x + -y$.

Now we state the proposition:

(31) $\mathfrak{born}\, (x + y) \subseteq \mathfrak{born}\, x \oplus \mathfrak{born}\, y$.

Let X, Y be sets. Let us note that $X \oplus Y$ is surreal-membered. Now we state the propositions:

(32) Transitive Law of Addition for Surreal Number, Conway Ch. 1 Th. 5:
$x \leqslant y$ if and only if $x + z \leqslant y + z$.

(33) Let us consider sets X_1, X_2, Y. Then $(X_1 \cup X_2) \oplus Y = (X_1 \oplus Y) \cup (X_2 \oplus Y)$.

(34) Let us consider sets X, Y_1, Y_2. Then $X \oplus (Y_1 \cup Y_2) = (X \oplus Y_1) \cup (X \oplus Y_2)$.

(35) Let us consider sets X_1, X_2, Y_1, Y_2. Suppose $X_1 \lessdot X_2$ and $Y_1 \lessdot Y_2$. Then $X_1 \oplus Y_1 \lessdot X_2 \oplus Y_2$. The theorem is a consequence of (32).

(36) $\{x\} \oplus \{y\} = \{x + y\}$.

(37) Associativity of Addition for Surreal Number, Conway Ch. 1 Th. 3(iii):
$(x + y) + z = x + (y + z)$.
Proof: Define $\mathcal{P}$[ordinal number] $\equiv$ for every surreal numbers x, y, z such that $(\mathfrak{born}\, x \oplus \mathfrak{born}\, y) \oplus \mathfrak{born}\, z \subseteq \$_1$ holds $(x + y) + z = x + (y + z)$. For every ordinal number δ such that for every ordinal number γ such that $\gamma \in \delta$ holds $\mathcal{P}[\gamma]$ holds $\mathcal{P}[\delta]$. For every ordinal number δ, $\mathcal{P}[\delta]$. □

(38) Additive Identity for Surreal Number, Conway Ch. 1 Th. 3(i):
$x + \mathbf{0_{No}} = x$.
Proof: Set $y = \mathbf{0_{No}}$. Define $\mathcal{P}$[ordinal number] $\equiv$ for every surreal number x such that $\mathfrak{born}\, x = \$_1$ holds $x + y = x$. For every ordinal number δ such that for every ordinal number γ such that $\gamma \in \delta$ holds $\mathcal{P}[\gamma]$ holds $\mathcal{P}[\delta]$. For every ordinal number δ, $\mathcal{P}[\delta]$. □

Let us consider x. Let us note that $x + \mathbf{0_{No}}$ reduces to x. Now we state the proposition:

(39) Property of The Aditive Inverse for Surreal Number, Conway Ch. 1 Th. 4(iii):
$x - x \approx \mathbf{0_{No}}$.
Proof: Set $y = \mathbf{0_{No}}$. Define $\mathcal{P}$[ordinal number] $\equiv$ for every surreal number x such that $\mathfrak{born}\, x = \$_1$ holds $x + -x \approx y$. For every ordinal number δ such that for every ordinal number γ such that $\gamma \in \delta$ holds $\mathcal{P}[\gamma]$ holds $\mathcal{P}[\delta]$ by (7), (28), [8, (43)], [9, (1)]. For every ordinal number δ, $\mathcal{P}[\delta]$. □

(40) Conway Ch. 1 Th. 4(i):
$-(x + y) = -x + -y$.
Proof: Define $\mathcal{P}$[ordinal number] $\equiv$ for every surreal numbers x, y such that $\mathfrak{born}\, x \oplus \mathfrak{born}\, y \subseteq \$_1$ holds $-(x + y) = -x + -y$. For every ordinal number δ such that for every ordinal number γ such that $\gamma \in \delta$ holds $\mathcal{P}[\gamma]$ holds $\mathcal{P}[\delta]$. For every ordinal number δ, $\mathcal{P}[\delta]$. □

(41) $x + y \leqslant z$ if and only if $x \leqslant z - y$.
PROOF: If $x+y \leqslant z$, then $x \leqslant z-y$. $x+y \leqslant z+-y+y$. $x+y \leqslant z+(-y+y)$. $y - y \approx \mathbf{0_{No}}$. $z + (-y + y) \leqslant z + \mathbf{0_{No}} = z$. □

(42) $x + y < z$ if and only if $x < z - y$.
PROOF: If $x + y < z$, then $x < z - y$. $z + -y \leqslant x + y + -y$. $z + -y \leqslant x + (y + -y)$. $y - y \approx \mathbf{0_{No}}$. $x + (y + -y) \leqslant x + \mathbf{0_{No}} = x$. □

(43) If $x \leqslant y$ and $z \leqslant t$, then $x + z \leqslant y + t$. The theorem is a consequence of (32).

(44) If $x \leqslant y$ and $z < t$, then $x + z < y + t$. The theorem is a consequence of (42), (39), (32), and (37).

(45) $x < y$ if and only if $\mathbf{0_{No}} < y - x$. The theorem is a consequence of (42).

(46) $x < y$ if and only if $x - y < \mathbf{0_{No}}$. The theorem is a consequence of (41).

(47) If $x - y \approx \mathbf{0_{No}}$, then $x \approx y$. The theorem is a consequence of (39), (37), and (43).

Let x be an object. Assume x is surreal. The functor $-'x$ yielding a surreal number is defined by

(Def. 10) for every surreal number x_1 such that $x_1 = x$ holds $it = -x_1$.

Let a be a surreal number. We identify $-'x$ with $-a$. Let x, y be objects. Assume x is surreal and y is surreal. The functor $x+'y$ yielding a surreal number is defined by

(Def. 11) for every surreal numbers x_1, y_1 such that $x_1 = x$ and $y_1 = y$ holds $it = x_1 + y_1$.

Let a, b be surreal numbers. We identify $x +' y$ with $a + b$.

5. The Product of Superreal Numbers

Let α be an ordinal number. The functor $\mathrm{mult}_{\mathbf{No}}(\alpha)$ yielding a many sorted set indexed by Triangle α is defined by

(Def. 12) there exists a $\subseteq$-monotone, function yielding transfinite sequence S such that $\mathrm{dom}\, S = \mathrm{succ}\, \alpha$ and $it = S(\alpha)$ and for every ordinal number β such that $\beta \in \mathrm{succ}\, \alpha$ there exists a many sorted set S_5 indexed by Triangle β such that $S(\beta) = S_5$ and for every object x such that $x \in$ Triangle β holds $S_5(x) = \langle \{((\bigcup \mathrm{rng}(S{\restriction}\beta))(\langle x_6, \mathrm{R}_x \rangle) +' (\bigcup \mathrm{rng}(S{\restriction}\beta))(\langle \mathrm{L}_x, y_4 \rangle)) +' -'(\bigcup \mathrm{rng}(S{\restriction}\beta))(\langle x_6, y_4 \rangle)$, where x_6 is an element of $\mathrm{L}_{\mathrm{L}_x}$, y_4 is an element of $\mathrm{L}_{\mathrm{R}_x} : x_6 \in \mathrm{L}_{\mathrm{L}_x}$ and $y_4 \in \mathrm{L}_{\mathrm{R}_x}\} \cup \{((\bigcup \mathrm{rng}(S{\restriction}\beta))(\langle x_7, \mathrm{R}_x \rangle) +' (\bigcup \mathrm{rng}(S{\restriction}\beta))(\langle \mathrm{L}_x, y_5 \rangle)) +' -'(\bigcup \mathrm{rng}(S{\restriction}\beta))(\langle x_7, y_5 \rangle)$, where x_7 is an element of $\mathrm{R}_{\mathrm{L}_x}$, y_5 is an element of $\mathrm{R}_{\mathrm{R}_x} : x_7 \in \mathrm{R}_{\mathrm{L}_x}$ and $y_5 \in \mathrm{R}_{\mathrm{R}_x}\}$, $\{((\bigcup \mathrm{rng}(S{\restriction}\beta))(\langle x_6, \mathrm{R}_x \rangle) +' (\bigcup \mathrm{rng}(S{\restriction}\beta))(\langle \mathrm{L}_x, y_5 \rangle)) +' -'(\bigcup \mathrm{rng}(S{\restriction}\beta))(\langle x_6, y_5 \rangle)$, where x_6 is an element of $\mathrm{L}_{\mathrm{L}_x}, y_5$ is an element of $\mathrm{R}_{\mathrm{R}_x}$: $x_6 \in \mathrm{L}_{\mathrm{L}_x}$ and $y_5 \in \mathrm{R}_{\mathrm{R}_x}\} \cup \{((\bigcup \mathrm{rng}(S \restriction \beta))(\langle x_7, \mathrm{R}_x \rangle) +' (\bigcup \mathrm{rng}(S \restriction \beta))(\langle \mathrm{L}_x, y_4 \rangle)) +' -'(\bigcup \mathrm{rng}(S \restriction \beta))(\langle x_7, y_4 \rangle)$, where x_7 is an element of $\mathrm{R}_{\mathrm{L}_x}, y_4$ is an element of $\mathrm{L}_{\mathrm{R}_x}$: $x_7 \in \mathrm{R}_{\mathrm{L}_x}$ and $y_4 \in \mathrm{L}_{\mathrm{R}_x}\}\rangle$.

Let x, y be surreal numbers. The functor $x \cdot y$ yielding a set is defined by the term

(Def. 13) $(\mathrm{mult}_{\mathbf{No}}(\mathfrak{born}\, x \oplus \mathfrak{born}\, y))(\langle x, y \rangle)$.

Now we state the proposition:

(48) Let us consider a $\subseteq$-monotone, function yielding transfinite sequence S. Suppose for every ordinal number β such that $\beta \in \mathrm{dom}\, S$ there exists a many sorted set S_5 indexed by Triangle β such that $S(\beta) = S_5$ and for every object x such that $x \in$ Triangle β holds $S_5(x) = \langle \{((\bigcup \mathrm{rng}(S \restriction \beta))(\langle x_6, \mathrm{R}_x \rangle) +' (\bigcup \mathrm{rng}(S \restriction \beta))(\langle \mathrm{L}_x, y_4 \rangle)) +' -'(\bigcup \mathrm{rng}(S \restriction \beta))(\langle x_6, y_4 \rangle)$, where x_6 is an element of $\mathrm{L}_{\mathrm{L}_x}, y_4$ is an element of $\mathrm{L}_{\mathrm{R}_x}$: $x_6 \in \mathrm{L}_{\mathrm{L}_x}$ and $y_4 \in \mathrm{L}_{\mathrm{R}_x}\} \cup \{((\bigcup \mathrm{rng}(S \restriction \beta))(\langle x_7, \mathrm{R}_x \rangle) +' (\bigcup \mathrm{rng}(S \restriction \beta))(\langle \mathrm{L}_x, y_5 \rangle)) +' -'(\bigcup \mathrm{rng}(S \restriction \beta))(\langle x_7, y_5 \rangle)$, where x_7 is an element of $\mathrm{R}_{\mathrm{L}_x}, y_5$ is an element of $\mathrm{R}_{\mathrm{R}_x}$: $x_7 \in \mathrm{R}_{\mathrm{L}_x}$ and $y_5 \in \mathrm{R}_{\mathrm{R}_x}\}, \{((\bigcup \mathrm{rng}(S \restriction \beta))(\langle x_6, \mathrm{R}_x \rangle) +' (\bigcup \mathrm{rng}(S \restriction \beta))(\langle \mathrm{L}_x, y_5 \rangle)) +' -'(\bigcup \mathrm{rng}(S \restriction \beta))(\langle x_6, y_5 \rangle)$, where x_6 is an element of $\mathrm{L}_{\mathrm{L}_x}, y_5$ is an element of $\mathrm{R}_{\mathrm{R}_x}$: $x_6 \in \mathrm{L}_{\mathrm{L}_x}$ and $y_5 \in \mathrm{R}_{\mathrm{R}_x}\} \cup \{((\bigcup \mathrm{rng}(S \restriction \beta))(\langle x_7, \mathrm{R}_x \rangle) +' (\bigcup \mathrm{rng}(S \restriction \beta))(\langle \mathrm{L}_x, y_4 \rangle)) +' -'(\bigcup \mathrm{rng}(S \restriction \beta))(\langle x_7, y_4 \rangle)$, where x_7 is an element of $\mathrm{R}_{\mathrm{L}_x}, y_4$ is an element of $\mathrm{L}_{\mathrm{R}_x}$: $x_7 \in \mathrm{R}_{\mathrm{L}_x}$ and $y_4 \in \mathrm{L}_{\mathrm{R}_x}\}\rangle$. Let us consider an ordinal number α. If $\alpha \in \mathrm{dom}\, S$, then $\mathrm{mult}_{\mathbf{No}}(\alpha) = S(\alpha)$.

PROOF: Define δ(ordinal number) $=$ Triangle $\$_1$. Define $\mathcal{H}$(object, $\subseteq$-monotone, function yielding transfinite sequence) $= \langle \{((\bigcup \mathrm{rng}\, \$_2)(\langle x_6, \mathrm{R}_{\$_1} \rangle) +' (\bigcup \mathrm{rng}\, \$_2)(\langle \mathrm{L}_{\$_1}, y_4 \rangle)) +' -'(\bigcup \mathrm{rng}\, \$_2)(\langle x_6, y_4 \rangle)$, where x_6 is an element of $\mathrm{L}_{\mathrm{L}_{\$_1}}, y_4$ is an element of $\mathrm{L}_{\mathrm{R}_{\$_1}}$: $x_6 \in \mathrm{L}_{\mathrm{L}_{\$_1}}$ and $y_4 \in \mathrm{L}_{\mathrm{R}_{\$_1}}\} \cup \{((\bigcup \mathrm{rng}\, \$_2)(\langle x_7, \mathrm{R}_{\$_1} \rangle) +' (\bigcup \mathrm{rng}\, \$_2)(\langle \mathrm{L}_{\$_1}, y_5 \rangle)) +' -'(\bigcup \mathrm{rng}\, \$_2)(\langle x_7, y_5 \rangle)$, where x_7 is an element of $\mathrm{R}_{\mathrm{L}_{\$_1}}, y_5$ is an element of $\mathrm{R}_{\mathrm{R}_{\$_1}}$: $x_7 \in \mathrm{R}_{\mathrm{L}_{\$_1}}$ and $y_5 \in \mathrm{R}_{\mathrm{R}_{\$_1}}\}, \{((\bigcup \mathrm{rng}\, \$_2)(\langle x_6, \mathrm{R}_{\$_1} \rangle) +' (\bigcup \mathrm{rng}\, \$_2)(\langle \mathrm{L}_{\$_1}, y_5 \rangle)) +' -'(\bigcup \mathrm{rng}\, \$_2)(\langle x_6, y_5 \rangle)$, where x_6 is an element of $\mathrm{L}_{\mathrm{L}_{\$_1}}, y_5$ is an element of $\mathrm{R}_{\mathrm{R}_{\$_1}}$: $x_6 \in \mathrm{L}_{\mathrm{L}_{\$_1}}$ and $y_5 \in \mathrm{R}_{\mathrm{R}_{\$_1}}\} \cup \{((\bigcup \mathrm{rng}\, \$_2)(\langle x_7, \mathrm{R}_{\$_1} \rangle) +' (\bigcup \mathrm{rng}\, \$_2)(\langle \mathrm{L}_{\$_1}, y_4 \rangle)) +' -'(\bigcup \mathrm{rng}\, \$_2)(\langle x_7, y_4 \rangle)$, where x_7 is an element of $\mathrm{R}_{\mathrm{L}_{\$_1}}, y_4$ is an element of $\mathrm{L}_{\mathrm{R}_{\$_1}}$: $x_7 \in \mathrm{R}_{\mathrm{L}_{\$_1}}$ and $y_4 \in \mathrm{L}_{\mathrm{R}_{\$_1}}\}\rangle$. Consider S_1 being a $\subseteq$-monotone, function yielding transfinite sequence such that $\mathrm{dom}\, S_1 = \mathrm{succ}\, \alpha$ and $\mathrm{mult}_{\mathbf{No}}(\alpha) = S_1(\alpha)$ and for every ordinal number β such that $\beta \in \mathrm{succ}\, \alpha$ there exists a many sorted set S_5 indexed by $\delta(\beta)$ such that $S_1(\beta) = S_5$ and for every object x such that $x \in \delta(\beta)$ holds $S_5(x) = \mathcal{H}(x, S_1 \restriction \beta)$. $S \restriction \mathrm{succ}\, \alpha = S_1 \restriction \mathrm{succ}\, \alpha$. □

Let x, y be surreal numbers and X, Y be sets. The functor $\mathrm{comp}(X, x, y, Y)$ yielding a set is defined by

(Def. 14) $o \in it$ iff there exist surreal numbers x_1, y_1 such that $o = (x_1 \cdot y +' x \cdot y_1) +' -'x_1 \cdot y_1$ and $x_1 \in X$ and $y_1 \in Y$.

Now we state the propositions:

(49) Let us consider a set X. Then $\mathrm{comp}(X, x, y, \emptyset) = \emptyset$.

(50) Let us consider surreal numbers x, y. Then $x \cdot y = \langle \mathrm{comp}(\mathrm{L}_x, x, y, \mathrm{L}_y) \cup \mathrm{comp}(\mathrm{R}_x, x, y, \mathrm{R}_y), \mathrm{comp}(\mathrm{L}_x, x, y, \mathrm{R}_y) \cup \mathrm{comp}(\mathrm{R}_x, x, y, \mathrm{L}_y)\rangle$.

PROOF: Set $B_3 = \mathfrak{b}\mathrm{orn}\, x$. Set $B_5 = \mathfrak{b}\mathrm{orn}\, y$. Set $\alpha = B_3 \oplus B_5$. Define $\mathcal{H}$(object, $\subseteq$-monotone, function yielding transfinite sequence) $=$ $\langle\{((\bigcup \mathrm{rng}\,\$_2)(\langle x_6, \mathrm{R}_{\$_1}\rangle) +' (\bigcup \mathrm{rng}\,\$_2)(\langle \mathrm{L}_{\$_1}, y_4\rangle)) +' -'(\bigcup \mathrm{rng}\,\$_2)(\langle x_6, y_4\rangle)$, where x_6 is an element of $\mathrm{L}_{\mathrm{L}_{\$_1}}$, y_4 is an element of $\mathrm{L}_{\mathrm{R}_{\$_1}}$: $x_6 \in \mathrm{L}_{\mathrm{L}_{\$_1}}$ and $y_4 \in \mathrm{L}_{\mathrm{R}_{\$_1}}\} \cup \{((\bigcup \mathrm{rng}\,\$_2)(\langle x_7, \mathrm{R}_{\$_1}\rangle) +' (\bigcup \mathrm{rng}\,\$_2)(\langle \mathrm{L}_{\$_1}, y_5\rangle)) +' -'(\bigcup \mathrm{rng}\,\$_2)(\langle x_7, y_5\rangle)$, where x_7 is an element of $\mathrm{R}_{\mathrm{L}_{\$_1}}$, y_5 is an element of $\mathrm{R}_{\mathrm{R}_{\$_1}}$: $x_7 \in \mathrm{R}_{\mathrm{L}_{\$_1}}$ and $y_5 \in \mathrm{R}_{\mathrm{R}_{\$_1}}\}$, $\{((\bigcup \mathrm{rng}\,\$_2)(\langle x_6, \mathrm{R}_{\$_1}\rangle) +' (\bigcup \mathrm{rng}\,\$_2)(\langle \mathrm{L}_{\$_1}, y_5\rangle)) +' -'(\bigcup \mathrm{rng}\,\$_2)(\langle x_6, y_5\rangle)$, where x_6 is an element of $\mathrm{L}_{\mathrm{L}_{\$_1}}$, y_5 is an element of $\mathrm{R}_{\mathrm{R}_{\$_1}}$: $x_6 \in \mathrm{L}_{\mathrm{L}_{\$_1}}$ and $y_5 \in \mathrm{R}_{\mathrm{R}_{\$_1}}\} \cup \{((\bigcup \mathrm{rng}\,\$_2)(\langle x_7, \mathrm{R}_{\$_1}\rangle) +' (\bigcup \mathrm{rng}\,\$_2)(\langle \mathrm{L}_{\$_1}, y_4\rangle)) +' -'(\bigcup \mathrm{rng}\,\$_2)(\langle x_7, y_4\rangle)$, where x_7 is an element of $\mathrm{R}_{\mathrm{L}_{\$_1}}$, y_4 is an element of $\mathrm{L}_{\mathrm{R}_{\$_1}}$: $x_7 \in \mathrm{R}_{\mathrm{L}_{\$_1}}$ and $y_4 \in \mathrm{L}_{\mathrm{R}_{\$_1}}\}\rangle$. Consider S being a $\subseteq$-monotone, function yielding transfinite sequence such that $\mathrm{dom}\, S = \mathrm{succ}\, \alpha$ and $\mathrm{mult}_{\mathbf{No}}(\alpha) = S(\alpha)$ and for every ordinal number β such that $\beta \in \mathrm{succ}\, \alpha$ there exists a many sorted set S_5 indexed by $\mathrm{Triangle}\, \beta$ such that $S(\beta) = S_5$ and for every object x such that $x \in \mathrm{Triangle}\, \beta$ holds $S_5(x) = \mathcal{H}(x, S{\restriction}\beta)$. Consider S_3 being a many sorted set indexed by $\mathrm{Triangle}\, \alpha$ such that $S(\alpha) = S_3$ and for every object x such that $x \in \mathrm{Triangle}\, \alpha$ holds $S_3(x) = \mathcal{H}(x, S{\restriction}\alpha)$. Set $U = \bigcup \mathrm{rng}(S{\restriction}\alpha)$. For every surreal-membered sets X, Y such that $X \subseteq \mathrm{L}_x \cup \mathrm{R}_x$ and $Y \subseteq \mathrm{L}_y \cup \mathrm{R}_y$ holds $\{(U(\langle x_6, y\rangle) +' U(\langle x, y_4\rangle)) +' -'U(\langle x_6, y_4\rangle)$, where x_6 is an element of X, y_4 is an element of Y : $x_6 \in X$ and $y_4 \in Y\} = \mathrm{comp}(X, x, y, Y)$. $\square$

(51) (i) for every x and y, $x \cdot y$ is a surreal number, and

(ii) for every x and y, $x \cdot y = y \cdot x$, and

(iii) for every surreal numbers x_1, x_2, y, x_4, x_5 such that $x_1 \approx x_2$ and $x_4 = x_1 \cdot y$ and $x_5 = x_2 \cdot y$ holds $x_4 \approx x_5$, and

(iv) for every surreal numbers x_1, x_2, y_1, y_2, x_{12}, x_{21}, x_{11}, x_{22} such that $x_{11} = x_1 \cdot y_1$ and $x_{12} = x_1 \cdot y_2$ and $x_{21} = x_2 \cdot y_1$ and $x_{22} = x_2 \cdot y_2$ and $x_1 < x_2$ and $y_1 < y_2$ holds $x_{12} + x_{21} < x_{11} + x_{22}$.

PROOF: Define $\mathcal{P}$[ordinal number, surreal number, surreal number] $\equiv$ if $\mathfrak{b}\mathrm{orn}\,\$_2 \oplus \mathfrak{b}\mathrm{orn}\,\$_3 \subseteq \$_1$, then $\$_2 \cdot \$_3 = \$_3 \cdot \$_2$. Define $\mathcal{S}$[ordinal number, surreal number, surreal number] $\equiv$ if $\mathfrak{b}\mathrm{orn}\,\$_2 \oplus \mathfrak{b}\mathrm{orn}\,\$_3 \subseteq \$_1$, then $\$_2 \cdot \$_3$ is a surreal number. Define $\mathcal{T}$[ordinal number, surreal number, surreal number, surreal number] $\equiv$ for every surreal numbers x_4, x_5 such that $\mathfrak{b}\mathrm{orn}\,\$_2 \oplus \mathfrak{b}\mathrm{orn}\,\$_4 \subseteq \$_1$

and $\mathfrak{born}\,\$_3 \oplus \mathfrak{born}\,\$_4 \subseteq \$_1$ and $\$_2 \approx \$_3$ and $x_4 = \$_2 \cdot \$_4$ and $x_5 = \$_3 \cdot \$_4$ holds $x_4 \approx x_5$. Define $\mathcal{V}$[ordinal number, surreal number, surreal number, surreal number, surreal number] $\equiv$ for every surreal numbers x_{12}, x_{21}, x_{11}, x_{22} such that $\mathfrak{born}\,\$_2 \oplus \mathfrak{born}\,\$_4 \subseteq \$_1$ and $\mathfrak{born}\,\$_3 \oplus \mathfrak{born}\,\$_4 \subseteq \$_1$ and $\mathfrak{born}\,\$_2 \oplus \mathfrak{born}\,\$_5 \subseteq \$_1$ and $\mathfrak{born}\,\$_3 \oplus \mathfrak{born}\,\$_5 \subseteq \$_1$ and $x_{11} = \$_2 \cdot \$_4$ and $x_{12} = \$_2 \cdot \$_5$ and $x_{21} = \$_3 \cdot \$_4$ and $x_{22} = \$_3 \cdot \$_5$ and $\$_2 < \$_3 < \$_5$ holds $x_{12} + x_{21} < x_{11} + x_{22}$. Define $\mathcal{F}$[ordinal number] $\equiv$ for every x and y, $\mathcal{P}[\$_1, x, y]$. Define $\mathcal{G}$[ordinal number] $\equiv$ for every x and y, $\mathcal{S}[\$_1, x, y]$. Define $\mathcal{H}$[ordinal number] $\equiv$ for every surreal numbers x_1, x_2, y, $\mathcal{T}[\$_1, x_1, x_2, y]$. Define $\mathcal{I}$[ordinal number] $\equiv$ for every surreal numbers x_1, x_2, y_1, y_2, $\mathcal{V}[\$_1, x_1, x_2, y_1, y_2]$. Define θ[ordinal number] $\equiv \mathcal{F}[\$_1]$ and $\mathcal{G}[\$_1]$ and $\mathcal{H}[\$_1]$ and $\mathcal{I}[\$_1]$. For every ordinal number δ such that for every ordinal number γ such that $\gamma \in \delta$ holds $\theta[\gamma]$ holds $\theta[\delta]$. For every ordinal number E, $\theta[E]$. For every surreal numbers x_1, x_2, y, x_4, x_5 such that $x_1 \approx x_2$ and $x_4 = x_1 \cdot y$ and $x_5 = x_2 \cdot y$ holds $x_4 \approx x_5$. □

Let a, b be surreal numbers. Observe that $a \cdot b$ is surreal. Let a, b be surreal numbers. One can check that the functor $a \cdot b$ is commutative. Let x, y be surreal numbers and X, Y be sets. Observe that $\mathrm{comp}(X, x, y, Y)$ is surreal-membered. Let us observe that the functor $\mathrm{comp}(X, x, y, Y)$ is defined by

(Def. 15) $o \in it$ iff there exist surreal numbers x_1, y_1 such that $o = x_1 \cdot y + x \cdot y_1 - x_1 \cdot y_1$ and $x_1 \in X$ and $y_1 \in Y$.

Now we state the propositions:

(52) $\mathrm{comp}(\{z\}, x, y, \{t\}) = \{z \cdot y + x \cdot t - z \cdot t\}$.

(53) Let us consider sets X, Y. Then $\mathrm{comp}(X, x, y, Y) = \mathrm{comp}(Y, y, x, X)$.

(54) Conway Ch. 1 Th. 8(i):
Let us consider surreal numbers x_1, x_2, y. If $x_1 \approx x_2$, then $x_1 \cdot y \approx x_2 \cdot y$.

(55) Conway Ch. 1 Th. 8(iii):
Let us consider surreal numbers x_1, x_2, y_1, y_2. Suppose $x_1 < x_2$ and $y_1 < y_2$. Then $x_1 \cdot y_2 + x_2 \cdot y_1 < x_1 \cdot y_1 + x_2 \cdot y_2$.

(56) Conway Ch. 1 Th. 7(i):
$x \cdot (\mathbf{0_{No}}) = \mathbf{0_{No}}$. The theorem is a consequence of (49) and (50).

(57) Multiplicative Identity for Surreal Number, Conway Ch. 1 Th. 7(ii):
$x \cdot (\mathbf{1_{No}}) = x$.
Proof: Define $\mathcal{P}$[ordinal number] $\equiv$ for every x such that $\mathfrak{born}\,x \subseteq \$_1$ holds $x \cdot (\mathbf{1_{No}}) = x$. For every ordinal number δ such that for every ordinal number γ such that $\gamma \in \delta$ holds $\mathcal{P}[\gamma]$ holds $\mathcal{P}[\delta]$. For every ordinal number δ, $\mathcal{P}[\delta]$. □

Let us consider x. Observe that $x \cdot (\mathbf{0_{No}})$ reduces to $\mathbf{0_{No}}$ and $x \cdot (\mathbf{1_{No}})$ reduces to x. Now we state the proposition:

(58) CONWAY CH. 1 TH. 7(IV):

(i) $x \cdot (-y) = -x \cdot y$, and

(ii) $(-x) \cdot y = -x \cdot y$, and

(iii) $(-x) \cdot (-y) = x \cdot y$.

Let us consider sets X, Y_1, Y_2. Now we state the propositions:

(59) If $Y_1 \subseteq Y_2$, then $\mathrm{comp}(X, x, y, Y_1) \subseteq \mathrm{comp}(X, x, y, Y_2)$.

(60) $\mathrm{comp}(X, x, y, Y_1 \cup Y_2) = \mathrm{comp}(X, x, y, Y_1) \cup \mathrm{comp}(X, x, y, Y_2)$. The theorem is a consequence of (59).

(61) Let us consider sets X, Y. Suppose for every x such that $x \in X$ there exists y such that $y \in Y$ and $x \approx y$. Then $X \lessdot Y$.

Let us consider sets X_1, X_2. Now we state the propositions:

(62) If $X_1 \lessdot X_2$, then $\ominus X_1 \lessdot \ominus X_2$. The theorem is a consequence of (10).

(63) $\ominus(X_1 \oplus X_2) = \ominus X_1 \oplus \ominus X_2$. The theorem is a consequence of (40).

(64) Let us consider a surreal-membered set X. Then $X \oplus \{\mathbf{0_{No}}\} = X$.

(65) If $x \approx y$, then $-x \approx -y$.

(66) Let us consider surreal numbers x_1, x_2, y_1, y_2. If $x_1 \approx x_2$ and $y_1 \approx y_2$, then $x_1 + y_1 \approx x_2 + y_2$.

(67) DISTRIBUTIVITY OF MULTIPLICATION OVER ADDITION FOR SURREAL NUMBERS, CONWAY CH. 1 TH. 7(V):
$x \cdot (y + z) \approx x \cdot y + x \cdot z$.
PROOF: Define $\mathcal{P}[\text{ordinal number}] \equiv$ for every surreal numbers x, y, z such that $(\mathfrak{born}\, x \oplus \mathfrak{born}\, y) \oplus \mathfrak{born}\, z \subseteq \$_1$ holds $x \cdot (y + z) \approx x \cdot y + x \cdot z$. For every ordinal number δ such that for every ordinal number γ such that $\gamma \in \delta$ holds $\mathcal{P}[\gamma]$ holds $\mathcal{P}[\delta]$. For every ordinal number δ, $\mathcal{P}[\delta]$. □

(68) Let us consider sets X_1, X_2, Y. Then $\mathrm{comp}(X_1 \cup X_2, x, y, Y) = \mathrm{comp}(X_1, x, y, Y) \cup \mathrm{comp}(X_2, x, y, Y)$. The theorem is a consequence of (53) and (60).

(69) ASSOCIATIVITY OF MULTIPLICATION FOR SURREAL NUMBERS, CONWAY CH. 1 TH. 7(VI):
$(x \cdot y) \cdot z \approx x \cdot (y \cdot z)$.
PROOF: Define $\mathcal{P}[\text{ordinal number}] \equiv$ for every surreal numbers x, y, z such that $(\mathfrak{born}\, x \oplus \mathfrak{born}\, y) \oplus \mathfrak{born}\, z \subseteq \$_1$ holds $(x \cdot y) \cdot z \approx x \cdot (y \cdot z)$. For every ordinal number δ such that for every ordinal number γ such that $\gamma \in \delta$ holds $\mathcal{P}[\gamma]$ holds $\mathcal{P}[\delta]$. For every ordinal number δ, $\mathcal{P}[\delta]$. □

(70) If $\mathbf{0_{No}} < x$ and $y < z$, then $y \cdot x < z \cdot x$. The theorem is a consequence of (51).

(71) If $x < \mathbf{0_{No}}$ and $y < z$, then $z \cdot x < y \cdot x$. The theorem is a consequence of (51).

(72) Conway Ch. 1 Th. 9:
$\mathbf{0_{No}} < x \cdot y$ if and only if $x < \mathbf{0_{No}}$ and $y < \mathbf{0_{No}}$ or $\mathbf{0_{No}} < x$ and $\mathbf{0_{No}} < y$. The theorem is a consequence of (51), (10), (58), and (23).

(73) If $\mathbf{0_{No}} < z$ and $x \cdot z < y \cdot z$, then $x < y$. The theorem is a consequence of (51) and (70).

(74) $x \cdot y < \mathbf{0_{No}}$ if and only if $x < \mathbf{0_{No}} < y$ or $\mathbf{0_{No}} < x$ and $y < \mathbf{0_{No}}$. The theorem is a consequence of (23), (10), (58), and (72).

(75) If $\mathbf{0_{No}} \leqslant x$ and $y \leqslant z$, then $y \cdot x \leqslant z \cdot x$. The theorem is a consequence of (51) and (70).

(76) $(x+y) \cdot (x+y) \approx x \cdot x + y \cdot y + (x \cdot y + y \cdot x)$. The theorem is a consequence of (67), (43), and (37).

(77) $x \cdot y \approx \mathbf{0_{No}}$ if and only if $x \approx \mathbf{0_{No}}$ or $y \approx \mathbf{0_{No}}$.

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