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Integration of Game Theoretic and Tree Theoretic Approaches to Conway Numbers

Karol Pąk
Faculty of Computer Science
University of Białystok
Poland

Summary. In this article, we develop our formalised concept of Conway numbers as outlined in [9]. We focus mainly pre-order properties, birthday arithmetic contained in the Chapter 1, *Properties of Order and Equality* of John Conway's seminal book. We also propose a method for the selection of class representatives respecting the relation defined by the pre-ordering in order to facilitate combining the results obtained for the original and tree-theoretic definitions of Conway numbers.

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Introduction

We present a formal analysis of the contents of Chapter 1, *Properties of Order and Equality* of John Conway's seminal book. This section focuses on the pre-order structure of Conway numbers.

Then, using the developed concept of Conway numbers, we thoroughly analyse the properties of surreal birthday arithmetic. We prove the *The Simplicity Theorem* (see Theorem 11 on p. 23 [3]) which can be expressed informally as follows *when x is given as a number, it is always the* simplest *number lying between the L_x and the R_x, where* simplest *means earliest created*. It also makes it easier to manipulate birthday numbers in the context of pre-ordering surreal numbers.

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In the final part, we select the representatives of the equivalence classes that are defined by the relation equivalence relation $\approx$ on surreal numbers such that $x \approx y$ iff $x \leqslant y$ and $y \leqslant x$. Representatives have a minimum-birthday as well as minimal-birthday as well as the left and right components of each representative having the smallest cardinality and such representatives as members.

The formalisation is mainly based on [3, 4, 5, 6], but also uses selected ideas proposed in [1, 2, 10].

1. Preorder of Surreal Numbers

From now on α, β, γ, θ denote ordinal numbers, X denotes a set, o denotes an object, and x, y, z, t, r, l denote surreal numbers.

The functor $\mathbf{1_{No}}$ yielding a surreal number is defined by the term

(Def. 1) $\langle\{\mathbf{0_{No}}\}, \emptyset\rangle$.

Now we state the propositions:

(1) If $y \in \mathrm{L}_x \cup \mathrm{R}_x$, then $\mathfrak{born}\, y \in \mathfrak{born}\, x$.

(2) $\mathrm{L}_x \neq \{x\} \neq \mathrm{R}_x$. The theorem is a consequence of (1).

(3) Preorder of Surreal Numbers – Reflexivity, Conway Ch. 1 Th. 0(iii):
$x \leqslant x$.
Proof: Define $\mathcal{P}$[ordinal number] $\equiv$ for every surreal number x such that $x \in \mathrm{Day}\$_1$ holds $x \leqslant x$. For every ordinal number δ such that for every ordinal number γ such that $\gamma \in \delta$ holds $\mathcal{P}[\gamma]$ holds $\mathcal{P}[\delta]$. For every ordinal number δ, $\mathcal{P}[\delta]$. $\square$

(4) Preorder of Surreal Numbers – Transitivity, Conway Ch. 1 Th. 1:
If $x \leqslant y \leqslant z$, then $x \leqslant z$.
Proof: Define $\mathcal{P}$[ordinal number] $\equiv$ for every surreal numbers x, y, z such that $x \leqslant y \leqslant z$ and $(\mathfrak{born}\, x \oplus \mathfrak{born}\, y) \oplus \mathfrak{born}\, z \subseteq \$_1$ holds $x \leqslant z$. For every ordinal number δ such that for every ordinal number γ such that $\gamma \in \delta$ holds $\mathcal{P}[\gamma]$ holds $\mathcal{P}[\delta]$. For every ordinal number δ, $\mathcal{P}[\delta]$. $\square$

(5) $\mathrm{L}_x \preceq \{x\} \preceq \mathrm{R}_x$.
Proof: Define $\mathcal{P}$[ordinal number] $\equiv$ for every surreal number x such that $\mathfrak{born}\, x \subseteq \$_1$ holds $\mathrm{L}_x \preceq \{x\} \preceq \mathrm{R}_x$. For every ordinal number δ such that for every ordinal number γ such that $\gamma \in \delta$ holds $\mathcal{P}[\gamma]$ holds $\mathcal{P}[\delta]$. For every ordinal number δ, $\mathcal{P}[\delta]$. $\square$

(6) Preorder of Surreal Numbers – Total, Conway Ch. 1 Th. 2(ii):
If $y \not\leqslant x$, then $x \leqslant y$. The theorem is a consequence of (5) and (4).

(7) If α is finite, then Dayα is finite.
Proof: Define $\mathcal{P}$[natural number] $\equiv$ Day$\$_1$ is finite. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every natural number n, $\mathcal{P}[n]$. $\square$

(8) If $\mathfrak{b}$orn x is finite, then L_x is finite and R_x is finite.
Proof: Day$\mathfrak{b}$orn x is finite. $\mathrm{L}_x \cup \mathrm{R}_x \subseteq$ Day$\mathfrak{b}$orn x. $\square$

Let us consider x and y. Let us note that the predicate $x \leqslant y$ is reflexive and connected. We introduce the notation $y \geqslant x$ as a synonym of $x \leqslant y$.

2. Equivalence Relation of Preorder

Let us consider x and y. We say that $x \approx y$ if and only if

(Def. 2) $x \leqslant y \leqslant x$.

Note that the predicate is reflexive and symmetric. Now we state the propositions:

(9) If $x \leqslant y < z$, then $x < z$.

(10) If $x \approx y$ and $y \approx z$, then $x \approx z$.

(11) Conway Ch. 1 Th. 2(i):
$\mathrm{L}_x \ll \{x\} \ll \mathrm{R}_x$.
Proof: $\mathrm{L}_x \ll \{x\}$. $\square$

(12) Let us consider a non empty, surreal-membered set S. Suppose S is finite. Then there exist surreal numbers M_3, M_2 such that
  (i) M_3, $M_2 \in S$, and
  (ii) for every x such that $x \in S$ holds $M_3 \leqslant x \leqslant M_2$.

Proof: Define $\mathcal{P}$[natural number] $\equiv$ for every non empty, surreal-membered set S such that $\$_1 = \overline{\overline{S}}$ there exist surreal numbers M_3, M_2 such that M_3, $M_2 \in S$ and for every x such that $x \in S$ holds $M_3 \leqslant x \leqslant M_2$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$ by [8, (55)]. For every natural number n, $\mathcal{P}[n]$. $\square$

(13) Suppose $x < y$. Then
  (i) there exists a surreal number x_2 such that $x_2 \in \mathrm{R}_x$ and $x < x_2 \leqslant y$, or
  (ii) there exists a surreal number y_3 such that $y_3 \in \mathrm{L}_y$ and $x \leqslant y_3 < y$.

The theorem is a consequence of (11).

(14) Suppose $\mathrm{L}_y \ll \{x\} \ll \mathrm{R}_y$. Then $\langle \mathrm{L}_x \cup \mathrm{L}_y, \mathrm{R}_x \cup \mathrm{R}_y \rangle$ is a surreal number.
Proof: Consider α being an ordinal number such that $x \in$ Dayα. Consider β being an ordinal number such that $y \in$ Dayβ. Set $X = \mathrm{L}_x \cup \mathrm{L}_y$. Set

$Y = \mathrm{R}_x \cup \mathrm{R}_y$. $X \ll Y$. For every object x such that $x \in X \cup Y$ there exists an ordinal number θ such that $\theta \in \alpha \cup \beta$ and $x \in \mathrm{Day}\theta$. □

(15) Suppose $\mathrm{L}_y \ll \{x\} \ll \mathrm{R}_y$ and $z = \langle \mathrm{L}_x \cup \mathrm{L}_y, \mathrm{R}_x \cup \mathrm{R}_y \rangle$. Then $x \approx z$. The theorem is a consequence of (11).

Now we state the propositions:

(16) The Simplicity Theorem for Surreal Numbers:
Suppose $\mathrm{L}_y \ll \{x\} \ll \mathrm{R}_y$ and for every z such that $\mathrm{L}_y \ll \{z\} \ll \mathrm{R}_y$ holds $\mathfrak{born}\, x \subseteq \mathfrak{born}\, z$. Then $x \approx y$.
Proof: Set $X = \mathrm{L}_x \cup \mathrm{L}_y$. Set $Y = \mathrm{R}_x \cup \mathrm{R}_y$. Reconsider $z = \langle X, Y \rangle$ as a surreal number. $\mathrm{L}_x \ll \{x\} \ll \mathrm{R}_x$. $\mathrm{L}_y \ll \{y\} \ll \mathrm{R}_y$. $\mathrm{L}_z \ll \{z\} \ll \mathrm{R}_z$. $\mathrm{L}_x \ll \{z\}$. $\{x\} \ll \mathrm{R}_z$. $\mathrm{L}_y \ll \{z\}$. $x \approx z$. $\{y\} \ll \mathrm{R}_z$. $\{z\} \ll \mathrm{R}_y$. $\mathrm{L}_z \ll \{y\}$. □

(17) If $X \ll \{x\}$ and $x \leqslant y$, then $X \ll \{y\}$. The theorem is a consequence of (4).

(18) If $\{x\} \ll X$ and $y \leqslant x$, then $\{y\} \ll X$. The theorem is a consequence of (4).

(19) If $x \approx y$, then $\langle \mathrm{L}_x \cup \mathrm{L}_y, \mathrm{R}_x \cup \mathrm{R}_y \rangle$ is a surreal number. The theorem is a consequence of (11), (17), (18), and (14).

(20) If $x \approx y$ and $z = \langle \mathrm{L}_x \cup \mathrm{L}_y, \mathrm{R}_x \cup \mathrm{R}_y \rangle$, then $x \approx z$. The theorem is a consequence of (11), (17), (18), and (15).

(21) $\{x\} \ll \{y\}$ if and only if $x < y$.

(22) $\langle \{x\}, \{y\} \rangle$ is a surreal number if and only if $x < y$. The theorem is a consequence of (21).

(23) Let us consider a surreal number M_2. Suppose for every y such that $y \in \mathrm{L}_x$ holds $y \leqslant M_2$ and $M_2 \in \mathrm{L}_x$. Then

  (i) $\langle \{M_2\}, \mathrm{R}_x \rangle$ is a surreal number, and

  (ii) for every y such that $y = \langle \{M_2\}, \mathrm{R}_x \rangle$ holds $y \approx x$ and $\mathfrak{born}\, y \subseteq \mathfrak{born}\, x$.

Proof: $\{M_2\} \ll \mathrm{R}_x$. For every object o such that $o \in \{M_2\} \cup \mathrm{R}_x$ there exists θ such that $\theta \in \mathfrak{born}\, x$ and $o \in \mathrm{Day}\theta$. For every surreal number x_1 such that $x_1 \in \mathrm{L}_x$ there exists a surreal number y_1 such that $y_1 \in \mathrm{L}_y$ and $x_1 \leqslant y_1$. For every surreal number x_1 such that $x_1 \in \mathrm{L}_y$ there exists a surreal number y_1 such that $y_1 \in \mathrm{L}_x$ and $x_1 \leqslant y_1$. □

(24) Let us consider a surreal number M_3. Suppose for every y such that $y \in \mathrm{R}_x$ holds $M_3 \leqslant y$ and $M_3 \in \mathrm{R}_x$. Then

  (i) $\langle \mathrm{L}_x, \{M_3\} \rangle$ is a surreal number, and

(ii) for every y such that $y = \langle \mathrm{L}_x, \{M_3\} \rangle$ holds $y \approx x$ and $\mathfrak{born}\, y \subseteq \mathfrak{born}\, x$.

PROOF: $\mathrm{L}_x \ll \{M_3\}$. For every object o such that $o \in \mathrm{L}_x \cup \{M_3\}$ there exists θ such that $\theta \in \mathfrak{born}\, x$ and $o \in \mathrm{Day}\theta$. For every surreal number x_1 such that $x_1 \in \mathrm{R}_y$ there exists a surreal number y_1 such that $y_1 \in \mathrm{R}_x$ and $y_1 \leqslant x_1$. For every surreal number x_1 such that $x_1 \in \mathrm{R}_x$ there exists a surreal number y_1 such that $y_1 \in \mathrm{R}_y$ and $y_1 \leqslant x_1$. □

(25) If $x \leqslant y$ and $z = \langle \{x, y\}, X \rangle$ and $t = \langle \{y\}, X \rangle$, then $z \approx t$. The theorem is a consequence of (23).

(26) If $z = \langle \{x, y\}, X \rangle$, then $\langle \{x\}, X \rangle$ is a surreal number.
PROOF: Set $b = \mathfrak{born}\, z$. $\{x\} \ll X$. For every object o such that $o \in \{x\} \cup X$ there exists θ such that $\theta \in b$ and $o \in \mathrm{Day}\theta$. □

(27) If $x \leqslant y$ and $z = \langle X, \{x, y\} \rangle$ and $t = \langle X, \{x\} \rangle$, then $z \approx t$. The theorem is a consequence of (24).

(28) If $z = \langle X, \{x, y\} \rangle$, then $\langle X, \{x\} \rangle$ is a surreal number.
PROOF: Set $b = \mathfrak{born}\, z$. $X \ll \{x\}$. For every object o such that $o \in X \cup \{x\}$ there exists θ such that $\theta \in b$ and $o \in \mathrm{Day}\theta$. □

Let X, Y be sets. We say that $X \lessdot Y$ if and only if

(Def. 3) for every surreal number x such that $x \in X$ there exist surreal numbers y_1, y_2 such that $y_1, y_2 \in Y$ and $y_1 \leqslant x \leqslant y_2$.

One can verify that the predicate is reflexive.

We say that $X \leftrightarrow Y$ if and only if

(Def. 4) $X \lessdot Y$ and $Y \lessdot X$.

One can verify that the predicate is reflexive and symmetric.

Now we state the propositions:

(29) Let us consider sets X_1, X_2, Y_1, Y_2. Suppose $X_1 \leftrightarrow X_2$ and $Y_1 \leftrightarrow Y_2$ and $x = \langle X_1, Y_1 \rangle$ and $y = \langle X_2, Y_2 \rangle$. Then $x \approx y$.

(30) Let us consider sets X, Y. If $X \subseteq Y$, then $X \lessdot Y$.

(31) Let us consider sets X_1, X_2, Y_1, Y_2. If $X_1 \lessdot X_2$ and $Y_1 \lessdot Y_2$, then $X_1 \cup Y_1 \lessdot X_2 \cup Y_2$.

(32) If $x \approx y$, then $\{x\} \lessdot \{y\}$.

3. Representative of Equivalence Class With a Unique Set of Properties

Let x be a surreal number. The functor $\mathfrak{born}_{\approx} x$ yielding an ordinal number is defined by

(Def. 5) there exists a surreal number y such that $\mathfrak{born}\, y = it$ and $y \approx x$ and for every surreal number y such that $y \approx x$ holds $it \subseteq \mathfrak{born}\, y$.

The functor $\mathfrak{Born}_{\approx} x$ yielding a surreal-membered set is defined by

(Def. 6) $y \in it$ iff $y \approx x$ and $y \in \mathrm{Day}\, \mathfrak{born}_{\approx} x$.

One can check that $\mathfrak{Born}_{\approx} x$ is non empty. Let α be a non empty, surreal-membered set. We say that x is α-smallest if and only if

(Def. 7) $x \in \alpha$ and for every y such that $y \in \alpha$ and $y \approx x$ holds $\overline{\overline{\mathrm{L}_x}} \oplus \overline{\overline{\mathrm{R}_x}} \subseteq \overline{\overline{\mathrm{L}_y}} \oplus \overline{\overline{\mathrm{R}_y}}$.

Observe that there exists a surreal number which is α-smallest. Now we state the propositions:

(33) If $x \approx y$, then $\mathfrak{born}_{\approx} x = \mathfrak{born}_{\approx} y$. The theorem is a consequence of (4).

(34) If $x \approx y$, then $\mathfrak{Born}_{\approx} x = \mathfrak{Born}_{\approx} y$.

(35) If $y \in \mathfrak{Born}_{\approx} x$, then $\mathfrak{born}\, y = \mathfrak{born}_{\approx} y = \mathfrak{born}_{\approx} x$. The theorem is a consequence of (33).

(36) $\langle \emptyset, \mathrm{Day}\alpha \rangle, \langle \mathrm{Day}\alpha, \emptyset \rangle \in (\mathrm{Day}\,\mathrm{succ}\, \alpha) \setminus (\mathrm{Day}\alpha)$. The theorem is a consequence of (11).

From now on n denotes a natural number. Let α be a set. The functor made of α yielding a surreal-membered set is defined by

(Def. 8) $o \in it$ iff o is surreal and $\mathrm{L}_o \cup \mathrm{R}_o \subseteq \alpha$.

Let α be an ordinal number. The functor $\mathrm{unique}_{\mathbf{No}}\mathrm{op}(\alpha)$ yielding a transfinite sequence is defined by

(Def. 9) $\mathrm{dom}\, it = \mathrm{succ}\, \alpha$ and for every ordinal number β such that $\beta \in \mathrm{succ}\, \alpha$ holds $it(\beta) \subseteq \mathrm{Day}\beta$ and for every x, $x \in it(\beta)$ iff $x \in \bigcup \mathrm{rng}(it \restriction \beta)$ or $\beta = \mathfrak{born}_{\approx} x$ and there exists a non empty, surreal-membered set Y such that $Y = \mathfrak{Born}_{\approx} x \cap$ made of $\bigcup \mathrm{rng}(it \restriction \beta)$ and $x =$ the Y-smallest surreal number.

Let us consider o. One can verify that $(\mathrm{unique}_{\mathbf{No}}\mathrm{op}(\alpha))(o)$ is surreal-membered. Now we state the propositions:

(37) Suppose $\alpha \subseteq \beta$. Then $\mathrm{unique}_{\mathbf{No}}\mathrm{op}(\beta) \restriction \mathrm{succ}\, \alpha = \mathrm{unique}_{\mathbf{No}}\mathrm{op}(\alpha)$.
Proof: Define $\mathcal{P}$[transfinite sequence, ordinal number, surreal number] $\equiv$ $\$_3 \in \bigcup \mathrm{rng}\, \$_1$ or $\$_2 = \mathfrak{born}_{\approx} \$_3$ and there exists a non empty, surreal-membered set Y such that $Y = \mathfrak{Born}_{\approx} \$_3 \cap$ made of $\bigcup \mathrm{rng}\, \$_1$ and $\$_3 =$

the Y-smallest surreal number. Define $\mathcal{H}$(transfinite sequence) $= \{e$, where e is an element of Daydom $\$_1$: for every x such that $x = e$ holds $\mathcal{P}[\$_1, \mathrm{dom}\,\$_1, x]\}$. Set $S_1 = \mathrm{unique}_{\mathbf{No}}\mathrm{op}(\alpha)$. Set $S = \mathrm{unique}_{\mathbf{No}}\mathrm{op}(\beta)$. Set $S_2 = S \restriction \mathrm{succ}\,\alpha$. $\mathrm{dom}\,S_1 = \mathrm{succ}\,\alpha$ and for every ordinal number β and for every transfinite sequence L_1 such that $\beta \in \mathrm{succ}\,\alpha$ and $L_1 = S_1 \restriction \beta$ holds $S_1(\beta) = \mathcal{H}(L_1)$. $\mathrm{dom}\,S_2 = \mathrm{succ}\,\alpha$ and for every ordinal number γ and for every transfinite sequence L_2 such that $\gamma \in \mathrm{succ}\,\alpha$ and $L_2 = S_2 \restriction \gamma$ holds $S_2(\gamma) = \mathcal{H}(L_2)$. $S_1 = S_2$. $\square$

(38) Suppose $x \in (\mathrm{unique}_{\mathbf{No}}\mathrm{op}(\alpha))(\beta)$. Then

(i) $\mathfrak{born}_{\approx} x = \mathfrak{born}\, x \subseteq \beta$, and

(ii) $x \in (\mathrm{unique}_{\mathbf{No}}\mathrm{op}(\alpha))(\mathfrak{born}\, x)$, and

(iii) $x \notin \bigcup \mathrm{rng}(\mathrm{unique}_{\mathbf{No}}\mathrm{op}(\alpha) \restriction \mathfrak{born}\, x)$.

PROOF: Set $M = \mathrm{unique}_{\mathbf{No}}\mathrm{op}(\alpha)$. Define $\mathcal{M}$[ordinal number] $\equiv x \in M(\$_1)$ and $\$_1 \in \mathrm{succ}\,\alpha$. Consider δ being an ordinal number such that $\mathcal{M}[\delta]$ and for every ordinal number E such that $\mathcal{M}[E]$ holds $\delta \subseteq E$. $x \notin \bigcup \mathrm{rng}(M \restriction \delta)$. Consider Y being a non empty, surreal-membered set such that $Y = \mathfrak{Born}_{\approx} x \cap$ made of $\bigcup \mathrm{rng}(M \restriction \delta)$ and $x =$ the Y-smallest surreal number. $\square$

(39) If $\theta \subseteq \alpha \subseteq \beta$, then $(\mathrm{unique}_{\mathbf{No}}\mathrm{op}(\alpha))(\theta) = (\mathrm{unique}_{\mathbf{No}}\mathrm{op}(\beta))(\theta)$. The theorem is a consequence of (37).

(40) Suppose $\alpha \subseteq \beta$ and $\beta \in \mathrm{succ}\,\gamma$. Then $(\mathrm{unique}_{\mathbf{No}}\mathrm{op}(\gamma))(\alpha) \subseteq (\mathrm{unique}_{\mathbf{No}}\mathrm{op}(\gamma))(\beta)$.

Let x be a surreal number. The functor $\mathrm{Unique}_{\mathbf{No}}(x)$ yielding a surreal number is defined by

(Def. 10) $it \approx x$ and $it \in (\mathrm{unique}_{\mathbf{No}}\mathrm{op}(\mathfrak{born}_{\approx} x))(\mathfrak{born}_{\approx} x)$.

Now we state the propositions:

(41) If $x \approx y$, then $\mathrm{Unique}_{\mathbf{No}}(x) = \mathrm{Unique}_{\mathbf{No}}(y)$. The theorem is a consequence of (33) and (4).

(42) $\mathbf{0}_{\mathbf{No}} = \mathrm{Unique}_{\mathbf{No}}(\mathbf{0}_{\mathbf{No}})$. The theorem is a consequence of (38).

Let x be a surreal number. We say that x is unique surreal if and only if

(Def. 11) $x = \mathrm{Unique}_{\mathbf{No}}(x)$.

One can verify that $\mathbf{0}_{\mathbf{No}}$ is unique surreal and there exists a surreal number which is unique surreal. Now we state the propositions:

(43) If x is an unique surreal number and $o \in \mathrm{L}_x \cup \mathrm{R}_x$, then o is an unique surreal number. The theorem is a consequence of (38), (1), and (39).

(44) If L_x is non empty and finite and x is an unique surreal number, then $\overline{\overline{\mathrm{L}_x}} = 1$. The theorem is a consequence of (12), (38), and (23).

(45) If R_x is non empty and finite and x is an unique surreal number, then $\overline{\overline{\mathrm{R}_x}} = 1$. The theorem is a consequence of (12), (38), and (24).

(46) $\overline{\overline{\mathrm{L}_x}} \oplus \overline{\overline{\mathrm{R}_x}} = 0$ if and only if $x = \mathbf{0_{No}}$.

(47) $\overline{\overline{\mathrm{L}_x}} \oplus \overline{\overline{\mathrm{R}_x}} = 1$ if and only if there exists a surreal number y such that $x = \langle\emptyset, \{y\}\rangle$ or $x = \langle\{y\}, \emptyset\rangle$.

PROOF: If $\overline{\overline{\mathrm{L}_x}} \oplus \overline{\overline{\mathrm{R}_x}} = 1$, then there exists a surreal number y such that $x = \langle\emptyset, \{y\}\rangle$ or $x = \langle\{y\}, \emptyset\rangle$ by [7, (86),(76)]. □

Let X be a set. We say that X is unique surreal-membered if and only if

(Def. 12) if $o \in X$, then o is an unique surreal number.

Note that every set which is empty is also unique surreal-membered. Let x be an unique surreal number. One can verify that $\mathrm{L}_x \cup \mathrm{R}_x$ is unique surreal-membered and $\{x\}$ is unique surreal-membered. Let X, Y be unique surreal-membered sets. One can check that $X \cup Y$ is unique surreal-membered. Let x be a surreal number. One can check that $\mathrm{Unique}_{\mathbf{No}}(x)$ is unique surreal. Now we state the propositions:

(48) If x is an unique surreal number, then $\mathfrak{born}\, x = \mathfrak{born}_{\approx} x$. The theorem is a consequence of (38).

(49) Suppose for every z such that $z \in \mathfrak{Born}_{\approx} x$ and $\mathrm{L}_z \cup \mathrm{R}_z$ is unique surreal-membered and $x \neq z$ holds $\overline{\overline{\mathrm{L}_x}} \oplus \overline{\overline{\mathrm{R}_x}} \in \overline{\overline{\mathrm{L}_z}} \oplus \overline{\overline{\mathrm{R}_z}}$ and $x \in \mathfrak{Born}_{\approx} x$ and $\mathrm{L}_x \cup \mathrm{R}_x$ is unique surreal-membered. Then x is an unique surreal number.

PROOF: Set $c = \mathrm{Unique}_{\mathbf{No}}(x)$. Set $\beta = \mathfrak{born}_{\approx} x$. $\mathfrak{born}_{\approx} c = \beta$ and $\mathfrak{Born}_{\approx} c = \mathfrak{Born}_{\approx} x$. $\mathfrak{born}_{\approx} c = \mathfrak{born}\, c$. $c \notin \bigcup \mathrm{rng}(\mathrm{unique}_{\mathbf{No}}\mathrm{op}(\beta) \restriction \beta)$. Consider Y being a non empty, surreal-membered set such that $Y = \mathfrak{Born}_{\approx} c \cap$ made of $\bigcup \mathrm{rng}(\mathrm{unique}_{\mathbf{No}}\mathrm{op}(\beta) \restriction \beta)$ and $c =$ the Y-smallest surreal number. $x \in \mathfrak{Born}_{\approx} c$. $\mathrm{L}_x \cup \mathrm{R}_x \subseteq \bigcup \mathrm{rng}(\mathrm{unique}_{\mathbf{No}}\mathrm{op}(\beta) \restriction \beta)$. □

(50) If x is an unique surreal number and y is an unique surreal number and $x \approx y$, then $x = y$. The theorem is a consequence of (41).

(51) Let us consider a surreal number c. Suppose $\mathfrak{born}\, c = \mathfrak{born}_{\approx} c$ and $\mathrm{L}_c \ll \{x\} \ll \mathrm{R}_c$. Then $\mathfrak{born}\, c \subseteq \mathfrak{born}\, x$.

PROOF: Define $\mathcal{P}[\text{ordinal number}] \equiv$ there exists y such that $\mathrm{L}_c \ll \{y\} \ll \mathrm{R}_c$ and $\mathfrak{born}\, y = \$_1$. Consider α such that $\mathcal{P}[\alpha]$ and for every β such that $\mathcal{P}[\beta]$ holds $\alpha \subseteq \beta$. Consider y such that $\mathrm{L}_c \ll \{y\} \ll \mathrm{R}_c$ and $\mathfrak{born}\, y = \alpha$. $\mathfrak{born}_{\approx} c = \mathfrak{born}_{\approx} y$. □

(52) Let us consider unique surreal numbers c, x. Suppose $\mathrm{L}_c \ll \{x\} \ll \mathrm{R}_c$ and $x \neq c$. Then $\mathfrak{born}\, c \in \mathfrak{born}\, x$. The theorem is a consequence of (48), (51), (50), (13), (1), (11), (17), (18), and (3).

(53) Suppose $\mathfrak{born}\, x = \mathfrak{born}_{\approx} x$ and $\mathfrak{born}\, x$ is not limit ordinal. Then there exist surreal numbers y, z such that

(i) $x \approx z$, and

(ii) $z = \langle\, \mathrm{L}_y \cup \{y\},\, \mathrm{R}_y \,\rangle$ or $z = \langle\, \mathrm{L}_y,\, \mathrm{R}_y \cup \{y\} \,\rangle$.

Proof: Consider β being an ordinal number such that $\mathfrak{born}\, x = \mathrm{succ}\, \beta$. Define $\mathcal{L}[\text{object}] \equiv$ for every z such that $z = \$_1$ holds $\mathfrak{born}\, z \in \beta$ and $z < x$. Consider L being a set such that $o \in L$ iff $o \in \mathrm{Day}\beta$ and $\mathcal{L}[o]$. Define $\mathcal{R}[\text{object}] \equiv$ for every z such that $z = \$_1$ holds $\mathfrak{born}\, z \in \beta$ and $x < z$. Consider R being a set such that $o \in R$ iff $o \in \mathrm{Day}\beta$ and $\mathcal{R}[o]$. $L \ll R$. For every object o such that $o \in L \cup R$ there exists θ such that $\theta \in \beta$ and $o \in \mathrm{Day}\theta$. Reconsider $L_3 = \langle L, R \rangle$ as a surreal number. $L_3 \not\approx x$. $\square$

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