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Multidimensional Measure Space and Integration[1]

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Summary. This paper introduces multidimensional measure spaces and the integration of functions on these spaces in Mizar. Integrals on the multidimensional Cartesian product measure space are defined and appropriate formal apparatus to deal with this notion is provided as well.

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Introduction

In this paper, using the Mizar system [1], [11], we introduce multidimensional measure spaces and the integration ([14], [2]) of functions on these spaces (for interesting survey of formalizations of real analysis in another proof-assistants like ACL2 [10], Isabelle/HOL [9], Coq [3], see [4]). It is the continuation of the mechanisation of this topic as developed in [5] and [8]. In constructing measures on multidimensional spaces [12], we constructed a finite sequence of Cartesian product spaces of sets in Section 1. In Section 2, using Fubini's Theorem [6], we have constructed measures on general multidimensional spaces by introducing

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measures one by one into the finite sequence of direct product spaces obtained in Section 1. In Section 3, integrals on the m-dimensional Cartesian product measure space obtained in Section 2 are presented, and the concept of sequentially integrable, which is useful in considering integrability [7] for functions on multidimensional spaces, is introduced and its effectiveness is shown.

1. Preliminaries

Let m, n be non zero natural numbers and X be a non-empty, m-elements finite sequence. Assume $n \leqslant m$. The functor $\mathrm{ElmFin}(X, n)$ yielding a non empty set is defined by the term

(Def. 1) $X(n)$.

Let m be a natural number. A family of σ-fields of X is an m-elements finite sequence defined by

(Def. 2) for every natural number i such that $i \in \mathrm{Seg}\, m$ holds $it(i)$ is a σ-field of subsets of $X(i)$.

Now we state the proposition:

(1) Let us consider non zero natural numbers m, n, a non-empty, m-elements finite sequence X, and a family of σ-fields S of X. If $n \leqslant m$, then $S(n)$ is a σ-field of subsets of $\mathrm{ElmFin}(X, n)$.

Let m be a non zero natural number and X be a non-empty, m-elements finite sequence. The functor $\prod_{\mathrm{FinS}} X$ yielding a non-empty, m-elements finite sequence is defined by

(Def. 3) $it(1) = X(1)$ and for every non zero natural number i such that $i < m$ holds $it(i+1) = it(i) \times X(i+1)$.

The functor $\prod_{\mathrm{FS}} X$ yielding a set is defined by the term

(Def. 4) $(\prod_{\mathrm{FinS}} X)(m)$.

Observe that $\prod_{\mathrm{FS}} X$ is non empty. Now we state the proposition:

(2) Let us consider a non zero natural number m, a natural number k, and a non-empty, m-elements finite sequence X. If $k \leqslant m$, then $X \restriction k$ is a non-empty, k-elements finite sequence.

Let m, n be non zero natural numbers and X be a non-empty, m-elements finite sequence. Assume $n \leqslant m$. The functor $\mathrm{SubFin}(X, n)$ yielding a non-empty, n-elements finite sequence is defined by the term

(Def. 5) $X \restriction n$.

Let S be a family of σ-fields of X. Assume $n \leqslant m$. The functor $\mathrm{SubFin}(S, n)$ yielding a family of σ-fields of $\mathrm{SubFin}(X, n)$ is defined by the term

(Def. 6) $S \restriction n$.

Assume $n \leqslant m$. The functor $\mathrm{ElmFin}(S, n)$ yielding a σ-field of subsets of $\mathrm{ElmFin}(X, n)$ is defined by the term

(Def. 7) $S(n)$.

Let m be a non zero natural number. Note that a family of σ-fields of X is a family of semialgebras of X. Let S be a family of σ-fields of X.

A family of σ-measures of S is an m-elements finite sequence defined by

(Def. 8) for every natural number i such that $i \in \mathrm{Seg}\, m$ there exists a σ-field S_3 of subsets of $X(i)$ such that $S_3 = S(i)$ and $it(i)$ is a σ-measure on S_3.

Let m, n be non zero natural numbers and M be a family of σ-measures of S. Assume $n \leqslant m$. The functor $\mathrm{SubFin}(M, n)$ yielding a family of σ-measures of $\mathrm{SubFin}(S, n)$ is defined by the term

(Def. 9) $M \restriction n$.

Assume $n \leqslant m$. The functor $\mathrm{ElmFin}(M, n)$ yielding a σ-measure on $\mathrm{ElmFin}(S, n)$ is defined by the term

(Def. 10) $M(n)$.

Now we state the proposition:

(3) Let us consider non zero natural numbers m, i, j, k, and a non-empty, m-elements finite sequence X. Suppose $i \leqslant j \leqslant k \leqslant m$. Then $(\prod_{\mathrm{FinS}} \mathrm{SubFin}(X, j))(i) = (\prod_{\mathrm{FinS}} \mathrm{SubFin}(X, k))(i)$.
PROOF: Define $\mathcal{P}$[natural number] $\equiv$ if $1 \leqslant \$_1 \leqslant j$, then $(\prod_{\mathrm{FinS}} \mathrm{SubFin}(X, j))(\$_1) = (\prod_{\mathrm{FinS}} \mathrm{SubFin}(X, k))(\$_1)$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every natural number n, $\mathcal{P}[n]$. $\square$

Let us consider non zero natural numbers m, n and a non-empty, m-elements finite sequence X. Now we state the propositions:

(4) If $n \leqslant m$, then $(\prod_{\mathrm{FinS}} X)(n) = (\prod_{\mathrm{FinS}} \mathrm{SubFin}(X, n))(n)$. The theorem is a consequence of (3).

(5) If $n < m$, then $(\prod_{\mathrm{FinS}} X)(n+1) = (\prod_{\mathrm{FinS}} \mathrm{SubFin}(X, n))(n) \times \mathrm{ElmFin}(X, n+1)$. The theorem is a consequence of (4).

(6) Let us consider a non zero natural number n, and a non-empty, $(n+1)$-elements finite sequence X. Then $\prod_{\mathrm{FS}} X = \prod_{\mathrm{FS}} \mathrm{SubFin}(X, n) \times \mathrm{ElmFin}(X, n+1)$. The theorem is a consequence of (4).

Let us consider non zero natural numbers m, n, k and a non-empty, m-elements finite sequence X. Now we state the propositions:

(7) If $k \leqslant n \leqslant m$, then $\mathrm{SubFin}(X, k) = \mathrm{SubFin}(\mathrm{SubFin}(X, n), k)$.

(8) If $k \leqslant n \leqslant m$, then $\mathrm{ElmFin}(X, k) = \mathrm{ElmFin}(\mathrm{SubFin}(X, n), k)$.

Let us consider non zero natural numbers m, n and a non-empty, m-elements finite sequence X. Now we state the propositions:

(9) If $n < m$, then $\prod_{\mathrm{FS}} \mathrm{SubFin}(X, n+1) = \prod_{\mathrm{FS}} \mathrm{SubFin}(X, n) \times \mathrm{ElmFin}(X, n+1)$. The theorem is a consequence of (8), (6), and (7).

(10) If $n < m$, then $(\prod_{\mathrm{FinS}} \mathrm{SubFin}(X, n+1))(n+1) = (\prod_{\mathrm{FinS}} \mathrm{SubFin}(X, n))(n) \times \mathrm{ElmFin}(X, n+1)$. The theorem is a consequence of (9).

(11) Let us consider non zero natural numbers n, i, a non-empty, $(n+1)$-elements finite sequence X, and a family of σ-fields S of X. Suppose $i \leqslant n$. Then $\prod_{\mathrm{FS}} \mathrm{SubFin}(X, i) = \prod_{\mathrm{FS}} \mathrm{SubFin}(\mathrm{SubFin}(X, n), i)$. The theorem is a consequence of (7).

(12) Let us consider non zero natural numbers m, n, k, a non-empty, m-elements finite sequence X, and a family of σ-fields S of X. Suppose $k \leqslant n \leqslant m$. Then $\mathrm{ElmFin}(S, k) = \mathrm{ElmFin}(\mathrm{SubFin}(S, n), k)$.

(13) Let us consider non zero natural numbers m, n, k, a non-empty, m-elements finite sequence X, a non-empty, n-elements finite sequence Y, and a family of σ-fields S of X. Suppose $n \leqslant m$ and $Y = X{\restriction}n$. Then $\mathrm{SubFin}(S, n)$ is a family of σ-fields of Y.

Proof: For every natural number i such that $i \in \mathrm{Seg}\, n$ holds $(\mathrm{SubFin}(S, n))(i)$ is a σ-field of subsets of $Y(i)$. □

(14) Let us consider non zero natural numbers m, n, k, a non-empty, m-elements finite sequence X, and a family of σ-fields S of X. Suppose $k \leqslant n \leqslant m$. Then $\mathrm{SubFin}(S, k) = \mathrm{SubFin}(\mathrm{SubFin}(S, n), k)$.

(15) Let us consider a non zero natural number m, and a non-empty, m-elements finite sequence X. Then there exists a function F from $\prod_{\mathrm{FS}} X$ into $\prod X$ such that F is one-to-one and onto.

Proof: Define $\mathcal{P}[\text{non zero natural number}] \equiv$ for every non-empty, $\$_1$-elements finite sequence X, there exists a function F from $\prod_{\mathrm{FS}} X$ into $\prod X$ such that F is one-to-one and onto. $\mathcal{P}[1]$ by [13, (2)]. For every non zero natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every non zero natural number n, $\mathcal{P}[n]$. □

(16) Let us consider non zero natural numbers m, n, a non-empty, m-elements finite sequence X, and a family P of semialgebras of $\prod_{\mathrm{FinS}} X$. Suppose $n \leqslant m$. Then $P(n)$ is a semialgebra of sets of $\prod_{\mathrm{FS}} \mathrm{SubFin}(X, n)$. The theorem is a consequence of (4).

Let us consider non zero natural numbers m, n, k, a non-empty, m-elements finite sequence X, a family of σ-fields S of X, and a family of σ-measures M of S. Now we state the propositions:

(17) If $k \leqslant n \leqslant m$, then $\mathrm{ElmFin}(M, k) = \mathrm{ElmFin}(\mathrm{SubFin}(M, n), k)$.

(18) If $k \leqslant n \leqslant m$, then $\mathrm{SubFin}(M, k) = \mathrm{SubFin}(\mathrm{SubFin}(M, n), k)$.

2. Construction of m-dimensional Measure Space

Let m be a non zero natural number, X be a non-empty, m-elements finite sequence, and S be a family of σ-fields of X. The functor $\sigma\mathrm{FldFS}_{\mathrm{Prod}}(S)$ yielding a family of σ-fields of $\prod_{\mathrm{FinS}} X$ is defined by

(Def. 11) $it(1) = S(1)$ and for every non zero natural number i such that $i < m$ there exists a σ-field S_3 of subsets of $\prod_{\mathrm{FS}} \mathrm{SubFin}(X, i)$ such that $S_3 = it(i)$ and $it(i+1) = \sigma(\mathrm{MeasRect}(S_3, \mathrm{ElmFin}(S, i+1)))$.

Now we state the proposition:

(19) Let us consider non zero natural numbers m, n, a non-empty, m-elements finite sequence X, and a family of σ-fields S of X. Suppose $n \leqslant m$. Then $(\sigma\mathrm{FldFS}_{\mathrm{Prod}}(S))(n)$ is a σ-field of subsets of $(\prod_{\mathrm{FinS}} X)(n)$.

Let m be a non zero natural number, X be a non-empty, m-elements finite sequence, and S be a family of σ-fields of X. The functor $\prod_{\mathrm{Field}} S$ yielding a σ-field of subsets of $\prod_{\mathrm{FS}} X$ is defined by the term

(Def. 12) $(\sigma\mathrm{FldFS}_{\mathrm{Prod}}(S))(m)$.

Now we state the propositions:

(20) Let us consider non zero natural numbers m, n, k, a non-empty, m-elements finite sequence X, and a family of σ-fields S of X. Suppose $k \leqslant n \leqslant m$. Then $(\sigma\mathrm{FldFS}_{\mathrm{Prod}}(S))(k) = (\sigma\mathrm{FldFS}_{\mathrm{Prod}}(\mathrm{SubFin}(S, n)))(k)$.
Proof: Define $\mathcal{P}[\text{natural number}] \equiv$ if $1 \leqslant \$_1 \leqslant n$, then $(\sigma\mathrm{FldFS}_{\mathrm{Prod}}(S))(\$_1) = (\sigma\mathrm{FldFS}_{\mathrm{Prod}}(\mathrm{SubFin}(S, n)))(\$_1)$. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$. For every natural number i, $\mathcal{P}[i]$. $\square$

(21) Let us consider non zero natural numbers m, n, a non-empty, m-elements finite sequence X, and a family of σ-fields S of X. Suppose $n < m$. Then $\prod_{\mathrm{Field}} \mathrm{SubFin}(S, n+1) = \sigma(\mathrm{MeasRect}(\prod_{\mathrm{Field}} \mathrm{SubFin}(S, n), \mathrm{ElmFin}(S, n+1)))$. The theorem is a consequence of (8), (12), (7), and (20).

Let m be a non zero natural number, X be a non-empty, m-elements finite sequence, S be a family of σ-fields of X, and M be a family of σ-measures of S. The functor $\sigma\mathrm{MesFS}_{\mathrm{Prod}}(M)$ yielding a family of σ-measures of $\sigma\mathrm{FldFS}_{\mathrm{Prod}}(S)$ is defined by

(Def. 13) $it(1) = M(1)$ and for every non zero natural number i such that $i < m$ there exists a σ-measure M_3 on $\prod_{\mathrm{Field}} \mathrm{SubFin}(S, i)$ such that $M_3 = it(i)$ and $it(i+1) = \mathrm{Prod}\,\sigma\text{-}\mathrm{Meas}(M_3, \mathrm{ElmFin}(M, i+1))$.

Now we state the proposition:

(22) Let us consider non zero natural numbers m, n, a non-empty, m-elements finite sequence X, a family of σ-fields S of X, and a family of σ-measures

M of S. Suppose $n \leqslant m$. Then $(\sigma\mathrm{MesFS}_{\mathrm{Prod}}(M))(n)$ is a σ-measure on $\prod_{\mathrm{Field}} \mathrm{SubFin}(S, n)$.

PROOF: Set $P_1 = \sigma\mathrm{MesFS}_{\mathrm{Prod}}(M)$. Define $\mathcal{L}$[natural number] $\equiv$ if $1 \leqslant \$_1 \leqslant m$, then there exists a non zero natural number k such that $k = \$_1$ and $P_1(\$_1)$ is a σ-measure on $\prod_{\mathrm{Field}} \mathrm{SubFin}(S, k)$. For every natural number i such that $\mathcal{L}[i]$ holds $\mathcal{L}[i+1]$. For every natural number n, $\mathcal{L}[n]$. $\square$

Let m be a non zero natural number, X be a non-empty, m-elements finite sequence, S be a family of σ-fields of X, and M be a family of σ-measures of S. The functor $\mathrm{Measure}_{\mathrm{Prod}}(M)$ yielding a σ-measure on $\prod_{\mathrm{Field}} S$ is defined by the term

(Def. 14) $(\sigma\mathrm{MesFS}_{\mathrm{Prod}}(M))(m)$.

We say that M is σ-finite if and only if

(Def. 15) for every natural number i such that $i \in \mathrm{Seg}\, m$ there exists a non empty set X_2 and there exists a σ-field S_3 of subsets of X_2 and there exists a σ-measure M_3 on S_3 such that $X_2 = X(i)$ and $S_3 = S(i)$ and $M_3 = M(i)$ and M_3 is σ-finite.

Now we state the propositions:

(23) Let us consider non zero natural numbers m, n, k, a non-empty, m-elements finite sequence X, a family of σ-fields S of X, and a family of σ-measures M of S. Suppose $k \leqslant n \leqslant m$. Then $(\sigma\mathrm{MesFS}_{\mathrm{Prod}}(\mathrm{SubFin}(M, n)))(k) = (\sigma\mathrm{MesFS}_{\mathrm{Prod}}(\mathrm{SubFin}(M, k)))(k)$. The theorem is a consequence of (7), (14), (8), (12), and (17).

(24) Let us consider non zero natural numbers m, n, a non-empty, m-elements finite sequence X, a family of σ-fields S of X, and a family of σ-measures M of S. Suppose $n \leqslant m$. Then $(\sigma\mathrm{MesFS}_{\mathrm{Prod}}(M))(n) = \mathrm{Measure}_{\mathrm{Prod}}(\mathrm{SubFin}(M, n))$.

PROOF: Define $\mathcal{P}$[natural number] $\equiv$ if $1 \leqslant \$_1 \leqslant m$, then there exists a non zero natural number k such that $k = \$_1$ and $(\sigma\mathrm{MesFS}_{\mathrm{Prod}}(M))(\$_1) = \mathrm{Measure}_{\mathrm{Prod}}(\mathrm{SubFin}(M, k))$. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$. For every natural number i, $\mathcal{P}[i]$. $\square$

(25) Let us consider a non zero natural number n, a non-empty, $(n+1)$-elements finite sequence X, a family of σ-fields S of X, and a family of σ-measures M of S. Then $\mathrm{Measure}_{\mathrm{Prod}}(M) = \mathrm{Prod}\,\sigma\text{-Meas}(\mathrm{Measure}_{\mathrm{Prod}}(\mathrm{SubFin}(M, n)), \mathrm{ElmFin}(M, n+1))$. The theorem is a consequence of (24).

(26) Let us consider a non empty set X, a field S of subsets of X, a set sequence E of S, and a natural number i. Then (the partial unions of E)$(i) \in S$.

PROOF: Define $\mathcal{P}$[natural number] $\equiv$ (the partial unions of E)$(\$_1) \in S$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every natural number n, $\mathcal{P}[n]$. $\square$

(27) Let us consider non empty sets X, Y, a σ-field S_1 of subsets of X, a σ-field S_2 of subsets of Y, a σ-measure M_1 on S_1, and a σ-measure M_2 on S_2. Suppose M_1 is σ-finite and M_2 is σ-finite. Then $\mathrm{ProdMeas}(M_1, M_2)$ is σ-finite.

Proof: Set $M = \mathrm{ProdMeas}(M_1, M_2)$. Consider E_1 being a set sequence of S_1 such that for every natural number n, $M_1(E_1(n)) < +\infty$ and $\bigcup E_1 = X$. Consider E_2 being a set sequence of S_2 such that for every natural number n, $M_2(E_2(n)) < +\infty$ and $\bigcup E_2 = Y$. Set $F_1 =$ the partial unions of E_1. Set $F_2 =$ the partial unions of E_2. Define $\mathcal{G}(\text{natural number}) = (F_1(\$_1) \times F_2(\$_1))(\in \sigma(\mathrm{MeasRect}(S_1, S_2)))$. Consider E being a function from $\mathbb{N}$ into $\sigma(\mathrm{MeasRect}(S_1, S_2))$ such that for every element i of $\mathbb{N}$, $E(i) = \mathcal{G}(i)$.

For every natural number i, $E(i) = F_1(i) \times F_2(i)$. For every natural number i, $E(i) \in \sigma(\mathrm{MeasRect}(S_1, S_2))$. For every object z, $z \in \bigcup E$ iff $z \in X \times Y$. Define $\mathcal{Q}[\text{natural number}] \equiv M_1(F_1(\$_1)), M_2(F_2(\$_1)) \in \mathbb{R}$. For every natural number i such that $\mathcal{Q}[i]$ holds $\mathcal{Q}[i + 1]$. For every natural number i, $\mathcal{Q}[i]$. For every natural number i, $M(E(i)) < +\infty$. $\square$

(28) Let us consider a non zero natural number n, a non-empty, $(n + 1)$-elements finite sequence X, a family of σ-fields S of X, and a family of σ-measures M of S. Then $\mathrm{Measure}_{\mathrm{Prod}}(M) = \mathrm{ProdMeas}(\mathrm{Measure}_{\mathrm{Prod}}(\mathrm{SubFin}(M, n)), \mathrm{ElmFin}(M, n + 1))$. The theorem is a consequence of (25).

(29) Let us consider a non zero natural number m, a non-empty, m-elements finite sequence X, a family of σ-fields S of X, and a family of σ-measures M of S. Suppose M is σ-finite. Then $\mathrm{Measure}_{\mathrm{Prod}}(M)$ is σ-finite.

Proof: Define $\mathcal{P}[\text{natural number}] \equiv$ for every non zero natural number n for every non-empty, n-elements finite sequence X for every family of σ-fields S of X for every family of σ-measures M of S such that M is σ-finite and $\$_1 = n$ holds $\mathrm{Measure}_{\mathrm{Prod}}(M)$ is σ-finite. $\mathcal{P}[1]$. For every non zero natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i + 1]$. For every non zero natural number k, $\mathcal{P}[k]$. $\square$

Let us consider non zero natural numbers m, n, a non-empty, m-elements finite sequence X, a family of σ-fields S of X, and a family of σ-measures M of S. Now we state the propositions:

(30) If $n \leqslant m$ and M is σ-finite, then $\mathrm{SubFin}(M, n)$ is σ-finite.

Proof: Set $X_6 = \mathrm{SubFin}(X, n)$. Set $S_6 = \mathrm{SubFin}(S, n)$. Set $M_6 = \mathrm{SubFin}(M, n)$. For every natural number j such that $j \in \mathrm{Seg}\, n$ there exists a non empty set X_3 and there exists a σ-field S_4 of subsets of X_3 and there exists a σ-measure M_4 on S_4 such that $X_3 = X_6(j)$ and $S_4 = S_6(j)$ and $M_4 = M_6(j)$ and M_4 is σ-finite. $\square$

(31) If $n \leqslant m$ and M is σ-finite, then $\mathrm{ElmFin}(M, n)$ is σ-finite.

3. Integrability of Functions on $(n+1)$-dimensional Space

Now we state the propositions:

(32) Let us consider a non zero natural number n, a non-empty, $(n+1)$-elements finite sequence X, a family of σ-fields S of X, a family of σ-measures M of S, and a partial function f from $\prod_{\mathrm{FS}} X$ to $\overline{\mathbb{R}}$. Suppose f is integrable on $\mathrm{Measure}_{\mathrm{Prod}}(M)$. Then there exists a partial function g from $\prod_{\mathrm{FS}} \mathrm{SubFin}(X, n) \times \mathrm{ElmFin}(X, n+1)$ to $\overline{\mathbb{R}}$ such that

- (i) $f = g$, and
- (ii) g is integrable on $\mathrm{ProdMeas}(\mathrm{Measure}_{\mathrm{Prod}}(\mathrm{SubFin}(M, n)), \mathrm{ElmFin}(M, n+1))$, and
- (iii) $\int f \, \mathrm{d}\, \mathrm{Measure}_{\mathrm{Prod}}(M) = \int g \, \mathrm{d}\, \mathrm{ProdMeas}(\mathrm{Measure}_{\mathrm{Prod}}(\mathrm{SubFin}(M, n)), \mathrm{ElmFin}(M, n+1))$.

The theorem is a consequence of (28), (6), and (21).

(33) Let us consider a non zero natural number n, a non-empty, $(n+1)$-elements finite sequence X, a family of σ-fields S of X, a family of σ-measures M of S, a partial function f from $\prod_{\mathrm{FS}} X$ to $\overline{\mathbb{R}}$, and a partial function g from $\prod_{\mathrm{FS}} \mathrm{SubFin}(X, n) \times \mathrm{ElmFin}(X, n+1)$ to $\overline{\mathbb{R}}$.

Suppose M is σ-finite and f is integrable on $\mathrm{Measure}_{\mathrm{Prod}}(M)$ and $f = g$ and for every element y of $\mathrm{ElmFin}(X, n+1)$, $(\mathrm{Integral1}(\mathrm{Measure}_{\mathrm{Prod}}(\mathrm{SubFin}(M, n)), |g|))(y) < +\infty$. Then

- (i) for every element y of $\mathrm{ElmFin}(X, n+1)$, $\mathrm{ProjPMap2}(g, y)$ is integrable on $\mathrm{Measure}_{\mathrm{Prod}}(\mathrm{SubFin}(M, n))$, and
- (ii) for every element V of $\mathrm{ElmFin}(S, n+1)$, $\mathrm{Integral1}(\mathrm{Measure}_{\mathrm{Prod}}(\mathrm{SubFin}(M, n)), g)$ is V-measurable, and
- (iii) $\mathrm{Integral1}(\mathrm{Measure}_{\mathrm{Prod}}(\mathrm{SubFin}(M, n)), g)$ is integrable on $\mathrm{ElmFin}(M, n+1)$, and
- (iv) $\int g \, \mathrm{d}\, \mathrm{ProdMeas}(\mathrm{Measure}_{\mathrm{Prod}}(\mathrm{SubFin}(M, n)), \mathrm{ElmFin}(M, n+1)) = \int \mathrm{Integral1}(\mathrm{Measure}_{\mathrm{Prod}}(\mathrm{SubFin}(M, n)), g) \, \mathrm{d}\, \mathrm{ElmFin}(M, n+1)$, and
- (v) $\mathrm{Integral1}(\mathrm{Measure}_{\mathrm{Prod}}(\mathrm{SubFin}(M, n)), g) \in$ the L^1 functionsof $\mathrm{ElmFin}(M, n+1)$.

Proof: There exists a partial function g_0 from $\prod_{\mathrm{FS}} \mathrm{SubFin}(X, n) \times \mathrm{ElmFin}(X, n+1)$ to $\overline{\mathbb{R}}$ such that $f = g_0$ and g_0 is integrable on $\mathrm{ProdMeas}(\mathrm{Measure}_{\mathrm{Prod}}(\mathrm{SubFin}(M, n)), \mathrm{ElmFin}(M, n+1))$ and $\int f \, \mathrm{d}\, \mathrm{Measure}_{\mathrm{Prod}}(M) = \int g_0$

$\mathrm{d}\,\mathrm{ProdMeas}(\mathrm{Measure}_{\mathrm{Prod}}(\mathrm{SubFin}(M,n)), \mathrm{ElmFin}(M,n{+}1))$. For every natural number j such that $j \in \mathrm{Seg}\,n$ there exists a non empty set X_3 and there exists a σ-field S_4 of subsets of X_3 and there exists a σ-measure m_1 on S_4 such that $X_3 = (\mathrm{SubFin}(X,n))(j)$ and $S_4 = (\mathrm{SubFin}(S,n))(j)$ and $m_1 = (\mathrm{SubFin}(M,n))(j)$ and m_1 is σ-finite. $\mathrm{Measure}_{\mathrm{Prod}}(\mathrm{SubFin}(M,n))$ is σ-finite. $\square$

Let n be a non zero natural number, X be a non-empty, $(n+1)$-elements finite sequence, f be a partial function from $\prod_{\mathrm{FS}} X$ to $\overline{\mathbb{R}}$, and x be an element of $\prod_{\mathrm{FS}} \mathrm{SubFin}(X,n)$. The functor $\mathrm{ProjPMap1}(f,x)$ yielding a partial function from $\mathrm{ElmFin}(X,n+1)$ to $\overline{\mathbb{R}}$ is defined by

(Def. 16) there exists a partial function g from $\prod_{\mathrm{FS}} \mathrm{SubFin}(X,n) \times \mathrm{ElmFin}(X,n{+}1)$ to $\overline{\mathbb{R}}$ such that $f = g$ and $\mathit{it} = \mathrm{ProjPMap1}(g,x)$.

Now we state the propositions:

(34) Let us consider a non zero natural number n, a non-empty, $(n+1)$-elements finite sequence X, a family of σ-fields S of X, and a family of σ-measures M of S. Then $\prod_{\mathrm{Field}} S = \sigma(\mathrm{MeasRect}(\prod_{\mathrm{Field}} \mathrm{SubFin}(S,n), \mathrm{Elm}{-}\mathrm{Fin}(S,n+1)))$. The theorem is a consequence of (21).

(35) Let us consider a non zero natural number n, a non-empty, $(n+1)$-elements finite sequence X, a family of σ-fields S of X, a family of σ-measures M of S, a partial function f from $\prod_{\mathrm{FS}} X$ to $\overline{\mathbb{R}}$, and a partial function f_3 from $\prod_{\mathrm{FS}} \mathrm{SubFin}(X,n) \times \mathrm{ElmFin}(X,n+1)$ to $\overline{\mathbb{R}}$.

Suppose M is σ-finite and $f = f_3$ and f is integrable on $\mathrm{Measure}_{\mathrm{Prod}}(M)$ and for every element x of $\prod_{\mathrm{FS}} \mathrm{SubFin}(X,n)$, $(\mathrm{Integral2}(\mathrm{ElmFin}(M,n+1),|f_3|))(x) < +\infty$. Then

(i) $\int f \,\mathrm{d}\,\mathrm{Measure}_{\mathrm{Prod}}(M) = \int f_3 \,\mathrm{d}\,\mathrm{ProdMeas}(\mathrm{Measure}_{\mathrm{Prod}}(\mathrm{SubFin}(M,n)), \mathrm{ElmFin}(M,n+1))$, and

(ii) for every element x of $\prod_{\mathrm{FS}} \mathrm{SubFin}(X,n)$, $\mathrm{ProjPMap1}(f_3,x)$ is integrable on $\mathrm{ElmFin}(M,n+1)$, and

(iii) for every element U of $\prod_{\mathrm{Field}} \mathrm{SubFin}(S,n)$, $\mathrm{Integral2}(\mathrm{ElmFin}(M,n+1), f_3)$ is U-measurable, and

(iv) $\mathrm{Integral2}(\mathrm{ElmFin}(M,n+1), f_3)$ is integrable on $\mathrm{Measure}_{\mathrm{Prod}}(\mathrm{SubFin}(M,n))$, and

(v) $\int f_3 \,\mathrm{d}\,\mathrm{ProdMeas}(\mathrm{Measure}_{\mathrm{Prod}}(\mathrm{SubFin}(M,n)), \mathrm{ElmFin}(M,n+1)) = \int \mathrm{Integral2}(\mathrm{ElmFin}(M,n+1), f_3) \,\mathrm{d}\,\mathrm{Measure}_{\mathrm{Prod}}(\mathrm{SubFin}(M,n))$, and

(vi) $\mathrm{Integral2}(\mathrm{ElmFin}(M,n{+}1), f_3) \in \mathrm{the} L^1$ functions of $\mathrm{Measure}_{\mathrm{Prod}}(\mathrm{SubFin}(M,n))$.

The theorem is a consequence of (6), (28), (29), (30), (31), and (21).

(36) Let us consider a non zero natural number n, a non-empty, $(n+1)$-elements finite sequence X, a family of σ-fields S of X, a family of σ-measures M of S, a partial function f from $\prod_{\rm FS} X$ to $\overline{\mathbb{R}}$, a partial function f_1 from $\prod_{\rm FS} \mathrm{SubFin}(X, n) \times \mathrm{ElmFin}(X, n+1)$ to $\overline{\mathbb{R}}$, and a partial function f_2 from $\prod_{\rm FS} \mathrm{SubFin}(X, n+1)$ to $\overline{\mathbb{R}}$. Suppose M is σ-finite and $f = f_1$ and $f = f_2$ and f is integrable on $\mathrm{Measure}_{\rm Prod}(M)$ and for every element x of $\prod_{\rm FS} \mathrm{SubFin}(X, n)$, $(\mathrm{Integral2}(\mathrm{ElmFin}(M, n+1), |f_1|))(x) < +\infty$. Then $\int f_2 \, \mathrm{d}\, \mathrm{Measure}_{\rm Prod}(\mathrm{SubFin}(M, n+1)) = \int \mathrm{Integral2}(\mathrm{ElmFin}(M, n+1), f_1) \, \mathrm{d}\, \mathrm{Measure}_{\rm Prod}(\mathrm{SubFin}(M, n))$. The theorem is a consequence of (35).

(37) Let us consider a non zero natural number n, a non-empty, $(n+1)$-elements finite sequence X, a family of σ-fields S of X, a family of σ-measures M of S, a partial function f from $\prod_{\rm FS} X$ to $\overline{\mathbb{R}}$, an element E of $\prod_{\rm Field} S$, and a partial function g from $\prod_{\rm FS} \mathrm{SubFin}(X, n) \times \mathrm{ElmFin}(X, n+1)$ to $\overline{\mathbb{R}}$.

Suppose M is σ-finite and $E = \mathrm{dom}\, f$ and f is E-measurable and $f = g$. Then g is integrable on $\mathrm{ProdMeas}(\mathrm{Measure}_{\rm Prod}(\mathrm{SubFin}(M, n)), \mathrm{ElmFin}(M, n+1))$ iff $\int \mathrm{Integral2}(\mathrm{ElmFin}(M, n+1), |g|) \, \mathrm{d}\, \mathrm{Measure}_{\rm Prod}(\mathrm{SubFin}(M, n)) < +\infty$. The theorem is a consequence of (6), (34), (30), (29), and (31).

Let n be a non zero natural number, X be a non-empty, $(n+1)$-elements finite sequence, S be a family of σ-fields of X, M be a family of σ-measures of S, and f be a partial function from $\prod_{\rm FS} X$ to $\overline{\mathbb{R}}$. The functor $\mathrm{Integral}_{\rm FS}(M, f)$ yielding an $(n+1)$-elements finite sequence is defined by

(Def. 17) $it(1) = f$ and for every natural number i such that $1 \leqslant i < n+1$ there exists a non zero natural number k and there exists a partial function g from $\prod_{\rm FS} \mathrm{SubFin}(X, k) \times \mathrm{ElmFin}(X, k+1)$ to $\overline{\mathbb{R}}$ such that $k = n+1-i$ and $g = it(i)$ and $it(i+1) = \mathrm{Integral2}(\mathrm{ElmFin}(M, k+1), g)$.

We say that f is sequentially integrable on M if and only if

(Def. 18) for every non zero natural number k such that $k < n+1$ there exists a partial function G from $\prod_{\rm FS} \mathrm{SubFin}(X, k+1)$ to $\overline{\mathbb{R}}$ and there exists a partial function H from $\prod_{\rm FS} \mathrm{SubFin}(X, k)$ to $\overline{\mathbb{R}}$ such that $G = (\mathrm{Integral}_{\rm FS}(M, f))(n+1-k)$ and $H = (\mathrm{Integral}_{\rm FS}(\mathrm{SubFin}(M, k+1), |G|))(2)$ and for every element x of $\prod_{\rm FS} \mathrm{SubFin}(X, k)$, $H(x) < +\infty$.

Now we state the propositions:

(38) Let us consider a non zero natural number n, a non-empty, $(n+1)$-elements finite sequence X, a family of σ-fields S of X, a family of σ-measures M of S, and a partial function f from $\prod_{\rm FS} X$ to $\overline{\mathbb{R}}$.

Suppose M is σ-finite and f is sequentially integrable on M and f is integrable on $\mathrm{Measure}_{\rm Prod}(M)$. Let us consider a non zero natural number k. Suppose $k < n+1$. Then there exists a partial function g from

$\prod_{\rm FS} \mathrm{SubFin}(X, k+1)$ to $\overline{\mathbb{R}}$ such that

(i) $g = (\mathrm{Integral}_{\rm FS}(M, f))(n + 1 - k)$, and

(ii) g is integrable on $\mathrm{Measure}_{\rm Prod}(\mathrm{SubFin}(M, k + 1))$.

PROOF: Define $\mathcal{P}$[natural number] $\equiv$ if $1 \leqslant \$_1 < n + 1$, then there exists a non zero natural number j and there exists a partial function g from $\prod_{\rm FS} \mathrm{SubFin}(X, j + 1)$ to $\overline{\mathbb{R}}$ such that $j = n + 1 - \$_1$ and $g = (\mathrm{Integral}_{\rm FS}(M, f))(\$_1)$ and g is integrable on $\mathrm{Measure}_{\rm Prod}(\mathrm{SubFin}(M, j + 1))$. $\mathcal{P}[1]$. For every non zero natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k + 1]$. For every non zero natural number k, $\mathcal{P}[k]$. $\square$

(39) Let us consider a non zero natural number n, a non-empty, $(n + 1)$-elements finite sequence X, a family of σ-fields S of X, a family of σ-measures M of S, a partial function f from $\prod_{\rm FS} X$ to $\overline{\mathbb{R}}$, and a partial function g from $\prod_{\rm FS} \mathrm{SubFin}(X, n) \times \mathrm{ElmFin}(X, n+1)$ to $\overline{\mathbb{R}}$. Suppose $f = g$. Then

(i) $(\mathrm{Integral}_{\rm FS}(M, f))(1) = f$, and

(ii) $(\mathrm{Integral}_{\rm FS}(M, f))(2) = \mathrm{Integral2}(\mathrm{ElmFin}(M, n + 1), g)$.

(40) Let us consider a non zero natural number n, a non-empty, $(n + 1)$-elements finite sequence X, a family of σ-fields S of X, a family of σ-measures M of S, and a partial function f from $\prod_{\rm FS} X$ to $\overline{\mathbb{R}}$. Suppose M is σ-finite and f is sequentially integrable on M and f is integrable on $\mathrm{Measure}_{\rm Prod}(M)$. Let us consider a non zero natural number k.

Suppose $k < n$. Then there exists a partial function F_5 from $\prod_{\rm FS} \mathrm{SubFin}(X, k) \times \mathrm{ElmFin}(X, k + 1)$ to $\overline{\mathbb{R}}$ and there exists a partial function G_2 from $\prod_{\rm FS} \mathrm{SubFin}(X, k + 1)$ to $\overline{\mathbb{R}}$ and there exists a function F_4 from $\prod_{\rm FS} \mathrm{SubFin}(X, k)$ into $\overline{\mathbb{R}}$ such that $G_2 = F_5$ and $G_2 = (\mathrm{Integral}_{\rm FS}(M, f))(n+1-k)$ and $F_4 = (\mathrm{Integral}_{\rm FS}(M, f))(n+1-(k-1))$ and $F_4 = \mathrm{Integral2}(\mathrm{ElmFin}(M, k + 1), F_5)$ and G_2 is integrable on $\mathrm{Measure}_{\rm Prod}(\mathrm{SubFin}(M, k + 1))$ and $\int G_2 \, \mathrm{d}\, \mathrm{Measure}_{\rm Prod}(\mathrm{SubFin}(M, k+1)) = \int F_5 \, \mathrm{d}\, \mathrm{ProdMeas}(\mathrm{Measure}_{\rm Prod}(\mathrm{SubFin}(M, k)), \mathrm{ElmFin}(M, k+1))$ and for every element x of $\prod_{\rm FS} \mathrm{SubFin}(X, k)$, $\mathrm{ProjPMap1}(F_5, x)$ is integrable on $\mathrm{ElmFin}(M, k + 1)$.

For every element U of $\prod_{\rm Field} \mathrm{SubFin}(S, k)$, F_4 is U-measurable and F_4 is integrable on $\mathrm{Measure}_{\rm Prod}(\mathrm{SubFin}(M, k))$ and $\int F_5 \, \mathrm{d}\, \mathrm{ProdMeas}(\mathrm{Measure}_{\rm Prod}(\mathrm{SubFin}(M, k)), \mathrm{ElmFin}(M, k+1)) = \int F_4 \, \mathrm{d}\, \mathrm{Measure}_{\rm Prod}(\mathrm{SubFin}(M, k))$ and $F_4 \in$ the L^1 functions of $\mathrm{Measure}_{\rm Prod}(\mathrm{SubFin}(M, k))$ and $\int G_2 \, \mathrm{d}\, \mathrm{Measure}_{\rm Prod}(\mathrm{SubFin}(M, k+1)) = \int F_4 \, \mathrm{d}\, \mathrm{Measure}_{\rm Prod}(\mathrm{SubFin}(M, k))$.

The theorem is a consequence of (7), (8), (14), (12), (18), (17), (30), (38), (9), (6), (39), (35), and (36).

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