FORMALIZED MATHEMATICS
Vol. 31, No. 1, pp. 151–159, 2023
DOI: 10.2478/forma-2023-0014
e-ISSN: 1898–9934

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sciendo.com/journal/forma

# On Fuzzy Negations and Laws of Contraposition. Lattice of Fuzzy Negations

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**Summary.** This the next article in the series formalizing the book of Baczyński and Jayaram "Fuzzy Implications". We define the laws of contraposition connected with various fuzzy negations, and in order to make the cluster registration mechanism fully working, we construct some more non-classical examples of fuzzy implications. Finally, as the testbed of the reuse of lattice-theoretical approach, we introduce the lattice of fuzzy negations and show its basic properties.

MSC: 03B52 68V20

Keywords: fuzzy implication; contrapositive symmetry; fuzzy negation

MML identifier: FUZIMPL4, version: 8.1.14 5.76.1452

## Introduction

The main aim of this Mizar article was to implement a formal counterpart of the handbook of fuzzy implications [1]. This is the next submission in the series formalizing this volume, following, among others, [5]. We define the laws of contraposition with the connection to various fuzzy negations [6]. Developing the approach proposed in [7], we deal with the part of Chapter 1.5, pp. 20–23 [1].

In the first section we introduce Mizar attributes [2] which define contrapositive symmetry (also in its weaker, left- and right-side form) with respect to the given fuzzy negation, in Section 2 we recall the notion of fuzzy negation, taking into account the fact that if its converse is just the function (denoted in the Mizar formalism by $R^{\smile}$) implies their surjectivity or injectivity.

Section 3, 4, and 5 formalize complete proofs of lemmas and corollaries 1.5.3–1.5.9 from Chapter 1.5 [1]. The sixth section introduces two fuzzy implications introduced by Drewniak [3], which were not formalized in Mizar before: $I_{\rm I3}$ and $I_{\rm I4}$, needed to formulate Example 1.5.10. Section 7 shows how nine basic fuzzy implications are connected with contrapositive symmetry. Most of these properties, once proven formally, can be obtained by the Mizar checker without any additional references, only by virtue of cluster registrations mechanism. These registrations in the Mizar code can be treated as the formal counterpart of Table 1.9, p. 29 from Baczyński and Jayaram book, quoted below.

| Fuzzy implication $I$ | (CP) | (L-CP) | (R-CP) |
|---|---|---|---|
| $I_{\rm LK}$ | $N_{\rm C}$ | $N_{\rm C}$ | $N_{\rm C}$ |
| $I_{\rm GD}$ | $\times$ | $\times$ | $N_{\rm D1}$ |
| $I_{\rm RC}$ | $N_{\rm C}$ | $N_{\rm C}$ | $N_{\rm C}$ |
| $I_{\rm KD}$ | $N_{\rm C}$ | $N_{\rm C}$ | $N_{\rm C}$ |
| $I_{\rm GG}$ | $\times$ | $\times$ | $N_{\rm D1}$ |
| $I_{\rm RS}$ | $N_{\rm C}$ | $N_{\rm C}$ | $N_{\rm C}$ |
| $I_{\rm YG}$ | $\times$ | $\times$ | $N_{\rm D1}$ |
| $I_{\rm WB}$ | $\times$ | $\times$ | $N_{\rm D2}$ |
| $I_{\rm FD}$ | $N_{\rm C}$ | $N_{\rm C}$ | $N_{\rm C}$ |

Additionally, in the final section we introduce the lattice of all fuzzy negations and show its basic properties [9], partially formulating and proving Theorem 1.4.3, p. 14. We wanted to avoid duplication of lattice-theoretical notions (ordering vs. lattice suprema and infima) [11], and the availability of min and max operations for various (formally distinct) classes of functions was an issue we had to cope with [12].

Our work makes a step towards the formalization of fuzzy sets and fuzzy numbers [4], [15] in the computerized proof assistant [8], [10]; see [13] and [14] for another interesting effort in this direction.

## 1. Laws of Contraposition

Let $L$ be a non empty 1-sorted structure and $a$, $b$ be elements of $L$. Let us note that the functor $\{a, b\}$ yields a subset of $L$. One can verify that there exists a fuzzy negation which is decreasing.

Let $N$ be a fuzzy negation and $I$ be a binary operation on $[0, 1]$. We say that $I$ satisfies contraposition property w.r.t. $N$ if and only if

(Def. 1) for every elements $x$, $y$ of $[0,1]$, $I(x,y) = I(N(y), N(x))$.

We say that $I$ satisfies left contraposition property w.r.t. $N$ if and only if

(Def. 2) for every elements $x$, $y$ of $[0,1]$, $I(N(x), y) = I(N(y), x)$.

We say that $I$ satisfies right contraposition property w.r.t. $N$ if and only if

(Def. 3) for every elements $x$, $y$ of $[0,1]$, $I(x, N(y)) = I(y, N(x))$.

## 2. Fuzzy Negations Revisited

Now we state the proposition:

(1) $N_C = (\mathrm{AffineMap}(-1,1)) \restriction [0,1]$.

Proof: Set $N = N_C$. Set $f = (\mathrm{AffineMap}(-1,1)) \restriction [0,1]$. For every object $x$ such that $x \in \mathrm{dom}\, N$ holds $f(x) = N(x)$. □

Note that $N_C$ is continuous and $N_C$ is strong and there exists a fuzzy negation which is strict and there exists a fuzzy negation which is strong. Every fuzzy negation which is satisfying (N3) is also decreasing and every fuzzy negation which is decreasing is also satisfying (N3).

Observe that every unary operation on $[0,1]$ is $\mathbb{R}$-defined and real-valued and every real-valued function which is $\mathbb{R}$-defined and decreasing is also one-to-one. Every unary operation on $[0,1]$ which is decreasing is also one-to-one and every fuzzy negation is non-increasing and every fuzzy negation which is strict is also one-to-one. Now we state the proposition:

(2) Let us consider a function $R$. If $R^{\smile}$ is a function, then $R$ is one-to-one.

Let us consider fuzzy negations $N_1$, $N_2$. Now we state the propositions:

(3) If $N_1{}^{\smile} = N_2$, then $N_1$ is one-to-one.

(4) If $N_1{}^{\smile} = N_2$, then $N_1$ is onto.

Proof: $N_2$ is one-to-one. For every object $y$ such that $y \in [0,1]$ there exists an object $x$ such that $x \in [0,1]$ and $y = N_1(x)$. □

(5) Let us consider a binary operation $I$ on $[0,1]$, a strict fuzzy negation $N$, and a fuzzy negation $N_1$. Suppose $N^{\smile} = N_1$. Then $I$ satisfies left contraposition property w.r.t. $N$ if and only if $I$ satisfies right contraposition property w.r.t. $N_1$.

Proof: $N$ is onto. If $I$ satisfies left contraposition property w.r.t. $N$, then $I$ satisfies right contraposition property w.r.t. $N_1$. For every elements $x$, $y$ of $[0,1]$, $I(N(x), y) = I(N(y), x)$. □

## 3. Proposition 1.5.3

Let us consider a binary operation $I$ on $[0, 1]$ and a strong fuzzy negation $N$. Now we state the propositions:

(6) If $I$ satisfies contraposition property w.r.t. $N$, then $I$ satisfies left contraposition property w.r.t. $N$.

(7) If $I$ satisfies left contraposition property w.r.t. $N$, then $I$ satisfies right contraposition property w.r.t. $N$.

(8) If $I$ satisfies right contraposition property w.r.t. $N$, then $I$ satisfies contraposition property w.r.t. $N$.

(9) $I$ satisfies contraposition property w.r.t. $N$ if and only if $I$ satisfies left contraposition property w.r.t. $N$.

(10) $I$ satisfies contraposition property w.r.t. $N$ if and only if $I$ satisfies right contraposition property w.r.t. $N$.

## 4. Lemma 1.5.4

Let us consider a binary operation $I$ on $[0, 1]$ and a fuzzy negation $N$. Now we state the propositions:

(11) If $I$ satisfies (I1) and contraposition property w.r.t. $N$, then $I$ satisfies (I2).

Proof: For every elements $x$, $y$, $z$ of $[0, 1]$ such that $y \leqslant z$ holds $I(x, y) \leqslant I(x, z)$. □

(12) If $I$ satisfies (I2) and contraposition property w.r.t. $N$, then $I$ satisfies (I1).

Proof: For every elements $x$, $y$, $z$ of $[0, 1]$ such that $x \leqslant y$ holds $I(x, z) \geqslant I(y, z)$. □

(13) If $I$ satisfies (LB) and contraposition property w.r.t. $N$, then $I$ satisfies (RB).

(14) If $I$ satisfies (RB) and contraposition property w.r.t. $N$, then $I$ satisfies (LB).

(15) If $I$ satisfies (NP) and contraposition property w.r.t. $N$, then $N = N_I$ and $N_I$ is strong.

(16) If $I$ satisfies (NP) and contraposition property w.r.t. $N$, then $I$ satisfies (I3), (I4), and (I5). The theorem is a consequence of (15).

(17) Let us consider a binary operation $I$ on $[0, 1]$. Suppose $I$ satisfies (NP). If $N_I$ is not strong, then for every fuzzy negation $N$, $I$ does not satisfy contraposition property w.r.t. $N$.

## 5. Lemma 1.5.6 and Corollaries

Let us consider a binary operation $I$ on $[0,1]$ and a strong fuzzy negation $N$. Now we state the propositions:

(18) If $N = N_I$, then if $I$ satisfies contraposition property w.r.t. $N$, then $I$ satisfies (NP).

(19) If $N = N_I$, then if $I$ satisfies (EP), then $I$ satisfies (I3), (I4), (I5), (NP), and contraposition property w.r.t. $N$. The theorem is a consequence of (18) and (16).

Let us consider a binary operation $I$ on $[0,1]$ and a fuzzy negation $N$. Now we state the propositions:

(20) If $I$ satisfies contraposition property w.r.t. $N$, then $I$ satisfies (I1) iff $I$ satisfies (I2).

(21) If $I$ satisfies contraposition property w.r.t. $N$, then $I$ satisfies (LB) iff $I$ satisfies (RB).

(22) If $I$ satisfies contraposition property w.r.t. $N$, then if $N$ is strong, then $I$ satisfies (NP) iff $N = N_I$.

(23) If $I$ satisfies contraposition property w.r.t. $N$, (I1), and (NP), then $I \in \mathcal{FI}$ and $N_I = N$ and $N$ is strong. The theorem is a consequence of (20), (16), and (15).

(24) Let us consider fuzzy implication $I$ satisfying (NP) and (EP). Then $N_I$ is strong if and only if $I$ satisfies contraposition property w.r.t. $(N_I)$.

## 6. Some Further Examples of Fuzzy Implications

The functor $I_{\mathrm{I3}}$ yielding a binary operation on $[0,1]$ is defined by

(Def. 4) for every elements $x$, $y$ of $[0,1]$, if $x = 0$ or $y \neq 0$, then $it(x,y) = 1$ and if $x \neq 0$ and $y = 0$, then $it(x,y) = 0$.

One can verify that $I_{\mathrm{I3}}$ is antitone w.r.t. 1st coordinate, isotone w.r.t. 2nd coordinate, 00-dominant, 11-dominant, and 10-weak. Now we state the proposition:

(25) $N_{I_{\mathrm{I3}}} = N_{\mathrm{D1}}$.

Let us note that $I_{\mathrm{I3}}$ satisfies (EP) but does not satisfy (NP) and $I_{\mathrm{I3}}$ satisfies contraposition property w.r.t. $(N_{I_{\mathrm{I3}}})$.

The functor $I_{\mathrm{I4}}$ yielding a binary operation on $[0,1]$ is defined by

(Def. 5) for every elements $x$, $y$ of $[0,1]$, if $x \neq 1$ or $y = 1$, then $it(x,y) = 1$ and if $x = 1$ and $y \neq 1$, then $it(x,y) = 0$.

One can verify that $I_{\mathrm{I4}}$ is antitone w.r.t. 1st coordinate, isotone w.r.t. 2nd coordinate, 00-dominant, 11-dominant, and 10-weak. Now we state the proposition:

(26) $N_{I_{\mathrm{I4}}} = N_{\mathrm{D2}}$.

Let us note that $I_{\mathrm{I4}}$ satisfies (EP) but does not satisfy (NP) and $I_{\mathrm{I4}}$ satisfies contraposition property w.r.t. $(N_{I_{\mathrm{I4}}})$.

## 7. Contrapositive Symmetry w.r.t. the Natural Negation

Let $I$ be a fuzzy implication. We say that $I$ satisfies contraposition property if and only if

(Def. 6) $I$ satisfies contraposition property w.r.t. $(N_I)$.

We say that $I$ satisfies left contraposition property if and only if

(Def. 7) $I$ satisfies left contraposition property w.r.t. $(N_I)$.

We say that $I$ satisfies right contraposition property if and only if

(Def. 8) $I$ satisfies right contraposition property w.r.t. $(N_I)$.

Observe that $I_{\mathrm{LK}}$ satisfies left contraposition property w.r.t. $(N_C)$, right contraposition property w.r.t. $(N_C)$, and contraposition property w.r.t. $(N_C)$ and $I_{\mathrm{LK}}$ satisfies left contraposition property, right contraposition property, and contraposition property. $I_{\mathrm{GD}}$ satisfies right contraposition property w.r.t. $(N_{\mathrm{D1}})$ and $I_{\mathrm{GD}}$ satisfies right contraposition property.

Note that $I_{\mathrm{RC}}$ satisfies contraposition property w.r.t. $(N_C)$, left contraposition property w.r.t. $(N_C)$, and right contraposition property w.r.t. $(N_C)$ and $I_{\mathrm{RC}}$ satisfies contraposition property, left contraposition property, and right contraposition property. $I_{\mathrm{KD}}$ satisfies contraposition property w.r.t. $(N_C)$ and $I_{\mathrm{KD}}$ satisfies left contraposition property w.r.t. $(N_C)$ and $I_{\mathrm{KD}}$ satisfies right contraposition property w.r.t. $(N_C)$ and $I_{\mathrm{KD}}$ satisfies contraposition property, left contraposition property, and right contraposition property.

Let us observe $I_{\mathrm{GG}}$ satisfies right contraposition property w.r.t. $(N_{\mathrm{D1}})$ and $I_{\mathrm{GG}}$ satisfies right contraposition property. Now we state the proposition:

(27) $I_{\mathrm{RS}}$ satisfies left contraposition property w.r.t. $(N_C)$.

One can check that $I_{\mathrm{RS}}$ satisfies contraposition property w.r.t. $(N_C)$, left contraposition property w.r.t. $(N_C)$, and right contraposition property w.r.t. $(N_C)$. Now we state the proposition:

(28) Let us consider a decreasing fuzzy negation $N$. Then $I_{\mathrm{RS}}$ satisfies contraposition property w.r.t. $N$.

Proof: Set $I = I_{\mathrm{RS}}$.

For every elements $x$, $y$ of $[0,1]$, $I(x,y) = I(N(y), N(x))$. □

Let us observe that $I_{\rm YG}$ satisfies right contraposition property w.r.t. $(N_{\rm D1})$ and $I_{\rm YG}$ satisfies right contraposition property. $I_{\rm WB}$ satisfies right contraposition property w.r.t. $(N_{\rm D2})$ and $I_{\rm WB}$ satisfies right contraposition property.

Note that $I_{\rm FD}$ satisfies contraposition property w.r.t. $(N_C)$, left contraposition property w.r.t. $(N_C)$, and right contraposition property w.r.t. $(N_C)$ and $I_{\rm FD}$ satisfies contraposition property, left contraposition property, and right contraposition property.

## 8. Fuzzy Lattice Revisited

Now we state the propositions:

(29) FuzzyLattice $[0,1]$ is a complete, Heyting, distributive lattice.

(30) the set of all $f$ where $f$ is a fuzzy negation $\subseteq [0,1]^{[0,1]}$.

Let $N_1$, $N_2$ be fuzzy negations. The functors: $\max(N_1, N_2)$ and $\min(N_1, N_2)$ yielding fuzzy negations are defined by conditions

(Def. 9) there exist functions $f$, $g$ from $[0,1]$ into $\mathbb{R}$ such that $f = N_1$ and $g = N_2$ and $\max(N_1, N_2) = \max(f, g)$,

(Def. 10) there exist functions $f$, $g$ from $[0,1]$ into $\mathbb{R}$ such that $f = N_1$ and $g = N_2$ and $\min(N_1, N_2) = \min(f, g)$,

respectively. The functor FuzzyNegations yielding a strict, full relational substructure of FuzzyLattice $[0,1]$ is defined by

(Def. 11) the carrier of *it* = the set of all $\mathcal{N}$ where $\mathcal{N}$ is a fuzzy negation.

Observe that FuzzyNegations is non empty, reflexive, transitive, and antisymmetric. Now we state the proposition:

(31) Let us consider fuzzy negations $N_1$, $N_2$.
Then $\max(N_1, N_2) = \mathbf{max}_{\mathbb{R}[0,1]}(N_1, N_2)$.
Proof: Set $A = [0,1]$. Set $\mathcal{F} = \max(N_1, N_2)$. Set $m = \mathbf{max}_{\mathbb{R}[0,1]}(N_1, N_2)$. Consider $f_1$ being a function such that $m = f_1$ and $\operatorname{dom} f_1 = A$ and $\operatorname{rng} f_1 \subseteq \mathbb{R}$. For every object $x$ such that $x \in [0,1]$ holds $\mathcal{F}(x) = m(x)$. □

Let us consider fuzzy negations $N_1$, $N_2$ and membership functions $f_2$, $g_2$ of $[0,1]$. Now we state the propositions:

(32) If $N_1 = f_2$ and $N_2 = g_2$, then $\max(N_1, N_2) = \max(f_2, g_2)$.

(33) If $N_1 = f_2$ and $N_2 = g_2$, then $\min(N_1, N_2) = \min(f_2, g_2)$.

(34) Let us consider fuzzy negations $N_1$, $N_2$.
Then $\min(N_1, N_2) = \mathbf{min}_{\mathbb{R}[0,1]}(N_1, N_2)$.
Proof: Set $A = [0,1]$. Set $\mathcal{F} = \min(N_1, N_2)$. Set $m = \mathbf{min}_{\mathbb{R}[0,1]}(N_1, N_2)$. Consider $f_1$ being a function such that $m = f_1$ and $\operatorname{dom} f_1 = A$ and $\operatorname{rng} f_1 \subseteq \mathbb{R}$. For every object $x$ such that $x \in [0,1]$ holds $\mathcal{F}(x) = m(x)$. □

Note that FuzzyNegations is join-inheriting and FuzzyNegations is meet-inheriting.

Let us consider elements $\mathcal{N}_1$, $\mathcal{N}_2$ of FuzzyNegations and fuzzy negations $N_1$, $N_2$. Now we state the propositions:

(35) If $N_1 = \mathcal{N}_1$ and $N_2 = \mathcal{N}_2$, then $\mathcal{N}_1 \sqcup \mathcal{N}_2 = \max(N_1, N_2)$. The theorem is a consequence of (32).

(36) If $N_1 = \mathcal{N}_1$ and $N_2 = \mathcal{N}_2$, then $\mathcal{N}_1 \sqcap \mathcal{N}_2 = \min(N_1, N_2)$. The theorem is a consequence of (33).

## References


[1] Michał Baczyński and Balasubramaniam Jayaram. *Fuzzy Implications*. Springer Publishing Company, Incorporated, 2008. doi:10.1007/978-3-540-69082-5.

[2] Grzegorz Bancerek, Czesław Byliński, Adam Grabowski, Artur Korniłowicz, Roman Matuszewski, Adam Naumowicz, and Karol Pąk. The role of the Mizar Mathematical Library for interactive proof development in Mizar. *Journal of Automated Reasoning*, 61(1):9–32, 2018. doi:10.1007/s10817-017-9440-6.

[3] Józef Drewniak. Invariant fuzzy implications. *Soft Computing*, 10:506–513, 2006.

[4] Didier Dubois and Henri Prade. *Fuzzy Sets and Systems: Theory and Applications*. Academic Press, New York, 1980.

[5] Adam Grabowski. Formal introduction to fuzzy implications. *Formalized Mathematics*, 25(**3**):241–248, 2017. doi:10.1515/forma-2017-0023.

[6] Adam Grabowski. On fuzzy negations generated by fuzzy implications. *Formalized Mathematics*, 28(**1**):121–128, 2020. doi:10.2478/forma-2020-0011.

[7] Adam Grabowski. Fuzzy implications in the Mizar system. In *30th IEEE International Conference on Fuzzy Systems, FUZZ-IEEE 2021, Luxembourg, July 11–14, 2021*, pages 1–6. IEEE, 2021. doi:10.1109/FUZZ45933.2021.9494593.

[8] Adam Grabowski. On the computer certification of fuzzy numbers. In M. Ganzha, L. Maciaszek, and M. Paprzycki, editors, *2013 Federated Conference on Computer Science and Information Systems (FedCSIS)*, Federated Conference on Computer Science and Information Systems, pages 51–54, 2013.

[9] Adam Grabowski. Lattice theory for rough sets – a case study with Mizar. *Fundamenta Informaticae*, 147(2–3):223–240, 2016. doi:10.3233/FI-2016-1406.

[10] Adam Grabowski and Takashi Mitsuishi. Initial comparison of formal approaches to fuzzy and rough sets. In Leszek Rutkowski, Marcin Korytkowski, Rafal Scherer, Ryszard Tadeusiewicz, Lotfi A. Zadeh, and Jacek M. Zurada, editors, *Artificial Intelligence and Soft Computing – 14th International Conference, ICAISC 2015, Zakopane, Poland, June 14-18, 2015, Proceedings, Part I*, volume 9119 of *Lecture Notes in Computer Science*, pages 160–171. Springer, 2015. doi:10.1007/978-3-319-19324-3_15.

[11] Adam Grabowski and Takashi Mitsuishi. Formalizing lattice-theoretical aspects of rough and fuzzy sets. In D. Ciucci, G. Wang, S. Mitra, and W.Z. Wu, editors, *Rough Sets and Knowledge Technology – 10th International Conference held as part of the International Joint Conference on Rough Sets (IJCRS), Tianjin, PR China, November 20–23, 2015, Proceedings*, volume 9436 of *Lecture Notes in Artificial Intelligence*, pages 347–356. Springer, 2015. doi:10.1007/978-3-319-25754-9_31.

[12] Adam Grabowski and Christoph Schwarzweller. On duplication in mathematical repositories. In Serge Autexier, Jacques Calmet, David Delahaye, Patrick D. F. Ion, Laurence Rideau, Renaud Rioboo, and Alan P. Sexton, editors, *Intelligent Computer Mathematics, 10th International Conference, AISC 2010, 17th Symposium, Calculemus 2010, and 9th International Conference, MKM 2010, Paris, France, July 5–10, 2010. Proceedings*, volume 6167 of *Lecture Notes in Computer Science*, pages 300–314. Springer, 2010. doi:10.1007/978-3-642-14128-7_26.

[13] Takashi Mitsuishi. Definition of centroid method as defuzzification. *Formalized Mathe-*

*matics*, 30(**2**):125–134, 2022. doi:10.2478/forma-2022-0010.

[14] Takashi Mitsuishi. Isosceles triangular and isosceles trapezoidal membership functions using centroid method. *Formalized Mathematics*, 31:59–66, 2023. doi:10.2478/forma-2023-0006.

[15] Lotfi Zadeh. Fuzzy sets. *Information and Control*, 8(3):338–353, 1965. doi:10.1016/S0019-9958(65)90241-X.

*Accepted November 21, 2023*