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Antiderivatives and Integration[1]

Noboru Endou
National Institute of Technology, Gifu College
2236-2 Kamimakuwa, Motosu, Gifu, Japan

Summary. In this paper, we introduce indefinite integrals [8] (antiderivatives) and proof integration by substitution in the Mizar system [2], [3]. In our previous article [15], we have introduced an indefinite-like integral, but it is inadequate because it must be an integral over the whole set of real numbers and in some sense it causes some duplication in the Mizar Mathematical Library [13]. For this reason, to define the antiderivative for a function, we use the derivative of an arbitrary interval as defined recently in [7]. Furthermore, antiderivatives are also used to modify the integration by substitution and integration by parts.

In the first section, we summarize the basic theorems on continuity and derivativity (for interesting survey of formalizations of real analysis in another proof-assistants like ACL2 [12], Isabelle/HOL [11], Coq [4], see [5]). In the second section, we generalize some theorems that were noticed during the formalization process. In the last section, we define the antiderivatives and formalize the integration by substitution and the integration by parts. We referred to [1] and [6] in our development.

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1. Basic Theorems on Continuity and Derivativity

From now on h, h_1 denote 0-convergent, non-zero sequences of real numbers and c, c_1 denote constant sequences of real numbers. Let us observe that every subset of $\mathbb{R}$ which is open interval is also open. Now we state the propositions:

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(1) Let us consider an interval I. If $\inf I \in I$, then $\inf I = \inf I$.

(2) Let us consider an interval subset I of $\mathbb{R}$. If $\sup I \in I$, then $\sup I = \sup I$.

(3) Let us consider real numbers a, b, and an interval I. If a, $b \in I$, then $[a, b] \subseteq I$.

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$ and real numbers a, b. Now we state the propositions:

(4) Suppose $a < b$ and $[a, b[\subseteq \operatorname{dom} f$ and $f \restriction [a, b[$ is continuous and f is differentiable on $]a, b[$ and $f'_{\restriction]a,b[}$ is right convergent in a. Then

  (i) f is right differentiable in a, and

  (ii) $f'_{+}(a) = \lim_{a^+} f'_{\restriction]a,b[}$.

PROOF: Consider e being a real number such that $a < e < b$. For every h and c such that $\operatorname{rng} c = \{a\}$ and $\operatorname{rng}(h + c) \subseteq \operatorname{dom} f$ and for every natural number n, $h(n) > 0$ holds $h^{-1} \cdot ((f_*(h + c)) - (f_* c))$ is convergent and $\lim(h^{-1} \cdot ((f_*(h + c)) - (f_* c))) = \lim_{a^+} f'_{\restriction]a,b[}$. □

(5) Suppose $a < b$ and $]a, b] \subseteq \operatorname{dom} f$ and $f \restriction]a, b]$ is continuous and f is differentiable on $]a, b[$ and $f'_{\restriction]a,b[}$ is left convergent in b. Then

  (i) f is left differentiable in b, and

  (ii) $f'_{-}(b) = \lim_{b^-} f'_{\restriction]a,b[}$.

PROOF: Consider e being a real number such that $a < e < b$. For every h and c such that $\operatorname{rng} c = \{b\}$ and $\operatorname{rng}(h + c) \subseteq \operatorname{dom} f$ and for every natural number n, $h(n) < 0$ holds $h^{-1} \cdot ((f_*(h + c)) - (f_* c))$ is convergent and $\lim(h^{-1} \cdot ((f_*(h + c)) - (f_* c))) = \lim_{b^-} f'_{\restriction]a,b[}$. □

(6) Let us consider real numbers a, b, x, a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and an interval I. Suppose $\inf I \leqslant a$ and $b \leqslant \sup I$ and $I \subseteq \operatorname{dom} f$ and $f \restriction I$ is continuous and $x \in]a, b[$. Then f is continuous in x.

(7) Let us consider an open subset X of $\mathbb{R}$, and partial functions f, F from $\mathbb{R}$ to $\mathbb{R}$. Suppose $X \subseteq \operatorname{dom} f$ and $f \restriction X$ is continuous. Let us consider a real number x. If $x \in X$, then f is continuous in x.

Let us consider real numbers a, b, x and a partial function f from $\mathbb{R}$ to $\mathbb{R}$. Now we state the propositions:

(8) Suppose $a \leqslant x < b$ and $]a, b[\subseteq \operatorname{dom} f$ and f is right convergent in x. Then

  (i) $f \restriction]a, b[$ is right convergent in x, and

  (ii) $\lim_{x^+}(f \restriction]a, b[) = \lim_{x^+} f$.

PROOF: For every real number r such that $x < r$ there exists a real number g such that $g < r$ and $x < g$ and $g \in \operatorname{dom}(f \restriction]a, b[)$. For every real number

r such that $0 < r$ there exists a real number d such that $x < d$ and for every real number x_1 such that $x_1 < d$ and $x < x_1$ and $x_1 \in \operatorname{dom}(f\restriction]a,b[)$ holds $|(f\restriction]a,b[)(x_1) - \lim_{x^+} f| < r$. $\square$

(9) Suppose $a < x \leqslant b$ and $]a,b[\subseteq \operatorname{dom} f$ and f is left convergent in x. Then

(i) $f\restriction]a,b[$ is left convergent in x, and

(ii) $\lim_{x^-}(f\restriction]a,b[) = \lim_{x^-} f$.

Proof: For every real number r such that $r < x$ there exists a real number g such that $r < g < x$ and $g \in \operatorname{dom}(f\restriction]a,b[)$. For every real number r such that $0 < r$ there exists a real number d such that $d < x$ and for every real number x_1 such that $d < x_1 < x$ and $x_1 \in \operatorname{dom}(f\restriction]a,b[)$ holds $|(f\restriction]a,b[)(x_1) - \lim_{x^-} f| < r$. $\square$

(10) Suppose $[a,b] \subseteq \operatorname{dom} f$ and $f\restriction[a,b]$ is continuous and $x \in [a,b[$. Then

(i) f is right convergent in x, and

(ii) $\lim_{x^+}(f\restriction]a,b[) = f(x)$.

Proof: For every real number r such that $x < r$ there exists a real number g such that $g < r$ and $x < g$ and $g \in \operatorname{dom} f$. For every real number r such that $0 < r$ there exists a real number s such that $x < s$ and for every real number x_1 such that $x_1 < s$ and $x < x_1$ and $x_1 \in \operatorname{dom} f$ holds $|f(x_1) - f(x)| < r$. For every real number r such that $0 < r$ there exists a real number s such that $x < s$ and for every real number x_1 such that $x_1 < s$ and $x < x_1$ and $x_1 \in \operatorname{dom}(f\restriction]a,b[)$ holds $|(f\restriction]a,b[)(x_1) - f(x)| < r$. $f\restriction]a,b[$ is right convergent in x and $\lim_{x^+}(f\restriction]a,b[) = \lim_{x^+} f$. $\square$

(11) Suppose $[a,b] \subseteq \operatorname{dom} f$ and $f\restriction[a,b]$ is continuous and $x \in]a,b]$. Then

(i) f is left convergent in x, and

(ii) $\lim_{x^-}(f\restriction]a,b[) = f(x)$.

Proof: For every real number r such that $r < x$ there exists a real number g such that $r < g < x$ and $g \in \operatorname{dom} f$. For every real number r such that $0 < r$ there exists a real number s such that $s < x$ and for every real number x_1 such that $s < x_1 < x$ and $x_1 \in \operatorname{dom} f$ holds $|f(x_1) - f(x)| < r$. For every real number r such that $0 < r$ there exists a real number s such that $s < x$ and for every real number x_1 such that $s < x_1 < x$ and $x_1 \in \operatorname{dom}(f\restriction]a,b[)$ holds $|(f\restriction]a,b[)(x_1) - f(x)| < r$. $f\restriction]a,b[$ is left convergent in x and $\lim_{x^-}(f\restriction]a,b[) = \lim_{x^-} f$. $\square$

Let us consider a real number x, a partial function f from $\mathbb{R}$ to $\mathbb{R}$, a non empty interval I, and a subset X of $\mathbb{R}$. Now we state the propositions:

(12) If $I \subseteq X$ and $x \in I$ and $x \neq \sup I$, then f is right differentiable in x iff $f\restriction X$ is right differentiable in x.

(13) If $I \subseteq X$ and $x \in I$ and $x \neq \inf I$, then f is left differentiable in x iff $f{\restriction}X$ is left differentiable in x.

(14) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, an open subset I of $\mathbb{R}$, and a subset X of $\mathbb{R}$. Suppose $I \subseteq X$. Then f is differentiable on I if and only if $f{\restriction}X$ is differentiable on I.

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, a non empty interval I, and a subset X of $\mathbb{R}$. Now we state the propositions:

(15) If $I \subseteq X$, then f is differentiable on interval I iff $f{\restriction}X$ is differentiable on interval I. The theorem is a consequence of (1), (12), (2), (13), and (14).

(16) If $I \subseteq X$ and f is differentiable on interval I, then $f'_I = (f{\restriction}X)'_I$. The theorem is a consequence of (15), (1), and (2).

(17) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and non empty intervals I, J. Suppose f is differentiable on interval I and $J \subseteq I$ and $\inf J < \sup J$. Then $f'_I{\restriction}J = f'_J$.

PROOF: For every element x of $\mathbb{R}$ such that $x \in \mathrm{dom}(f'_I{\restriction}J)$ holds $(f'_I{\restriction}J)(x) = f'_J(x)$. $\square$

2. Generalization of Previous Theorems

Now we state the propositions:

(18) Let us consider extended real numbers a, b. If $a < b$, then there exists a real number c such that $a < c < b$.

(19) Let us consider extended real numbers p, q, and a partial function f from $\mathbb{R}$ to $\mathbb{R}$. Suppose f is differentiable on $]p, q[$ and for every real number x such that $x \in\ ]p, q[$ holds $f'(x) = 0$. Then $f{\restriction}]p, q[$ is constant.

(20) Let us consider extended real numbers p, q, and partial functions f_1, f_2 from $\mathbb{R}$ to $\mathbb{R}$. Suppose f_1 is differentiable on $]p, q[$ and f_2 is differentiable on $]p, q[$ and for every real number x such that $x \in\ ]p, q[$ holds ${f_1}'(x) = {f_2}'(x)$. Then

   (i) $(f_1 - f_2){\restriction}]p, q[$ is constant, and

   (ii) there exists a real number r such that for every real number x such that $x \in\ ]p, q[$ holds $f_1(x) = f_2(x) + r$.

   The theorem is a consequence of (19).

Let us consider extended real numbers p, q and a partial function f from $\mathbb{R}$ to $\mathbb{R}$. Now we state the propositions:

(21) Suppose f is differentiable on $]p, q[$ and for every real number x such that $x \in\ ]p, q[$ holds $0 < f'(x)$. Then $f{\restriction}]p, q[$ is increasing.

(22) Suppose f is differentiable on $]p,q[$ and for every real number x such that $x \in]p,q[$ holds $f'(x) < 0$. Then $f\restriction]p,q[$ is decreasing.

(23) Suppose f is differentiable on $]p,q[$ and for every real number x such that $x \in]p,q[$ holds $0 \leqslant f'(x)$. Then $f\restriction]p,q[$ is non-decreasing.

(24) Suppose f is differentiable on $]p,q[$ and for every real number x such that $x \in]p,q[$ holds $f'(x) \leqslant 0$. Then $f\restriction]p,q[$ is non-increasing.

(25) Let us consider an open subset X of $\mathbb{R}$, a real number x_0, and a partial function f from $\mathbb{R}$ to $\mathbb{R}$. Suppose $x_0 \in X$ and f is differentiable on X. Then $f'(x_0) = (f\restriction X)'(x_0)$.

PROOF: Consider N being a neighbourhood of x_0 such that $N \subseteq \operatorname{dom}(f\restriction X)$ and there exists a linear function L and there exists a rest R such that $(f\restriction X)'(x_0) = L(1)$ and for every real number x such that $x \in N$ holds $(f\restriction X)(x) - (f\restriction X)(x_0) = L(x - x_0) + R(x - x_0)$. Consider L being a linear function, R being a rest such that $(f\restriction X)'(x_0) = L(1)$ and for every real number x such that $x \in N$ holds $(f\restriction X)(x) - (f\restriction X)(x_0) = L(x - x_0) + R(x - x_0)$. For every real number x such that $x \in N$ holds $f(x) - f(x_0) = L(x - x_0) + R(x - x_0)$. $\square$

(26) Let us consider real numbers a, b, and a partial function f from $\mathbb{R}$ to $\mathbb{R}$. Suppose $a < b$ and $[a,b] \subseteq \operatorname{dom} f$ and $f\restriction[a,b]$ is continuous. Then there exists a partial function F from $\mathbb{R}$ to $\mathbb{R}$ such that

(i) $]a,b[\subseteq \operatorname{dom} F$, and

(ii) for every real number x such that $x \in]a,b[$ holds $F(x) = \int\limits_a^x f(x)dx$, and

(iii) F is differentiable on $]a,b[$, and

(iv) $F'_{\restriction]a,b[} = f\restriction]a,b[$.

PROOF: Consider x_0 being a real number such that $a < x_0 < b$. Consider F being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $]a,b[\subseteq \operatorname{dom} F$ and for every real number x such that $x \in]a,b[$ holds $F(x) = \int\limits_a^x f(x)dx$ and F is differentiable in x_0 and $F'(x_0) = f(x_0)$. For every real number x such that $x \in]a,b[$ holds $F\restriction]a,b[$ is differentiable in x. For every element x of $\mathbb{R}$ such that $x \in \operatorname{dom} F'_{\restriction]a,b[}$ holds $F'_{\restriction]a,b[}(x) = (f\restriction]a,b[)(x)$. $\square$

(27) Let us consider real numbers a, b, and partial functions f, F from $\mathbb{R}$ to $\mathbb{R}$. Suppose $a < b$ and $[a,b] \subseteq \operatorname{dom} f$ and $f\restriction[a,b]$ is continuous and $]a,b[\subseteq \operatorname{dom} F$ and for every real number x such that $x \in]a,b[$ holds

$F(x) = \int\limits_a^x f(x)dx$. Then

(i) F is differentiable on $]a, b[$, and

(ii) $F'_{\restriction]a,b[} = f \restriction]a, b[$.

PROOF: Consider G being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $]a, b[\subseteq \operatorname{dom} G$ and for every real number x such that $x \in\ ]a, b[$ holds $G(x) = \int\limits_a^x f(x)dx$ and G is differentiable on $]a, b[$ and $G'_{\restriction]a,b[} = f \restriction]a, b[$. For every element x of $\mathbb{R}$ such that $x \in \operatorname{dom}(F \restriction]a, b[)$ holds $(F \restriction]a, b[)(x) = (G \restriction]a, b[)(x)$. □

3. Antiderivatives and Related Theorems

Let f, F be partial functions from $\mathbb{R}$ to $\mathbb{R}$ and I be a non empty interval. We say that F is antiderivative of f on I if and only if

(Def. 1) F is differentiable on interval I and $F'_I = f \restriction I$.

Now we state the propositions:

(28) Let us consider partial functions f, F, g, G from $\mathbb{R}$ to $\mathbb{R}$, and a non empty interval I. Suppose F is antiderivative of f on I and G is antiderivative of g on I. Then

(i) $F + G$ is antiderivative of $f + g$ on I, and

(ii) $F - G$ is antiderivative of $f - g$ on I.

(29) Let us consider partial functions f, F from $\mathbb{R}$ to $\mathbb{R}$, a non empty interval I, and a real number r. If F is antiderivative of f on I, then $r \cdot F$ is antiderivative of $r \cdot f$ on I.

Let us consider partial functions f, g, F, G from $\mathbb{R}$ to $\mathbb{R}$ and a non empty interval I. Now we state the propositions:

(30) If F is antiderivative of f on I and G is antiderivative of g on I, then $F \cdot G$ is antiderivative of $f \cdot G + F \cdot g$ on I.

(31) Suppose F is antiderivative of f on I and G is antiderivative of g on I and for every set x such that $x \in I$ holds $G(x) \neq 0$. Then $\frac{F}{G}$ is antiderivative of $\frac{f \cdot G - F \cdot g}{G \cdot G}$ on I.

(32) Let us consider real numbers a, b, and partial functions f, F from $\mathbb{R}$ to $\mathbb{R}$. Suppose $a \leqslant b$ and $[a, b] \subseteq \operatorname{dom} f$ and $f \restriction [a, b]$ is continuous and $[a, b] \subseteq \operatorname{dom} F$ and for every real number x such that $x \in [a, b]$ holds

$F(x) = \int\limits_{a}^{x} f(x)dx$. Let us consider a real number x. Suppose $x \in]a, b[$. Then

(i) F is differentiable in x, and

(ii) $F'(x) = f(x)$.

Proof: Set $O =]a, b[$. Define $\mathcal{G}_0(\text{real number}) = (\int\limits_{a}^{\$_1} f(x)dx)(\in \mathbb{R})$. Consider G_1 being a function from $\mathbb{R}$ into $\mathbb{R}$ such that for every element h of $\mathbb{R}$, $G_1(h) = \mathcal{G}_0(h)$. Reconsider $G = G_1 \restriction O$ as a partial function from $\mathbb{R}$ to $\mathbb{R}$. For every real number x such that $x \in O$ holds G is differentiable in x and $G'(x) = f(x)$ by (6), [9, (10),(11)]. For every real number x such that $x \in]a, b[$ holds F is differentiable in x and $F'(x) = f(x)$ by [14, (2)]. □

Let us consider real numbers a, b and partial functions f, F from $\mathbb{R}$ to $\mathbb{R}$. Now we state the propositions:

(33) Suppose $a \leqslant b$ and $[a, b] \subseteq \operatorname{dom} f$ and $f \restriction [a, b]$ is bounded and f is integrable on $[a, b]$ and $[a, b] = \operatorname{dom} F$ and for every real number x such that $x \in [a, b]$ holds $F(x) = \int\limits_{a}^{x} f(x)dx$. Then F is Lipschitzian.

Proof: Consider r_0 being a real number such that for every object x such that $x \in [a, b] \cap \operatorname{dom} f$ holds $|f(x)| \leqslant r_0$. Reconsider $r = \max(r_0, 1)$ as a real number. For every real numbers p, q such that p, $q \in [a, b]$ and $p \leqslant q$ holds f is integrable on $[p, q]$ and $f \restriction [p, q]$ is bounded. For every real numbers x_1, x_2 such that x_1, $x_2 \in \operatorname{dom} F$ holds $|F(x_1) - F(x_2)| \leqslant r \cdot |x_1 - x_2|$ by [10, (20),(23)]. □

(34) Suppose $a < b$ and $[a, b] \subseteq \operatorname{dom} f$ and $f \restriction [a, b]$ is continuous and $[a, b] \subseteq \operatorname{dom} F$ and for every real number x such that $x \in [a, b]$ holds $F(x) = \int\limits_{a}^{x} f(x)dx$. Then $F'_{\restriction]a,b[}$ is right convergent in a and left convergent in b.

Proof: For every real number x such that $x \in]a, b[$ holds $F \restriction]a, b[$ is differentiable in x. For every element x of $\mathbb{R}$ such that $x \in \operatorname{dom} F'_{\restriction]a,b[}$ holds $F'_{\restriction]a,b[}(x) = (f \restriction]a, b[)(x)$. For every real number r such that $a < r$ there exists a real number g such that $g < r$ and $a < g$ and $g \in \operatorname{dom} F'_{\restriction]a,b[}$. For every real number g_1 such that $0 < g_1$ there exists a real number r such that $a < r$ and for every real number r_1 such that $r_1 < r$ and $a < r_1$ and $r_1 \in \operatorname{dom} F'_{\restriction]a,b[}$ holds $|F'_{\restriction]a,b[}(r_1) - f(a)| < g_1$. For every real number r such that $r < b$ there exists a real number g such that $r < g < b$ and $g \in \operatorname{dom} F'_{\restriction]a,b[}$. For every real number g_1 such that $0 < g_1$ there exists a real number r such that $r < b$ and for every real number r_1 such that $r < r_1 < b$ and $r_1 \in \operatorname{dom} F'_{\restriction]a,b[}$ holds $|F'_{\restriction]a,b[}(r_1) - f(b)| < g_1$. $\square$

(35) Suppose $a < b$ and $[a,b] \subseteq \operatorname{dom} f$ and $f \restriction [a,b]$ is continuous and $[a,b] \subseteq \operatorname{dom} F$ and for every real number x such that $x \in [a,b]$ holds $F(x) = \int\limits_a^x f(x)dx$. Then

(i) F is right differentiable in a, and

(ii) $F'_+(a) = \lim_{a^+} F'_{\restriction]a,b[}$.

PROOF: For every real number x such that $x \in]a,b[$ holds $F \restriction]a,b[$ is differentiable in x. $F'_{\restriction]a,b[}$ is right convergent in a. For every real number x such that $x \in [a,b]$ holds $(F \restriction [a,b])(x) = \int\limits_a^x f(x)dx$. $F \restriction [a,b[$ is Lipschitzian. $\square$

(36) Suppose $a < b$ and $[a,b] \subseteq \operatorname{dom} f$ and $f \restriction [a,b]$ is continuous and $[a,b] \subseteq \operatorname{dom} F$ and for every real number x such that $x \in [a,b]$ holds $F(x) = \int\limits_a^x f(x)dx$. Then

(i) F is left differentiable in b, and

(ii) $F'_-(b) = \lim_{b^-} F'_{\restriction]a,b[}$.

PROOF: For every real number x such that $x \in]a,b[$ holds $F \restriction]a,b[$ is differentiable in x. $F'_{\restriction]a,b[}$ is left convergent in b. For every real number x such that $x \in [a,b]$ holds $(F \restriction [a,b])(x) = \int\limits_a^x f(x)dx$. $F \restriction]a,b]$ is Lipschitzian. $\square$

(37) Suppose $a < b$ and $[a,b] \subseteq \operatorname{dom} f$ and $f \restriction [a,b]$ is continuous and $[a,b] \subseteq \operatorname{dom} F$ and for every real number x such that $x \in [a,b]$ holds $F(x) = \int\limits_a^x f(x)dx$. Then

(i) F is differentiable on interval $[a,b]$, and

(ii) $F'_{[a,b]} = f \restriction [a,b]$.

PROOF: Reconsider $I = [a,b]$ as a non empty interval. If $\inf I \in I$, then F is right differentiable in $\inf I$. If $\sup I \in I$, then F is left differentiable in $\sup I$. For every real number x such that $x \in]a,b[$ holds $F \restriction]a,b[$ is differentiable in x. $F'_{\restriction]a,b[} = f \restriction]a,b[$. For every element x of $\mathbb{R}$ such that $x \in \operatorname{dom} F'_{[a,b]}$ holds $F'_{[a,b]}(x) = (f \restriction [a,b])(x)$. $\square$

(38) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and real numbers a, b. Then $\int\limits_{b}^{a} f(x)dx = -\int\limits_{a}^{b} f(x)dx$.

(39) Let us consider real numbers a, b, and partial functions f, F from $\mathbb{R}$ to $\mathbb{R}$. Suppose $a < b$ and $[a, b] \subseteq \operatorname{dom} f$ and $f{\restriction}[a, b]$ is continuous and $[a, b] \subseteq \operatorname{dom} F$ and for every real number x such that $x \in [a, b]$ holds $F(x) = \int\limits_{a}^{x} f(x)dx$. Let us consider a real number x. Suppose $x \in\ ]a, b[$. Then

(i) F is differentiable in x, and

(ii) $F'(x) = f(x)$.

The theorem is a consequence of (37).

(40) Let us consider real numbers a, b, and a partial function f from $\mathbb{R}$ to $\mathbb{R}$. Suppose $a < b$ and $[a, b] \subseteq \operatorname{dom} f$ and $f{\restriction}[a, b]$ is continuous. Then there exists a partial function F from $\mathbb{R}$ to $\mathbb{R}$ such that

(i) F is antiderivative of f on $[a, b]$, and

(ii) for every real number x such that $x \in [a, b]$ holds $F(x) = \int\limits_{a}^{x} f(x)dx$.

The theorem is a consequence of (37).

(41) Let us consider a real number c, partial functions f, F, G from $\mathbb{R}$ to $\mathbb{R}$, and a non empty interval I. Suppose $I \subseteq \operatorname{dom} f$ and F is antiderivative of f on I and $I \subseteq \operatorname{dom} G$ and for every real number x such that $x \in I$ holds $G(x) = F(x) + c$. Then G is antiderivative of f on I.
Proof: Reconsider $c_0 = c$ as an element of $\mathbb{R}$. Define $\mathcal{F}(\text{element of } \mathbb{R}) = c_0$. Consider F_0 being a function from $\mathbb{R}$ into $\mathbb{R}$ such that for every element x of $\mathbb{R}$, $F_0(x) = \mathcal{F}(x)$. $F{\restriction}I$ is differentiable on interval I. G is differentiable on interval I. $\square$

(42) Let us consider partial functions f, F from $\mathbb{R}$ to $\mathbb{R}$, and non empty intervals I, J. Suppose $\inf I < \sup I$ and $I \subseteq J$ and F is antiderivative of f on J. Then F is antiderivative of f on I.

(43) Let us consider real numbers a, b, a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and a partition D of $[a, b]$. Suppose $a < b$ and f is differentiable on interval $[a, b]$ and $f'_{[a,b]}$ is bounded. Then $\text{lower\_sum}(f'_{[a,b]} \restriction [a, b], D) \leqslant f(b) - f(a) \leqslant \text{upper\_sum}(f'_{[a,b]} \restriction [a, b], D)$.

(44) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, real numbers a, b, and a non empty interval I. Suppose a, $b \in I$ and $a < b$ and f is differentiable

on interval I and f'_I is integrable on $[a,b]$ and f'_I is bounded. Then

(i) $\int\limits_a^b f'_{[a,b]}(x)dx = f(b) - f(a)$, and

(ii) $\int\limits_a^b f'_I(x)dx = f(b) - f(a)$.

The theorem is a consequence of (3) and (17).

(45) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, a real number a, and a non empty interval I. Suppose f is differentiable on interval I and $a \in I$. Then $\int\limits_a^a f'_I(x)dx = 0$. The theorem is a consequence of (3).

(46) Let us consider partial functions f, F, G from $\mathbb{R}$ to $\mathbb{R}$, and a non empty interval I. Suppose F is antiderivative of f on I and G is antiderivative of f on I. Then there exists a real number c such that for every real number x such that $x \in I$ holds $F(x) = G(x) + c$. The theorem is a consequence of (42), (1), (2), and (18).

(47) Integration by substitution:
Let us consider real numbers a, b, p, q, and partial functions f, g from $\mathbb{R}$ to $\mathbb{R}$. Suppose $a < b$ and $p < q$ and $[a,b] \subseteq \operatorname{dom} f$ and $f{\restriction}[a,b]$ is continuous and g is differentiable on interval $[p,q]$ and $g'_{[p,q]}$ is integrable on $[p,q]$ and $g'_{[p,q]}$ is bounded and $\operatorname{rng}(g{\restriction}[p,q]) \subseteq [a,b]$ and $g(p) = a$ and $g(q) = b$. Then $\int\limits_a^b f(x)dx = \int\limits_p^q (f \cdot g \cdot g'_{[p,q]})(x)dx$. The theorem is a consequence of (37).

(48) Let us consider real numbers a, b, and partial functions f, g from $\mathbb{R}$ to $\mathbb{R}$. Suppose $a < b$ and f is differentiable on interval $[a,b]$ and g is differentiable on interval $[a,b]$ and $f'_{[a,b]}$ is integrable on $[a,b]$ and $f'_{[a,b]}$ is bounded and $g'_{[a,b]}$ is integrable on $[a,b]$ and $g'_{[a,b]}$ is bounded. Then

$$\int\limits_a^b (f'_{[a,b]} \cdot g)(x)dx = f(b) \cdot g(b) - f(a) \cdot g(a) - \int\limits_a^b (f \cdot g'_{[a,b]})(x)dx.$$

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