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Internal Direct Products and the Universal Property of Direct Product Groups

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Summary. This is a "quality of life" article concerning product groups, using the Mizar system [2], [4]. Like a Sonata, this article consists of three movements.

The first act, the slowest of the three, builds the infrastructure necessary for the rest of the article. We prove group homomorphisms map arbitrary finite products to arbitrary finite products, introduce a notion of "group yielding" families, as well as families of homomorphisms. We close the first act with defining the inclusion morphism of a subgroup into its parent group, and the projection morphism of a product group onto one of its factors.

The second act introduces the universal property of products and its consequences as found in, e.g., Kurosh [7]. Specifically, for the product of an arbitrary family of groups, we prove the center of a product group is the product of centers. More exciting, we prove for a product of a finite family groups, the commutator subgroup of the product is the product of commutator subgroups, but this is because in general: the direct sum of commutator subgroups is the subgroup of the commutator subgroup of the product group, and the commutator subgroup of the product is a subgroup of the product of derived subgroups. We conclude this act by proving a few theorems concerning the image and kernel of morphisms between product groups, as found in Hungerford [5], as well as quotients of product groups.

The third act introduces the notion of an internal direct product. Isaacs [6] points out (paraphrasing with Mizar terminology) that the internal direct product is a predicate but the external direct product is a [Mizar] functor. To our delight, we find the bulk of the "recognition theorem" (as stated by Dummit and Foote [3], Aschbacher [1], and Robinson [11]) are already formalized in the heroic work of Nakasho, Okazaki, Yamazaki, and Shimada [9], [8]. We generalize the notion of an internal product to a set of subgroups, proving it is equivalent to the internal product of a family of subgroups [10].

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1. Preliminaries

Now we state the propositions:

(1) Let us consider sets X, Y, Z, W. Suppose $Z \neq \emptyset$ and $W \neq \emptyset$. Let us consider a function f from $X \times Y$ into Z, and a function g from $X \times Y$ into W. If for every element a of X for every element b of Y, $f(a, b) = g(a, b)$, then $f = g$.

(2) Let us consider a finite set A. Then $\mathrm{CFS}(A)$ is a many sorted set indexed by $\mathrm{Seg}\,\overline{\overline{A}}$.

(3) Let us consider non empty sets X, Y, and a function f from X into Y. Suppose f is onto. Then there exists a function g from Y into X such that $f \cdot g = \mathrm{id}_Y$.

PROOF: Define $\mathcal{P}[\text{object}, \text{object}] \equiv \$_1 = f(\$_2)$. For every object y such that $y \in Y$ there exists an object x such that $x \in X$ and $\mathcal{P}[y, x]$. Consider g being a function from Y into X such that for every object y such that $y \in Y$ holds $\mathcal{P}[y, g(y)]$. For every element y of Y, $(f \cdot g)(y) = y$. □

Let I be a non empty set, A, B be many sorted sets indexed by I, f be a many sorted function from A into B, and i be an element of I. Let us observe that the functor $f(i)$ yields a function from $A(i)$ into $B(i)$. Let F_1, F_2 be 1-sorted yielding many sorted sets indexed by I.

A many sorted function from F_1 into F_2 is a many sorted function from the support of F_1 into the support of F_2. Let φ be a many sorted function from F_1 into F_2 and i be an element of I. Note that the functor $\varphi(i)$ yields a function from $F_1(i)$ into $F_2(i)$. Now we state the proposition:

(4) Let us consider a non empty set I, many sorted sets A, B indexed by I, and a many sorted set f indexed by I. Then f is a many sorted function from A into B if and only if for every element i of I, $f(i)$ is a function from $A(i)$ into $B(i)$.

Let I, X be sets. Observe that there exists a many sorted set indexed by I which is (2^X)-valued.

Let M be a (2^X)-valued many sorted set indexed by I. One can check that the functor $\bigcup M$ yields a subset of X. Let I be a set, J be a subset of I, and F be a many sorted set indexed by I. One can check that $F \restriction J$ is J-defined and total.

Let F be a 1-sorted yielding many sorted set indexed by I. Observe that $F \restriction J$ is 1-sorted yielding, J-defined, and total. Now we state the proposition:

(5) Let us consider a non empty set I, a many sorted set M indexed by I, and an object y. Then $y \in \operatorname{rng} M$ if and only if there exists an element i of I such that $y = M(i)$.

2. Sequences of Group Elements under Homomorphisms

Now we state the propositions:

(6) Let us consider groups G_1, G_2, a homomorphism φ from G_1 to G_2, a finite sequence F_1 of elements of the carrier of G_1, and a finite sequence F_2 of elements of the carrier of G_2. If $F_2 = \varphi \cdot F_1$, then $\prod F_2 = \varphi(\prod F_1)$.
Proof: Define $\mathcal{P}$[finite sequence of elements of the carrier of G_1] $\equiv \varphi(\prod \$_1) = \prod \varphi \cdot \$_1$. $\mathcal{P}[\varepsilon_\alpha]$, where α is the carrier of G_1. For every finite sequence p_0 of elements of the carrier of G_1 and for every element x of the carrier of G_1 such that $\mathcal{P}[p_0]$ holds $\mathcal{P}[p_0 \frown \langle x \rangle]$. For every finite sequence p_0 of elements of the carrier of G_1, $\mathcal{P}[p_0]$. □

(7) Let us consider groups G_1, G_2, a homomorphism φ from G_1 to G_2, and a finite sequence F_1 of elements of the carrier of G_1. Then there exists a finite sequence F_2 of elements of the carrier of G_2 such that

(i) $\operatorname{len} F_1 = \operatorname{len} F_2$, and

(ii) $F_2 = \varphi \cdot F_1$, and

(iii) $\prod F_2 = \varphi(\prod F_1)$.

Proof: Set $n_1 = \operatorname{len} F_1$. Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists a natural number k such that $k = \$_1$ and $\$_2 = \varphi(F_1(k))$. For every natural number k such that $k \in \operatorname{Seg} n_1$ there exists an object x such that $\mathcal{P}[k, x]$. Consider p being a finite sequence such that $\operatorname{dom} p = \operatorname{Seg} n_1$ and for every natural number k such that $k \in \operatorname{Seg} n_1$ holds $\mathcal{P}[k, p(k)]$. $p = \varphi \cdot F_1$. □

(8) Let us consider groups G_1, G_2, a homomorphism φ from G_1 to G_2, a finite sequence F_1 of elements of the carrier of G_1, and a finite sequence k_1 of elements of $\mathbb{Z}$. Then there exists a finite sequence F_2 of elements of the carrier of G_2 such that

(i) $\operatorname{len} F_1 = \operatorname{len} F_2$, and

(ii) $F_2 = \varphi \cdot F_1$, and

(iii) $\prod {F_2}^{k_1} = \varphi(\prod {F_1}^{k_1})$.

Proof: Consider F_2 being a finite sequence of elements of the carrier of G_2 such that $\operatorname{len} F_1 = \operatorname{len} F_2$ and $F_2 = \varphi \cdot F_1$ and $\prod F_2 = \varphi(\prod F_1)$. For every natural number k such that $k \in \operatorname{dom} {F_2}^{k_1}$ holds $(\varphi \cdot {F_1}^{k_1})(k) = {F_2}^{k_1}(k)$. □

3. Preliminary Work about Group-families and Group-yielding Many Sorted Sets

Let I_2 be a binary relation. We say that I_2 is group yielding if and only if

(Def. 1) for every object G such that $G \in \operatorname{rng} I_2$ holds G is a group.

One can check that every function which is group yielding is also 1-sorted yielding and every function which is group yielding is also multiplicative magma yielding. Now we state the proposition:

(9) Let us consider a set I. Then every associative, group-like multiplicative magma family of I is group yielding.

Let I be a set. One can check that there exists a many sorted set indexed by I which is group yielding and every multiplicative magma family of I which is associative and group-like is also group yielding and there exists a function which is group yielding. Now we state the proposition:

(10) Let us consider a non empty set I, a group yielding many sorted set F indexed by I, and an element i of I. Then $F(i)$ is a group.

Let I be a non empty set, i be an element of I, and F be a group yielding many sorted set indexed by I. Note that $F(i)$ is group-like, associative, unital, and non empty as a multiplicative magma. Now we state the proposition:

(11) Let us consider a set I, and a many sorted set F indexed by I. Then F is group yielding if and only if for every object i such that $i \in I$ holds $F(i)$ is a group.

Let I be a set. Let us observe that every multiplicative magma family of I which is group yielding is also group-like and associative and every group-like, associative multiplicative magma family of I is group yielding and every group yielding many sorted set indexed by I is group-like, associative, and multiplicative magma yielding.

From now on I denotes a non empty set, i denotes an element of I, F denotes a group family of I, and G denotes a group. Now we state the propositions:

(12) $\emptyset \longmapsto G$ is a group family of $\emptyset$.

(13) Let us consider a natural number n. Then $\operatorname{Seg} n \longmapsto G$ is a group family of $\operatorname{Seg} n$. The theorem is a consequence of (12).

Let G be a group and n be a natural number. One can verify that $\operatorname{Seg} n \longmapsto G$ is group yielding. Now we state the proposition:

(14) (The support of $F)(i) =$ the carrier of $F(i)$.

The scheme *GrFamSch* deals with a non empty set I_1 and a unary functor $\mathcal{A}$ yielding a group and states that

(Sch. 1) There exists a group family $\mathcal{F}$ of I_1 such that for every element i of I_1, $\mathcal{F}(i) = \mathcal{A}(i)$.

4. Subgroup-family of a Family of Groups

Let I be a set and F, I_2 be group families of I. We say that I_2 is F-subgroup yielding if and only if

(Def. 2) for every element i of I and for every group G such that $G = F(i)$ holds $I_2(i)$ is a subgroup of G.

Now we state the propositions:

(15) Let us consider a group family S of I. Then S is F-subgroup yielding if and only if for every element i of I, $S(i)$ is a subgroup of $F(i)$.

(16) Let us consider a set I. Then every group family of I is F-subgroup yielding.

Let I be a set and F be a group family of I. Let us observe that there exists a group family of I which is F-subgroup yielding.

A subgroup family of F is an F-subgroup yielding group family of I. Let I be a non empty set, S be a subgroup family of F, and i be an element of I. Let us observe that the functor $S(i)$ yields a subgroup of $F(i)$. From now on S denotes a subgroup family of F. Now we state the proposition:

(17) Let us consider a group family S of I. Then S is a subgroup family of F if and only if for every element i of I, $S(i)$ is a subgroup of $F(i)$.

The scheme *SubFamSch* deals with a non empty set I_1 and a group family $\mathcal{F}$ of I_1 and a unary functor $\mathcal{S}$ yielding a group and states that

(Sch. 2) There exists a subgroup family S of $\mathcal{F}$ such that for every element i of I_1, $S(i) = \mathcal{S}(\mathcal{F}(i))$

provided

- for every group G, $\mathcal{S}(G)$ is a subgroup of G.

Let I be a non empty set and I_2 be a group family of I. We say that I_2 is componentwise strict if and only if

(Def. 3) for every element i of I, $I_2(i)$ is strict.

One can check that there exists a group family of I which is componentwise strict. Now we state the proposition:

(18) Let us consider a non empty set I, a group family F of I, and a subgroup family I_2 of F. Then I_2 is componentwise strict if and only if for every element i of I, $I_2(i)$ is a strict subgroup of $F(i)$.

Let I be a non empty set and F be a group family of I. One can verify that there exists a subgroup family of F which is componentwise strict. Let S be a componentwise strict subgroup family of F and i be an element of I. Note that $S(i)$ is strict as a subgroup of $F(i)$.

The scheme *StrSubFamSch* deals with a non empty set I_1 and a group family $\mathcal{F}$ of I_1 and a unary functor $\mathcal{S}$ yielding a group and states that

(Sch. 3) There exists a componentwise strict subgroup family S of $\mathcal{F}$ such that for every element i of I_1, $S(i) = \mathcal{S}(\mathcal{F}(i))$

provided

- for every group G, $\mathcal{S}(G)$ is a strict subgroup of G.

Now we state the proposition:

(19) Let us consider subgroup families A, B of F. If for every element i of I, $A(i) = B(i)$, then $A = B$.

Let I be a non empty set and F be a group family of I. The functor $\{\mathbf{1}\}_F$ yielding a componentwise strict subgroup family of F is defined by

(Def. 4) for every element i of I, $it(i) = \{\mathbf{1}\}_{F(i)}$.

The functor Ω_F yielding a componentwise strict subgroup family of F is defined by

(Def. 5) for every element i of I, $it(i) = \Omega_{F(i)}$.

Let I_2 be a subgroup family of F. We say that I_2 is normal if and only if

(Def. 6) for every element i of I, $I_2(i)$ is a normal subgroup of $F(i)$.

Let us note that there exists a subgroup family of F which is componentwise strict and normal. Let S be a normal subgroup family of F and i be an element of I. One can check that $S(i)$ is normal as a subgroup of $F(i)$.

Let S be a componentwise strict subgroup family of F. Note that $S(i)$ is strict as a subgroup of $F(i)$ and $\{\mathbf{1}\}_F$ is normal and Ω_F is normal. Let N be a normal subgroup family of F. The functor ${}^F/_N$ yielding a group family of I is defined by

(Def. 7) for every element i of I, $it(i) = {}^{F(i)}/_{N(i)}$.

Observe that ${}^F/_N$ is componentwise strict. Now we state the propositions:

(20) There exists a componentwise strict, normal subgroup family S of F such that for every element i of I, $S(i) = F(i)^{\mathrm{c}}$.

PROOF: Define $\mathcal{A}(\text{group}) = \$_1{}^{\mathrm{c}}$. Consider S being a componentwise strict subgroup family of F such that for every element i of I, $S(i) = \mathcal{A}(F(i))$. For every element i of I, $S(i)$ is a normal subgroup of $F(i)$. □

(21) Let us consider a strict multiplicative magma M. Suppose there exists an object x such that the carrier of $M = \{x\}$. Then there exists a strict, trivial group G such that $M = G$.

(22) Let us consider an empty set I, and a multiplicative magma family F of I. Then $\prod F$ is a trivial group. The theorem is a consequence of (21).

5. Inclusion Morphism

Let G, H be groups. Assume H is a subgroup of G. The functor $\mathrm{incl}(H, G)$ yielding a homomorphism from H to G is defined by the term

(Def. 8) id_{α}, where α is the carrier of H.

Let G be a group and H be a subgroup of G. The functor $\overset{H}{\hookrightarrow}$ yielding a homomorphism from H to G is defined by the term

(Def. 9) $\mathrm{incl}(H, G)$.

Now we state the propositions:

(23) Let us consider a group H, and an element h of H. If H is a subgroup of G, then $(\mathrm{incl}(H, G))(h) = h$.

(24) Let us consider a subgroup H of G. Then

(i) $\mathrm{incl}(H, G)$ is one-to-one, and

(ii) $\mathrm{Im}\,\mathrm{incl}(H, G) =$ the multiplicative magma of H.

Proof: Set $f = \mathrm{incl}(H, G)$. $\mathrm{Ker}\, f = \{\mathbf{1}\}_H$. □

Let G be a group and H be a subgroup of G. Let us observe that $\mathrm{incl}(H, G)$ is one-to-one. Now we state the propositions:

(25) Let us consider groups H, K. Suppose H is a subgroup of G. Let us consider a homomorphism φ from G to K. Then $\varphi\restriction$(the carrier of H) $= \varphi \cdot (\mathrm{incl}(H, G))$.

Proof: $\mathrm{dom}(\varphi\restriction$(the carrier of H)) = the carrier of H. For every object x such that $x \in \mathrm{dom}(\varphi\restriction$(the carrier of H)) holds $(\varphi\restriction$(the carrier of H))$(x) = (\varphi \cdot (\mathrm{incl}(H, G)))(x)$. □

(26) Let us consider a group K, a subgroup H of G, and a homomorphism φ from G to K. Then $\varphi\restriction H = \varphi \cdot (\overset{H}{\hookrightarrow})$.

Proof: For every element h of H, $(\varphi\restriction H)(h) = (\varphi \cdot (\overset{H}{\hookrightarrow}))(h)$. □

(27) Let us consider a group G, and a strict subgroup H of G. Then $\mathrm{Im}(\overset{H}{\hookrightarrow}) = H$.

6. Families of Homomorphisms

Let G be a group, I be a non empty set, and F be a group family of I.

A homomorphism family of G and F is a many sorted function indexed by I defined by

(Def. 10) for every element i of I, $it(i)$ is a homomorphism from G to $F(i)$.

Let f be a homomorphism family of G and F and i be an element of I. One can check that the functor $f(i)$ yields a homomorphism from G to $F(i)$. In the sequel f denotes a homomorphism family of G and F. Now we state the proposition:

(28) $\langle i, f(i)\rangle \in f$.

Let I be a non empty set and F_1, F_2 be group families of I.

A homomorphism family of F_1 and F_2 is a many sorted function from F_1 into F_2 defined by

(Def. 11) for every element i of I, $it(i)$ is a homomorphism from $F_1(i)$ to $F_2(i)$.

Let i be an element of I and φ be a homomorphism family of F_1 and F_2. Note that $\varphi(i)$ is multiplicative as a function from $F_1(i)$ into $F_2(i)$. Now we state the proposition:

(29) Let us consider a non empty set I, group families A, B of I, and a many sorted set f indexed by I. Then f is a homomorphism family of A and B if and only if for every element i of I, $f(i)$ is a homomorphism from $A(i)$ to $B(i)$. The theorem is a consequence of (14).

The scheme *HomFamSch* deals with a non empty set I_1 and a group family D_1 of I_1 and a group family $\mathcal{C}$ of I_1 and a unary functor $\mathcal{A}$ yielding a function and states that

(Sch. 4) There exists a homomorphism family H of D_1 and $\mathcal{C}$ such that for every element i of I_1, $H(i) = \mathcal{A}(i)$

provided

- for every element i of I_1, $\mathcal{A}(i)$ is a homomorphism from $D_1(i)$ to $\mathcal{C}(i)$.

Now we state the proposition:

(30) Let us consider a group G, a non empty set I, a group family F of I, and a many sorted set f indexed by I. Then f is a homomorphism family of G and F if and only if for every element i of I, $f(i)$ is a homomorphism from G to $F(i)$.

The scheme *RHomFamSch* deals with a non empty set I_1 and a group D_1 and a group family $\mathcal{C}$ of I_1 and a unary functor $\mathcal{A}$ yielding a function and states that

(Sch. 5) There exists a homomorphism family H of D_1 and $\mathcal{C}$ such that for every element i of I_1, $H(i) = \mathcal{A}(i)$

provided

- for every element i of I_1, $\mathcal{A}(i)$ is a homomorphism from D_1 to $\mathcal{C}(i)$.

Now we state the proposition:

(31) Let us consider a non empty set I, group families A, B of I, and a many sorted set f indexed by I. Then f is a homomorphism family of A and B if and only if for every element i of I, $f(i)$ is a homomorphism from $A(i)$ to $B(i)$. The theorem is a consequence of (14).

7. Projection Morphisms from Product Group to Direct Factors

Now we state the proposition:

(32) Let us consider an element g of $\prod F$. Then $g(i)$ is an element of $F(i)$.

Let I be a non empty set, F be a group family of I, g be an element of $\prod F$, and i be an element of I. The functor $g_{/i}$ yielding an element of $F(i)$ is defined by the term

(Def. 12) $g(i)$.

We identify $g(i)$ with $g_{/i}$. The functor $\text{proj}(F, i)$ yielding a homomorphism from $\prod F$ to $F(i)$ is defined by

(Def. 13) for every element h of $\prod F$, $it(h) = h(i)$.

Now we state the proposition:

(33) $\text{proj}(F, i)$ is onto.

PROOF: For every object y such that $y \in$ the carrier of $F(i)$ there exists an object x such that $x \in$ the carrier of $\prod F$ and $y = (\text{proj}(F, i))(x)$. □

Let I be a non empty set, F be a group family of I, and i be an element of I. Let us observe that $\text{proj}(F, i)$ is onto. Now we state the propositions:

(34) proj(the support of F, i) is a function from $\prod$(the support of F) into the carrier of $F(i)$.

(35) Let us consider an element g of $\prod F$.
Then $(\text{proj}(F, i))(g) =$ (proj(the support of F, i))(g).

(36) $\text{proj}(F, i) =$ proj(the support of F, i). The theorem is a consequence of (34) and (35).

(37) Let us consider an element g of $\prod F$, and an element h of $F(i)$. Then $g +\cdot (i, h) \in \prod F$.

(38) Let us consider an element i of I, and an element g of $\prod F$. Then $g +\cdot (i, \mathbf{1}_{F(i)}) \in \text{Ker}\,\text{proj}(F, i)$. The theorem is a consequence of (37).

(39) Let us consider groups G_1, G_2, and a homomorphism f from G_1 to G_2. If for every element g of G_1, $f(g) = g$, then G_1 is a subgroup of G_2.
PROOF: The carrier of $G_1 \subseteq$ the carrier of G_2. Set $U_1 =$ the carrier of G_1. For every element a of U_1 and for every element b of U_1, (the multiplication of $G_1)(a, b) =$ ((the multiplication of $G_2) \restriction U_1)(a, b)$. (The multiplication of $G_2) \restriction U_1$ is a binary operation on U_1. □

(40) Let us consider elements i, j of I. Suppose $i \neq j$. Then $(\mathrm{proj}(F, j)) \cdot (\mathrm{1ProdHom}(F, i)) = F(i) \to \{\mathbf{1}\}_{F(j)}$.
PROOF: Set $U =$ the carrier of $F(i)$. $\mathrm{dom}(F(i) \to \{\mathbf{1}\}_{F(j)}) = U$ and $\mathrm{dom}((\mathrm{proj}(F, j)) \cdot (\mathrm{1ProdHom}(F, i))) = U$. For every element x of U, $((\mathrm{proj}(F, j)) \cdot (\mathrm{1ProdHom}(F, i)))(x) = (F(i) \to \{\mathbf{1}\}_{F(j)})(x)$. □

(41) $(\mathrm{proj}(F, i)) \cdot (\mathrm{1ProdHom}(F, i)) = \mathrm{id}_\alpha$, where α is the carrier of $F(i)$.
PROOF: Set $U =$ the carrier of $F(i)$. For every element x of U, $((\mathrm{proj}(F, i)) \cdot (\mathrm{1ProdHom}(F, i)))(x) = x$. □

8. Universal Property of Direct Products of Groups

Let us consider a homomorphism family f of G and F. Now we state the propositions:

(42) There exists a homomorphism φ from G to $\prod F$ such that for every element g of G for every element j of I, $(f(j))(g) = (\mathrm{proj}(F, j))(\varphi(g))$.
PROOF: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists an element g_0 of $\prod F$ such that $\$_2 = g_0$ and for every element j of I, $f(j)(\$_1) = g_0(j)$. Define $\mathcal{F} =$ the carrier of G. For every object x such that $x \in \mathcal{F}$ there exists an object y such that $y \in$ the carrier of $\prod F$ and $\mathcal{P}[x, y]$. Consider φ being a function from $\mathcal{F}$ into the carrier of $\prod F$ such that for every object x such that $x \in \mathcal{F}$ holds $\mathcal{P}[x, \varphi(x)]$. For every element g of G and for every element j of I, $\varphi(g)(j) = f(j)(g)$. For every elements a, b of G, $\varphi(a \cdot b) = \varphi(a) \cdot \varphi(b)$. For every element j of I, $(f(j))(g) = (\mathrm{proj}(F, j))(\varphi(g))$. □

(43) There exists a homomorphism φ from G to $\prod F$ such that for every element i of I, $f(i) = (\mathrm{proj}(F, i)) \cdot \varphi$.
PROOF: Consider φ being a homomorphism from G to $\prod F$ such that for every element g of G and for every element j of I, $(f(j))(g) = (\mathrm{proj}(F, j))$ $(\varphi(g))$. For every element g of G, $((\mathrm{proj}(F, i)) \cdot \varphi)(g) = f(i)(g)$. □

(44) Let us consider a homomorphism family f of G and F, and homomorphisms φ_1, φ_2 from G to $\prod F$. Suppose for every element i of I, $f(i) = (\mathrm{proj}(F, i)) \cdot \varphi_1$ and for every element i of I, $f(i) = (\mathrm{proj}(F, i)) \cdot \varphi_2$. Then $\varphi_1 = \varphi_2$.
PROOF: For every element g of G, $\varphi_1(g) = \varphi_2(g)$. □

Let G be a group, I be a non empty set, F be a group family of I, and f be a homomorphism family of G and F. The functor $\prod f$ yielding a homomorphism from G to $\prod F$ is defined by

(Def. 14) for every element g of G and for every element i of I, $f(i)(g) = it(g)(i)$.

Let us consider an element g of G. Now we state the propositions:

(45) for every element i of I, $(\prod f)(g)(i) = \mathbf{1}_{F(i)}$ if and only if $(\prod f)(g) = \mathbf{1}_{\prod F}$.

PROOF: If for every element i of I, $(\prod f)(g)(i) = \mathbf{1}_{F(i)}$, then $(\prod f)(g) = \mathbf{1}_{\prod F}$. □

(46) $g \in \operatorname{Ker} \prod f$ if and only if for every element i of I, $g \in \operatorname{Ker} f(i)$.

PROOF: If $g \in \operatorname{Ker} \prod f$, then for every element i of I, $g \in \operatorname{Ker} f(i)$. If for every element i of I, $g \in \operatorname{Ker} f(i)$, then $g \in \operatorname{Ker} \prod f$. □

(47) Let us consider groups G_1, G_2, G_3, a homomorphism f_1 from G_1 to G_2, a homomorphism f_2 from G_2 to G_3, and an element g of G_1. Then $g \in \operatorname{Ker} f_2 \cdot f_1$ if and only if $f_1(g) \in \operatorname{Ker} f_2$.

PROOF: If $g \in \operatorname{Ker} f_2 \cdot f_1$, then $f_1(g) \in \operatorname{Ker} f_2$. If $f_1(g) \in \operatorname{Ker} f_2$, then $g \in \operatorname{Ker} f_2 \cdot f_1$. □

(48) Let us consider groups G_1, G_2, G_3, a homomorphism f_1 from G_1 to G_2, and a homomorphism f_2 from G_2 to G_3. Then the carrier of $\operatorname{Ker} f_2 \cdot f_1 = {f_1}^{-1}$((the carrier of $\operatorname{Ker} f_2$)).

PROOF: For every element g of G_1 such that $g \in$ the carrier of $\operatorname{Ker} f_2 \cdot f_1$ holds $g \in {f_1}^{-1}$((the carrier of $\operatorname{Ker} f_2$)). For every element g of G_1 such that $g \in {f_1}^{-1}$((the carrier of $\operatorname{Ker} f_2$)) holds $g \in$ the carrier of $\operatorname{Ker} f_2 \cdot f_1$. □

(49) The carrier of $\operatorname{Ker} \prod f = \bigcap$ the set of all the carrier of $\operatorname{Ker} f(i)$ where i is an element of I.

PROOF: Set $\mathcal{F}$ = the set of all the carrier of $\operatorname{Ker} f(i)$ where i is an element of I. $\mathcal{F} \neq \emptyset$. For every object g, $g \in \operatorname{Ker} \prod f$ iff for every set A such that $A \in \mathcal{F}$ holds $g \in A$. For every object g, $g \in \operatorname{Ker} \prod f$ iff $g \in \bigcap \mathcal{F}$. For every object g, $g \in$ the carrier of $\operatorname{Ker} \prod f$ iff $g \in \bigcap \mathcal{F}$. □

(50) Let us consider a function f, a non empty set I, and a group family F of I. Suppose $\operatorname{dom} f = I$ and for every element i of I, $f(i) \in F(i)$. Then $f \in \prod F$. The theorem is a consequence of (14).

(51) Let us consider a group family S of I, and an element g of $\prod F$. Then $g \in \prod S$ if and only if for every element i of I, $(\operatorname{proj}(F, i))(g) \in S(i)$. The theorem is a consequence of (50).

(52) Let us consider group families F_1, F_2 of I. Suppose for every element i of I, $F_1(i)$ is a subgroup of $F_2(i)$. Then $\prod F_1$ is a subgroup of $\prod F_2$.

PROOF: Define $\mathcal{A}$(element of I) $=$ (incl(F_1(\$$_1$), F_2(\$$_1$))) $\cdot$ (proj(F_1, \$$_1$)). Consider f being a homomorphism family of $\prod F_1$ and F_2 such that for every element i of I, $f(i) = \mathcal{A}(i)$. For every element g of $\prod F_1$ and for every element i of I, $f(i)(g) = g(i)$. Consider φ being a homomorphism from $\prod F_1$ to $\prod F_2$ such that for every element g of $\prod F_1$ and for every element i of I, $(f(i))(g) = (\mathrm{proj}(F_2, i))(\varphi(g))$. For every element g of $\prod F_1$, $\varphi(g) = g$. □

Let I be a non empty set, F be a group family of I, and S be a subgroup family of F. The functor $\prod S$ yielding a strict subgroup of $\prod F$ is defined by the term

(Def. 15) $\prod S$.

Now we state the propositions:

(53) $\mathrm{Im}\,\mathrm{proj}(F, i) =$ the multiplicative magma of $F(i)$.

PROOF: For every object g such that $g \in$ the carrier of $F(i)$ holds $g \in$ the carrier of $\mathrm{Im}\,\mathrm{proj}(F, i)$. □

(54) Let us consider componentwise strict subgroup families F_1, F_2 of F. Suppose for every element i of I, $\mathrm{Im}\,\mathrm{proj}(F_1, i)$ is a subgroup of $\mathrm{Im}\,\mathrm{proj}(F_2, i)$. Then $\prod F_1$ is a strict subgroup of $\prod F_2$. The theorem is a consequence of (53) and (52).

(55) Let us consider a strict subgroup G of $\prod F$, and S. Suppose for every element i of I, $S(i) = \mathrm{Im}(\mathrm{proj}(F, i)) \cdot (\overset{G}{\hookrightarrow})$. Let us consider a homomorphism family f of G and S. Suppose for every element i of I, $f(i) = (\mathrm{proj}(F, i)) \cdot (\overset{G}{\hookrightarrow})$. Then $\prod f = \mathrm{id}_{\alpha}$, where α is the carrier of G.

PROOF: For every element g of G and for every element i of I, $((\mathrm{proj}(F, i)) \cdot (\overset{G}{\hookrightarrow}))(g) = ((\mathrm{proj}(F, i)) \cdot (\prod f))(g)$. For every element g of $\prod F$ such that $g \in G$ holds $(\prod f)(g) = g$. For every object x such that $x \in$ the carrier of G holds $(\prod f)(x) = x$. □

(56) Let us consider groups G_1, G_2, a homomorphism φ from G_1 to G_2, and an element x of G_1. Suppose $x \in$ the commutators of G_1. Then $\varphi(x) \in$ the commutators of G_2.

(57) Let us consider groups G_1, G_2, G_3, a homomorphism f_1 from G_1 to G_2, a homomorphism f_2 from G_2 to G_3, and an element g of G_1. Then $(f_2 \cdot f_1)(g) = f_2(f_1(g))$.

(58) Let us consider groups G_1, G_2, a subgroup H of G_2, a homomorphism f_1 from G_1 to G_2, and a homomorphism f_2 from G_1 to H. If $f_1 = f_2$, then $\mathrm{Im}\,f_1 = \mathrm{Im}\,f_2$.

PROOF: For every element g of G_2, $g \in \mathrm{Im}\,f_1$ iff $g \in \mathrm{Im}\,f_2$. □

(59) Let us consider elements a, b of $\prod F$, and i. Then $[a, b](i) = [a_{/i}, b_{/i}]$.

The scheme *SubFamEx* deals with a non empty set I_1 and a group family $\mathcal{F}$ of I_1 and a binary predicate $\mathcal{P}$ and states that

(Sch. 6) There exists a subgroup family S of $\mathcal{F}$ such that for every element i of I_1, $\mathcal{P}[i, S(i)]$

provided

- for every element i of I_1, there exists a subgroup j of $\mathcal{F}(i)$ such that $\mathcal{P}[i, j]$.

Now we state the propositions:

(60) Let us consider a many sorted set A indexed by I. Suppose for every element i of I, $A(i)$ is a subset of $F(i)$. Then $\prod A$ is a subset of $\prod F$.
Proof: For every object x such that $x \in \prod A$ holds $x \in$ the carrier of $\prod F$. □

(61) Let us consider a normal subgroup family S of F. Then $\prod S$ is a normal subgroup of $\prod F$.
Proof: For every element g of $\prod F$, $(\prod S)^g$ is a subgroup of $\prod S$. □

Let I be a non empty set, F be a group family of I, and S be a normal subgroup family of F. Note that $\prod S$ is normal as a subgroup of $\prod F$.

9. Commutator Subgroup and Center of Product Groups

Now we state the proposition:

(62) Let us consider a group family Z of I. If for every element i of I, $Z(i) = \mathrm{Z}(F(i))$, then $\mathrm{Z}(\prod F) = \prod Z$.
Proof: For every element a of $\prod F$, $a \in \prod Z$ iff for every element b of $\prod F$, $a \cdot b = b \cdot a$. For every element a of $\prod F$, $a \in \prod Z$ iff $a \in \mathrm{Z}(\prod F)$. For every element i of I, $Z(i)$ is a subgroup of $F(i)$. □

Let us consider a subgroup family D of F. Now we state the propositions:

(63) If for every element i of I, $D(i) = F(i)^{\mathrm{c}}$, then $(\prod F)^{\mathrm{c}}$ is a strict subgroup of $\prod D$.
Proof: For every elements a, b of $\prod F$, $[a, b] \in \prod D$. □

(64) If for every element i of I, $D(i) = F(i)^{\mathrm{c}}$, then $\mathrm{sum}\, D$ is a strict subgroup of $(\prod F)^{\mathrm{c}}$.
Proof: For every element g of $\prod F$ such that $g \in \mathrm{sum}\, D$ holds $g \in (\prod F)^{\mathrm{c}}$. □

(65) Let us consider a finite, non empty set I, a group family F of I, and a subgroup family D of F. Suppose for every element i of I, $D(i) = F(i)^{\mathrm{c}}$. Then $(\prod F)^{\mathrm{c}} = \prod D$. The theorem is a consequence of (64) and (63).

10. Quotients of Product Groups

Let I be a non empty set, F_1, F_2 be group families of I, and f be a homomorphism family of F_1 and F_2. The functor $\prod f$ yielding a homomorphism from $\prod F_1$ to $\prod F_2$ is defined by

(Def. 16) for every element i of I, $(\mathrm{proj}(F_2, i)) \cdot it = f(i) \cdot (\mathrm{proj}(F_1, i))$.

The functor $\mathrm{Ker}\, f$ yielding a componentwise strict, normal subgroup family of F_1 is defined by

(Def. 17) for every element i of I, $it(i) = \mathrm{Ker}(f(i)$ **qua** homomorphism from $F_1(i)$ to $F_2(i))$.

The functor $\mathrm{Im}\, f$ yielding a componentwise strict subgroup family of F_2 is defined by

(Def. 18) for every element i of I, $it(i) = \mathrm{Im}(f(i)$ **qua** homomorphism from $F_1(i)$ to $F_2(i))$.

Let us consider group families F_1, F_2 of I and a homomorphism family f of F_1 and F_2. Now we state the propositions:

(66) $\mathrm{Ker} \prod f = \prod \mathrm{Ker}\, f$.

Proof: For every element g of $\prod F_1$, $g \in \mathrm{Ker} \prod f$ iff $g \in \prod \mathrm{Ker}\, f$. $\square$

(67) $\mathrm{Im} \prod f = \prod \mathrm{Im}\, f$.

Proof: For every element g of $\prod F_2$, $g \in \mathrm{Im} \prod f$ iff $g \in \prod \mathrm{Im}\, f$. $\square$

(68) Let us consider a componentwise strict, normal subgroup family S of F. Then ${}^{\prod F}/_{\prod S}$ and $\prod({}^{F}/_{S})$ are isomorphic.

Proof: Define $\mathcal{A}$(element of I) $=$ the canonical homomorphism onto cosets of $S(\$_1)$. For every element i of I, $\mathcal{A}(i)$ is a homomorphism from $F(i)$ to $({}^{F}/_{S})(i)$. Consider f being a homomorphism family of F and ${}^{F}/_{S}$ such that for every element i of I, $f(i) = \mathcal{A}(i)$. $\mathrm{Ker}\, f = S$. $\mathrm{Ker} \prod f = \prod S$. $\mathrm{Im}\, f = {}^{F}/_{S}$. $\mathrm{Im} \prod f = \prod \mathrm{Im}\, f$. $\square$

11. Internal Direct Products

Let I be a set, G be a group, and I_2 be a homomorphism family of I and G. We say that I_2 is normal if and only if

(Def. 19) for every object i such that $i \in I$ holds $I_2(i)$ is a normal subgroup of G.

We say that I_2 is componentwise strict if and only if

(Def. 20) for every object i such that $i \in I$ holds $I_2(i)$ is a strict subgroup of G.

Let us consider a non empty set I, a group G, and a homomorphism family F of I and G. Now we state the propositions:

(69) F is normal if and only if for every element i of I, $F(i)$ is a normal subgroup of G.

(70) F is componentwise strict if and only if for every element i of I, $F(i)$ is a strict subgroup of G.

Let I be a set and G be a group. Note that there exists a homomorphism family of I and G which is componentwise strict and normal.

Let I be a non empty set, F be a homomorphism family of I and G, and i be an element of I. Note that the functor $F(i)$ yields a subgroup of G. Let F be a normal homomorphism family of I and G. One can check that $F(i)$ is normal as a subgroup of G. Now we state the propositions:

(71) Let us consider subgroups H_1, H_2 of G. Suppose $[H_1, H_2] = \{\mathbf{1}\}_G$. Let us consider elements a, b of G. If $a \in H_1$ and $b \in H_2$, then $a \cdot b = b \cdot a$.

(72) Let us consider a normal subgroup N of G, and elements a, b of G. If $a \in N$, then $a^b \in N$.

(73) Let us consider normal subgroups H, K of G. Suppose $H \cap K = \{\mathbf{1}\}_G$. Let us consider elements h, k of G. If $h \in H$ and $k \in K$, then $h \cdot k = k \cdot h$.
PROOF: $[h, k] \in H \cap K$. □

(74) Let us consider a normal homomorphism family F of I and G, and a subset A of G. Suppose $A = \bigcup\{$the carrier of $F(i)$, where i is an element of $I\}$. Then there exists a strict, normal subgroup N of G such that $N = \mathrm{gr}(A)$.
PROOF: Reconsider $N = \mathrm{gr}(A)$ as a strict subgroup of G. For every element i of I, the carrier of $F(i) \subseteq$ the carrier of N. For every element a of G, N^a is a subgroup of N. □

Let I be a set, J be a subset of I, and F be a group yielding many sorted set indexed by I. One can verify that $F{\upharpoonright}J$ is group yielding, J-defined, and total.

Now we state the proposition:

(75) Let us consider a set I, a homomorphism family F of I and G, and a set J. If $J \subseteq I$, then $F{\upharpoonright}J$ is a homomorphism family of J and G.
PROOF: For every object j such that $j \in J$ holds $(F{\upharpoonright}J)(j)$ is a subgroup of G. □

Let I be a set, G be a group, F be a homomorphism family of I and G, and J be a subset of I. Note that the functor $F{\upharpoonright}J$ yields a homomorphism family of J and G. One can check that $F{\upharpoonright}J$ is group yielding. Now we state the propositions:

(76) Let us consider a normal homomorphism family F of I and G, a subset A of G, and an element i of I. Suppose $A = \bigcup\{$the carrier of $F(j)$, where j is an element of $I : i \neq j\}$. Then there exists a strict, normal subgroup

N of G such that $N = \mathrm{gr}(A)$. The theorem is a consequence of (75), (69), and (74).

(77) Let us consider a non empty subset J of I, and a normal homomorphism family F of I and G. Then $F \restriction J$ is a normal homomorphism family of J and G.

PROOF: For every element j of J, $(F \restriction J)(j)$ is a normal subgroup of G. □

(78) Let us consider a set I, a subset J of I, and a normal homomorphism family F of I and G. Then $F \restriction J$ is a normal homomorphism family of J and G.

PROOF: For every object i such that $i \in J$ holds $(F \restriction J)(i)$ is a normal subgroup of G. □

Let I be a set, J be a subset of I, G be a group, and F be a normal homomorphism family of I and G. Let us note that $F \restriction J$ is normal as a homomorphism family of J and G. Now we state the proposition:

(79) Let us consider a set I, a subset J of I, and a componentwise strict homomorphism family F of I and G. Then $F \restriction J$ is a componentwise strict homomorphism family of J and G.

PROOF: For every object i such that $i \in J$ holds $(F \restriction J)(i)$ is a strict subgroup of G. □

Let I be a set, J be a subset of I, G be a group, and F be a componentwise strict homomorphism family of I and G. Let us note that $F \restriction J$ is componentwise strict as a homomorphism family of J and G. Now we state the propositions:

(80) Let us consider a set I, and a subset J of I. Suppose J is empty. Let us consider a normal homomorphism family F of I and G. Then the support of $F \restriction J = \emptyset \longmapsto 2^{\alpha}$, where α is the carrier of G.

(81) Let us consider a set I, a subset J of I, a normal homomorphism family F of I and G, and a subset A of G. Suppose $A = \bigcup(\text{the support of } F \restriction J)$. Then there exists a strict, normal subgroup N of G such that $N = \mathrm{gr}(A)$.

(82) Let us consider a set I, a normal homomorphism family F of I and G, and a subset A of G. Suppose $A = \bigcup(\text{the support of } F)$. Then there exists a strict, normal subgroup N of G such that $N = \mathrm{gr}(A)$. The theorem is a consequence of (81).

(83) Every componentwise strict homomorphism family of I and G is $(\mathrm{SubGr}\, G)$-valued. The theorem is a consequence of (5) and (70).

Let I be a non empty set and G be a group. Let us observe that every componentwise strict homomorphism family of I and G is $(\mathrm{SubGr}\, G)$-valued. Let I be a set and F be a 1-sorted yielding many sorted set indexed by I. An element of F is an element of the support of F. Now we state the proposition:

(84) Let us consider a group family F of I, an element g of F, and an element i of I. Then $g(i)$ is an element of $F(i)$. The theorem is a consequence of (14).

Let I be a non empty set, G be a group, and F be a homomorphism family of I and G. Observe that every element of F is (the carrier of G)-valued and every element of $\prod F$ is I-defined, relation-like, and function-like and every element of $\prod F$ is I-defined, (the carrier of G)-valued, and total. Now we state the proposition:

(85) Let us consider a set I, a group G, and a homomorphism family F of I and G. Then the support of F is (2^{α})-valued, where α is the carrier of G. The theorem is a consequence of (14).

Let I be a set, G be a group, and F be a homomorphism family of I and G. Observe that the support of F is $(2^{(\text{the carrier of } G)})$-valued. Now we state the propositions:

(86) Let us consider a group G, a finite subset S of SubGr G, and a natural number n. Suppose $n = \overline{\overline{S}}$. Then $\mathrm{CFS}(S)$ is a homomorphism family of Seg n and G.

Proof: For every object y such that $y \in \operatorname{rng} \mathrm{CFS}(S)$ holds y is a subgroup of G. $\mathrm{CFS}(S)$ is a group family of Seg n. For every object i such that $i \in \mathrm{Seg}\, n$ holds $(\mathrm{CFS}(S))(i)$ is a subgroup of G. □

(87) Let us consider a group G, a finite subset N of the normal subgroups of G, and a natural number n. Suppose $n = \overline{\overline{N}}$. Then $\mathrm{CFS}(N)$ is a normal homomorphism family of Seg n and G.

Proof: For every object i such that $i \in \mathrm{Seg}\, n$ holds $(\mathrm{CFS}(N))(i)$ is a normal subgroup of G. □

(88) Let us consider a group G, an empty set I, and a homomorphism family F of I and G. Then $\mathrm{gr}(\bigcup(\text{the support of } F)) = \{\mathbf{1}\}_G$.

Let G be a group, I be a set, F be a homomorphism family of I and G, and i be an object. Assume $i \in I$. The functor $F_{/i}$ yielding a subgroup of G is defined by the term

(Def. 21) $F(i)$.

We say that G is an internal product of F if and only if

(Def. 22) for every object i such that $i \in I$ holds $F(i)$ is a normal subgroup of G and the multiplicative magma of $G = \mathrm{gr}(\bigcup(\text{the support of } F))$ and for every object i such that $i \in I$ for every strict, normal subgroup N of G such that $N = \mathrm{gr}(\bigcup(\text{the support of } F{\restriction}I \setminus \{j$, where j is an element of $I : F(i) = F(j)\})))$ holds $F_{/i} \cap N = \{\mathbf{1}\}_G$.

Now we state the propositions:

(89) Let us consider a group G, an empty set I, and a homomorphism family F of I and G. Then G is an internal product of F if and only if G is trivial. The theorem is a consequence of (88).

(90) Let us consider a group G, a non empty set I, and a homomorphism family F of I and G. Then G is an internal product of F if and only if for every element i of I, $F(i)$ is a normal subgroup of G and the multiplicative magma of $G = \text{gr}(\bigcup(\text{the support of } F))$ and for every element i of I and for every subset J of I such that $J = I \setminus \{j$, where j is an element of $I : F(i) = F(j)\}$ for every strict, normal subgroup N of G such that $N = \text{gr}(\bigcup(\text{the support of } F \restriction J))$ holds $F(i) \cap N = \{\mathbf{1}\}_G$.

Let G be a group, I be a set, and F be a normal homomorphism family of I and G. One can check that G is an internal product of F if and only if the condition (Def. 23) is satisfied.

(Def. 23) the multiplicative magma of $G = \text{gr}(\bigcup(\text{the support of } F))$ and for every object i such that $i \in I$ for every strict, normal subgroup N of G such that $N = \text{gr}(\bigcup(\text{the support of } F \restriction I \setminus \{j$, where j is an element of $I : F(i) = F(j)\}))$ holds $F_{/i} \cap N = \{\mathbf{1}\}_G$.

Let us consider a group G, a non empty set I, and a normal homomorphism family F of I and G. Now we state the propositions:

(91) G is an internal product of F if and only if the multiplicative magma of $G = \text{gr}(\bigcup(\text{the support of } F))$ and for every element i of I and for every subset J of I such that $J = I \setminus \{j$, where j is an element of $I : F(i) = F(j)\}$ for every strict, normal subgroup N of G such that $N = \text{gr}(\bigcup(\text{the support of } F \restriction J))$ holds $F(i) \cap N = \{\mathbf{1}\}_G$. The theorem is a consequence of (90).

(92) Suppose F is one-to-one. Then G is an internal product of F if and only if the multiplicative magma of $G = \text{gr}(\bigcup(\text{the support of } F))$ and for every element i of I and for every subset J of I such that $J = I \setminus \{i\}$ for every strict, normal subgroup N of G such that $N = \text{gr}(\bigcup(\text{the support of } F \restriction J))$ holds $F(i) \cap N = \{\mathbf{1}\}_G$. The theorem is a consequence of (91).

(93) The celebrated "Recognition Theorem", see Aschbacher [1, (1.9)], Hungerford [5, (1.8.6)], Robinson [11, (1.4.7.ii)]:
Let us consider a strict group G, a non empty set I, and a normal homomorphism family F of I and G. Suppose F is one-to-one. Then G is an internal product of F if and only if F is an internal direct sum components of G and I.

Proof: For every element i of I and for every subset J of I, the support of $F \restriction J = (\text{the support of } F) \restriction J$. If G is an internal product of F, then F is an internal direct sum components of G and I. If F is an internal direct sum components of G and I, then G is an internal product of F. □

Let G be a group and $\mathcal{F}$ be a subset of SubGr G. We say that G is an internal product of $\mathcal{F}$ if and only if

(Def. 24) for every strict subgroup H of G such that $H \in \mathcal{F}$ holds H is a normal subgroup of G and there exists a subset A of G such that $A = \bigcup\{U_3$, where U_3 is a subset of G : there exists a strict subgroup H of G such that $H \in \mathcal{F}$ and $U_3 =$ the carrier of $H\}$ and the multiplicative magma of $G = \mathrm{gr}(A)$ and for every strict subgroup H of G such that $H \in \mathcal{F}$ for every subset A of G such that $A = \bigcup\{U_4$, where U_4 is a subset of G : there exists a strict subgroup K of G such that $K \in \mathcal{F}$ and $U_4 =$ the carrier of K and $K \neq H\}$ holds $H \cap \mathrm{gr}(A) = \{\mathbf{1}\}_G$.

Let H be a strict subgroup of G. We say that H is an internal product of $\mathcal{F}$ if and only if

(Def. 25) for every strict subgroup H_1 of G such that $H_1 \in \mathcal{F}$ holds H_1 is a normal subgroup of H and there exists a subset A of G such that $A = \bigcup\{U_3$, where U_3 is a subset of G : there exists a strict subgroup H of G such that $H \in \mathcal{F}$ and $U_3 =$ the carrier of $H\}$ and $H = \mathrm{gr}(A)$ and for every strict subgroup H_1 of G such that $H_1 \in \mathcal{F}$ for every subset A of G such that $A = \bigcup\{U_4$, where U_4 is a subset of G : there exists a strict subgroup K of G such that $K \in \mathcal{F}$ and $U_4 =$ the carrier of K and $K \neq H_1\}$ holds $H_1 \cap \mathrm{gr}(A) = \{\mathbf{1}\}_G$.

Now we state the propositions:

(94) G is a subgroup of Ω_G.

(95) Let us consider a group G, and a subgroup H of G. Suppose H is a normal subgroup of Ω_G. Then H is a normal subgroup of G. The theorem is a consequence of (94).

(96) Let us consider a group G, and a subset $\mathcal{F}$ of SubGr G. Then G is an internal product of $\mathcal{F}$ if and only if Ω_G is an internal product of $\mathcal{F}$. The theorem is a consequence of (95).

(97) Let us consider a group G, a non empty set I, a componentwise strict homomorphism family F of I and G, and a subset $\mathcal{F}$ of SubGr G. Suppose $\mathcal{F} = \mathrm{rng}\, F$. Then $\bigcup\{A$, where A is a subset of G : there exists a strict subgroup H of G such that $H \in \mathcal{F}$ and $A =$ the carrier of $H\} = \bigcup(\text{the support of } F)$. The theorem is a consequence of (5) and (14).

(98) Let us consider a group G, a non empty set I, a componentwise strict homomorphism family F of I and G, and a subset $\mathcal{F}$ of SubGr G. Suppose $\mathcal{F} = \mathrm{rng}\, F$. Let us consider a strict subgroup H of G, and an element i of I. Suppose $H = F(i)$. Let us consider a subset J of I. Suppose $J = I \setminus \{j$, where j is an element of $I : F(i) = F(j)\}$. Then $\bigcup\{A$, where A is a subset of G : there exists a strict subgroup K of G such that $K \in$

$\mathcal{F}$ and $A =$ the carrier of K and $K \neq H\} = \bigcup$(the support of $F \restriction J$).

PROOF: Set $X = \{A$, where A is a subset of G : there exists a strict subgroup K of G such that $K \in \mathcal{F}$ and $A =$ the carrier of K and $K \neq H\}$. For every object x, $x \in X$ iff $x \in$ rng(the support of $F \restriction J$). □

(99) Let us consider a group G, a non empty set I, a componentwise strict homomorphism family F of I and G, and a subset $\mathcal{F}$ of $\operatorname{SubGr} G$. Suppose $\mathcal{F} = \operatorname{rng} F$. Then G is an internal product of F if and only if G is an internal product of $\mathcal{F}$. The theorem is a consequence of (5), (97), (69), (81), (98), and (70).

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