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About Regular Graphs

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Summary. In this article regular graphs, both directed and undirected, are formalized in the Mizar system [7], [2], based on the formalization of graphs as described in [10]. The handshaking lemma is also proven.

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Introduction

Regular graphs, especially cubic graphs, are a cornerstone of graph theory (cf. [3], [12], [6]). In this article the concept of regular graphs is formalized (compare similar efforts using various proof-assistants [11], [5], [4]), along with some adjacent concepts, developing further some of the previous Mizar articles [8], [9]. In the first section, the directed analogue of `complete` from [1] is introduced. Sections 2 and 3 deal with the undirected and directed-regular graphs respectively. Section 4 is rather technical in nature; at its end $2m = \mathfrak{a}n$ is proven, where m and n denote the size and order of an $\mathfrak{a}$-regular graph, where $\mathfrak{a}$ can be any cardinal. Finally Section 5 introduces tools needed to formalize the combinatorial proof of the rather simple Handshaking Lemma.

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1. Directed-complete Graphs

Let G be a graph. We say that G is directed-complete if and only if

(Def. 1) for every vertices v, w of G such that $v \neq w$ there exists an object e such that e joins v to w in G.

Let c be a non empty cardinal number. The functors: canCompleteGraph(c) and canDCompleteGraph(c) yielding graphs are defined by terms

(Def. 2) createGraph(c, $\subseteq_c \setminus (\mathrm{id}_c)$),

(Def. 3) createGraph(c, $(c \times c) \setminus (\mathrm{id}_c)$),

respectively. Observe that the vertices of canCompleteGraph(c) reduces to c and the vertices of canDCompleteGraph(c) reduces to c.

Observe that every vertex of canCompleteGraph(c) is ordinal and every vertex of canDCompleteGraph(c) is ordinal and every vertex of canCompleteGraph (ω) is natural and every vertex of canDCompleteGraph(ω) is natural.

Let n be a non zero natural number. Observe that canCompleteGraph(n) is finite and canDCompleteGraph(n) is finite and every vertex of canCompleteGraph(n) is natural and every vertex of canDCompleteGraph(n) is natural.

Let c be a non empty cardinal number. One can verify that canCompleteGraph(c) is plain, c-vertex, simple, and complete and canDCompleteGraph(c) is plain, c-vertex, directed-simple, and directed-complete. Now we state the propositions:

(1) Let us consider a non empty cardinal number c, and a vertex v of canCompleteGraph(c). Then

(i) v.inNeighbors() $= v$, and

(ii) v.outNeighbors() $= c \setminus (\operatorname{succ} v)$.

(2) Let us consider a vertex v of canCompleteGraph(ω). Then

(i) v.inDegree() $= v$, and

(ii) v.outDegree() $= \omega$.

The theorem is a consequence of (1).

(3) Let us consider a non zero natural number n, and a vertex v of canCompleteGraph(n). Then

(i) v.inDegree() $= v$, and

(ii) v.outDegree() $= n - v - 1$.

The theorem is a consequence of (1).

Let c be a non empty cardinal number. Let us observe that there exists a graph which is simple, c-vertex, and complete and there exists a graph which is directed-simple, c-vertex, and directed-complete and every graph which is directed-complete is also complete and every graph which is trivial is also directed-complete and every graph which is non trivial and directed-complete is also non non-multi and non edgeless and there exists a graph which is non directed-complete. Now we state the propositions:

(4) Let us consider graphs G_1, G_2. Suppose $G_1 \approx G_2$ and G_1 is directed-complete. Then G_2 is directed-complete.

(5) Let us consider a graph G_1, and a subgraph G_2 of G_1 with loops removed. Then G_1 is directed-complete if and only if G_2 is directed-complete.

(6) Let us consider a graph G_1, and a subgraph G_2 of G_1 with directed-parallel edges removed. Then G_1 is directed-complete if and only if G_2 is directed-complete.

(7) Let us consider a graph G_1, and a directed-simple graph G_2 of G_1. Then G_1 is directed-complete if and only if G_2 is directed-complete. The theorem is a consequence of (6) and (5).

(8) Let us consider a graph G_1, and a graph G_2 given by reversing directions of the edges of G_1. Then G_1 is directed-complete if and only if G_2 is directed-complete.

Let G be a directed-complete graph. Let us note that every subgraph of G with loops removed is directed-complete and every subgraph of G with directed-parallel edges removed is directed-complete and every directed-simple graph of G is directed-complete and every graph given by reversing directions of the edges of G is directed-complete.

Let V be a set. Observe that every subgraph of G induced by V is directed-complete and every graph by adding a loop to each vertex of G in V is directed-complete. Let v, e, w be objects. Note that every supergraph of G extended by e between vertices v and w is directed-complete.

Let G be a non directed-complete graph. One can verify that every subgraph of G with loops removed is non directed-complete and every subgraph of G with directed-parallel edges removed is non directed-complete and every directed-simple graph of G is non directed-complete and every graph given by reversing directions of the edges of G is non directed-complete and every subgraph of G which is spanning is also non directed-complete.

Let us consider graphs G_1, G_2 and a partial graph mapping F from G_1 to G_2. Now we state the propositions:

(9) If F is directed-continuous and strong subgraph embedding, then if G_2 is directed-complete, then G_1 is directed-complete.

(10) If F is directed-isomorphism, then G_1 is directed-complete iff G_2 is directed-complete. The theorem is a consequence of (9).

Let G be a directed-complete graph. Observe that every graph which is G-directed-isomorphic is also directed-complete. Now we state the propositions:

(11) Let us consider a directed-complete graph G, and a vertex v of G. Then

(i) (the vertices of G) $\setminus \{v\} \subseteq v$.inNeighbors(), and

(ii) (the vertices of G) $\setminus \{v\} \subseteq v$.outNeighbors(), and

(iii) (the vertices of G) $\setminus \{v\} \subseteq v$.allNeighbors().

(12) Let us consider a loopless, directed-complete graph G, and a vertex v of G. Then

(i) v.inNeighbors() = (the vertices of G) $\setminus \{v\}$, and

(ii) v.outNeighbors() = (the vertices of G) $\setminus \{v\}$, and

(iii) v.allNeighbors() = (the vertices of G) $\setminus \{v\}$.

The theorem is a consequence of (11).

(13) Let us consider a directed-simple, directed-complete graph G, and a vertex v of G. Then

(i) v.inDegree() $+ 1 = G$.order(), and

(ii) v.outDegree() $+ 1 = G$.order().

The theorem is a consequence of (12).

(14) Let us consider a graph G_1, and a directed graph complement G_2 of G_1 with loops. Then the edges of $G_1 = G_1$.loops() if and only if G_2 is directed-complete.

Let G be an edgeless graph. Let us observe that every directed graph complement of G with loops is directed-complete. Now we state the proposition:

(15) Let us consider a graph G_1, and a directed graph complement G_2 of G_1 with loops. Then G_1 is directed-complete if and only if the edges of $G_2 = G_2$.loops().

One can verify that there exists a graph which is loopfull and directed-complete.

Let G be a loopfull, directed-complete graph. Let us observe that every directed graph complement of G with loops is edgeless. Now we state the proposition:

(16) Let us consider a graph G_1, and a directed graph complement G_2 of G_1. Then the edges of $G_1 = G_1$.loops() if and only if G_2 is directed-complete. The theorem is a consequence of (14).

Let G be an edgeless graph. Note that every directed graph complement of G is directed-complete. Now we state the proposition:

(17) Let us consider a graph G_1, and a directed graph complement G_2 of G_1. Then G_1 is directed-complete if and only if G_2 is edgeless. The theorem is a consequence of (15).

Let G be a directed-complete graph. One can verify that every directed graph complement of G is edgeless. Let G be a non directed-complete graph. One can check that every directed graph complement of G is non edgeless.

Let G_1 be a graph and G_2 be a directed graph complement of G_1 with loops. One can verify that every graph union of G_1 and G_2 is directed-complete. Let G_2 be a directed graph complement of G_1. Note that every graph union of G_1 and G_2 is directed-complete. Now we state the propositions:

(18) Let us consider a graph G. Then G is directed-complete if and only if $((\text{the vertices of } G) \times (\text{the vertices of } G)) \setminus (\mathrm{id}_\alpha) \subseteq \mathrm{VertDomRel}(G)$, where α is the vertices of G.

(19) Let us consider a non empty set V, and a binary relation E on V. Then $\mathrm{createGraph}(V, E)$ is directed-complete if and only if $(V \times V) \setminus (\mathrm{id}_V) \subseteq E$.

2. Regular Graphs

From now on c, c_1, c_2 denote cardinal numbers, G, G_1, G_2 denote graphs, and v denotes a vertex of G.

Let us consider c and G. We say that G is c-regular if and only if

(Def. 4) for every v, $v.\mathrm{degree}() = c$.

One can check that every graph which is c-regular is also with max degree and every graph which is $(c+1)$-vertex, simple, and complete is also c-regular and there exists a graph which is simple and c-regular. Now we state the propositions:

(20) Degree of regularity is unique:
If G is c_1-regular and c_2-regular, then $c_1 = c_2$.

(21) G is c-regular if and only if every component of G is c-regular.

Let us consider c. Let us observe that there exists a graph which is non c-regular. Let G be a c-regular graph. Note that every component of G is c-regular. Now we state the propositions:

(22) Let us consider a c-regular graph G. Then

(i) $\delta(G) = c$, and

(ii) $\Delta(G) = c$.

(23) If $\delta(G) = c$ and $\bar{\Delta}(G) = c$, then G is c-regular.

Let n be a natural number. Observe that every graph which is n-regular is also locally-finite and there exists a graph which is simple, vertex-finite, and n-regular. Now we state the proposition:

(24) G is edgeless if and only if G is 0-regular.

One can verify that every graph which is edgeless is also 0-regular and every graph which is 0-regular is also edgeless. Let c be a non empty cardinal number. Let us observe that every graph which is c-regular is also non edgeless. Now we state the propositions:

(25) Let us consider a simple, c-regular graph G. Then $c \subseteq G$.order().

(26) Let us consider a natural number n, and a simple, vertex-finite, n-regular graph G_1. Then every graph complement of G_1 is (G_1.order() $-'$ $(n+1)$)-regular.

(27) If there exists v such that v is isolated and G is c-regular, then $c = 0$.

(28) If there exists v such that v is endvertex and G is c-regular, then $c = 1$.

Let G be a 1-regular graph. Observe that every vertex of G is endvertex. Now we state the proposition:

(29) Let us consider a 1-regular graph G, and a trail T of G. Suppose T is not trivial. Then there exists an object e such that

(i) e joins T.first() and T.last() in G, and

(ii) $T = G$.walkOf(T.first(), e, T.last()).

One can verify that every graph which is 1-regular and connected is also 2-vertex, 1-edge, and complete and every graph which is simple, 2-vertex, and connected is also 1-regular. Now we state the propositions:

(30) Let us consider a partial graph mapping F from G_1 to G_2. Suppose F is isomorphism. Then G_1 is c-regular if and only if G_2 is c-regular.

(31) If $G_1 \approx G_2$ and G_1 is c-regular, then G_2 is c-regular.

(32) Let us consider a set E, and a graph G_2 given by reversing directions of the edges E of G_1. Then G_1 is c-regular if and only if G_2 is c-regular. The theorem is a consequence of (30).

Let G be a graph. We say that G is cubic if and only if

(Def. 5) G is 3-regular.

One can verify that every graph which is cubic is also 3-regular and every graph which is 3-regular is also cubic. Now we state the propositions:

(33) G is cubic if and only if for every v, v.degree() $= 3$.

(34) Let us consider a partial graph mapping F from G_1 to G_2. If F is isomorphism, then G_1 is cubic iff G_2 is cubic.

(35) If $G_1 \approx G_2$ and G_1 is cubic, then G_2 is cubic.

(36) Let us consider a set E, and a graph G_2 given by reversing directions of the edges E of G_1. Then G_1 is cubic if and only if G_2 is cubic.

Let G be a graph. We say that G is regular if and only if

(Def. 6) there exists a cardinal number c such that G is c-regular.

Now we state the proposition:

(37) G is regular if and only if $\delta(G) = \bar{\Delta}(G)$. The theorem is a consequence of (22) and (23).

Let G be a locally-finite graph. One can check that G is regular if and only if the condition (Def. 7) is satisfied.

(Def. 7) there exists a natural number n such that G is n-regular.

Let c be a cardinal number. Let us note that every graph which is c-regular is also regular and every graph which is cubic is also regular and every graph which is regular is also with max degree and there exists a graph which is simple, non edgeless, finite, and regular.

Let G be a regular graph. Note that every component of G is regular. Let G be a simple, finite, regular graph. One can verify that every graph complement of G is regular. Now we state the propositions:

(38) If there exists v such that v is isolated and G is regular, then G is edgeless. The theorem is a consequence of (27).

(39) If there exists v such that v is endvertex and G is regular, then G is 1-regular. The theorem is a consequence of (28).

(40) Let us consider a partial graph mapping F from G_1 to G_2. If F is isomorphism, then G_1 is regular iff G_2 is regular. The theorem is a consequence of (30).

(41) If $G_1 \approx G_2$ and G_1 is regular, then G_2 is regular. The theorem is a consequence of (40).

(42) Let us consider a set E, and a graph G_2 given by reversing directions of the edges E of G_1. Then G_1 is regular if and only if G_2 is regular. The theorem is a consequence of (40).

3. Directed-regular Graphs

Let us consider c and G. We say that G is c-directed-regular if and only if

(Def. 8) for every v, v.inDegree() $= c$ and v.outDegree() $= c$.

Let us note that every graph which is c-directed-regular is also with max indegree and with max outdegree and every graph which is $(c+1)$-vertex, directed-simple, and directed-complete is also c-directed-regular and there exists a graph which is directed-simple and c-directed-regular. Now we state the proposition:

(43) Degree of directed regularity is unique:
If G is c_1-directed-regular and c_2-directed-regular, then $c_1 = c_2$.

Let us consider c. One can check that there exists a graph which is non c-directed-regular. Let G be a c-directed-regular graph. Observe that every component of G is c-directed-regular. Now we state the propositions:

(44) Let us consider a c-directed-regular graph G. Then

(i) $\delta^-(G) = c$, and

(ii) $\delta^+(G) = c$, and

(iii) $\Delta^-(G) = c$, and

(iv) $\Delta^+(G) = c$.

(45) If $\delta^-(G) = c$ and $\delta^+(G) = c$ and $\bar{\Delta}^-(G) = c$ and $\bar{\Delta}^+(G) = c$, then G is c-directed-regular.

(46) Let us consider a natural number n. If G is n-directed-regular, then G is $(2 \cdot n)$-regular.

Let n be a natural number. One can check that every graph which is n-directed-regular is also $(2 \cdot n)$-regular and locally-finite and there exists a graph which is directed-simple, finite, and n-directed-regular.

Let c be an infinite cardinal number. Let us note that every graph which is c-directed-regular is also c-regular. Now we state the proposition:

(47) G is edgeless if and only if G is 0-directed-regular. The theorem is a consequence of (46).

One can verify that every graph which is edgeless is also 0-directed-regular and every graph which is 0-directed-regular is also edgeless.

Let c be a non empty cardinal number. Let us observe that every graph which is c-directed-regular is also non edgeless. Now we state the propositions:

(48) Let us consider a directed-simple, c-directed-regular graph G. Then $c \subseteq G.\text{order}()$.

(49) Let us consider a natural number n, and a directed-simple, vertex-finite, n-directed-regular graph G_1. Then every directed graph complement of G_1 is $(G_1.\text{order}() -' (n+1))$-directed-regular.

(50) If there exists v such that v is isolated and G is c-directed-regular, then $c = 0$.

Let us consider c. Let G be a c-directed-regular graph. Let us note that every vertex of G is non endvertex and every graph which is 2-edge, 2-vertex, and directed-simple is also 1-directed-regular and complete and every graph which is trivial and 1-edge is also 1-directed-regular. Now we state the propositions:

(51) Let us consider a partial graph mapping F from G_1 to G_2. Suppose F is directed-isomorphism. Then G_1 is c-directed-regular if and only if G_2 is c-directed-regular.

(52) If $G_1 \approx G_2$ and G_1 is c-directed-regular, then G_2 is c-directed-regular.

Let G be a graph. We say that G is directed-regular if and only if

(Def. 9) there exists a cardinal number c such that G is c-directed-regular.

Now we state the proposition:

(53) G is directed-regular if and only if $\delta^-(G) = \bar{\Delta}^-(G)$ and $\delta^+(G) = \bar{\Delta}^+(G)$ and $\delta^-(G) = \delta^+(G)$. The theorem is a consequence of (44) and (45).

Let G be a locally-finite graph. One can verify that G is directed-regular if and only if the condition (Def. 10) is satisfied.

(Def. 10) there exists a natural number n such that G is n-directed-regular.

Let c be a cardinal number. Note that every graph which is c-directed-regular is also directed-regular and every graph which is directed-regular is also with max degree and there exists a graph which is directed-simple, non edgeless, finite, and directed-regular.

Let G be a directed-regular graph. Observe that every component of G is directed-regular. Let G be a directed-simple, finite, directed-regular graph. Note that every directed graph complement of G is directed-regular. Let G be a directed-regular graph. Note that every vertex of G is non endvertex. Now we state the propositions:

(54) Let us consider a partial graph mapping F from G_1 to G_2. Suppose F is directed-isomorphism. Then G_1 is directed-regular if and only if G_2 is directed-regular. The theorem is a consequence of (51).

(55) If $G_1 \approx G_2$ and G_1 is directed-regular, then G_2 is directed-regular. The theorem is a consequence of (54).

4. Counting the Edges

Now we state the propositions:

(56) Let us consider a set P, and a cardinal number c. Suppose P is mutually-disjoint and for every set A such that $A \in P$ holds $\overline{\overline{A}} = c$. Then $\overline{\overline{\bigcup P}} = c \cdot \overline{\overline{P}}$.

(57) Let us consider a non empty set X, a partition P of X, and a cardinal number c. Suppose for every element x of X, $\overline{\overline{\mathrm{EqClass}(x, P)}} = c$. Then $\overline{\overline{X}} = c \cdot \overline{\overline{P}}$. The theorem is a consequence of (56).

Let f be a function and X be a set. One can verify that $\langle f, \mathrm{id}_X \rangle$ is one-to-one.

Let f be a one-to-one function. One can verify that $f^{\smile}$ is one-to-one and $\curvearrowleft f$ is one-to-one.

Let X be a set and f be a function. Let us observe that $\langle \mathrm{id}_X, f \rangle$ is one-to-one.

Now we state the proposition:

(58) Let us consider a c-regular graph G. Then $2 \cdot G$.size() $= c \cdot G$.order(). The theorem is a consequence of (56).

5. The Degree Map and Degree Sequence

Let G be a graph. The functors: G.degreeMap(), G.inDegreeMap(), and G.outDegreeMap() yielding many sorted sets indexed by the vertices of G are defined by conditions

(Def. 11) for every vertex v of G, G.degreeMap()$(v) = v$.degree(),

(Def. 12) for every vertex v of G, G.inDegreeMap()$(v) = v$.inDegree(),

(Def. 13) for every vertex v of G, G.outDegreeMap()$(v) = v$.outDegree(),

respectively. Let us observe that G.degreeMap() is cardinal yielding and G.inDegreeMap() is cardinal yielding and G.outDegreeMap() is cardinal yielding. Now we state the propositions:

(59) Let us consider a graph G. Then

(i) $\overline{\overline{G.\mathrm{degreeMap}()}} = G$.order(), and

(ii) $\overline{\overline{G.\mathrm{inDegreeMap}()}} = G$.order(), and

(iii) $\overline{\overline{G.\mathrm{outDegreeMap}()}} = G$.order().

(60) Let us consider a graph G, and a vertex v of G. Then $(G$.degreeMap()$)(v)$ $= (G$.inDegreeMap()$)(v) + (G$.outDegreeMap()$)(v)$.

Let G be a locally-finite graph. Note that G.degreeMap() is natural-valued and G.inDegreeMap() is natural-valued and G.outDegreeMap() is natural-valued.

The functors: G.degreeMap(), G.inDegreeMap(), and G.outDegreeMap() yield functions from the vertices of G into $\mathbb{N}$. Let G be a vertex-finite graph. Note that G.degreeMap() is finite and G.inDegreeMap() is finite and G.outDegreeMap() is finite. Now we state the proposition:

(61) Let us consider a cardinal number c, a trivial, c-edge graph G, and a vertex v of G. Then

(i) G.inDegreeMap() $= v \dot{\longmapsto} c$, and

(ii) G.outDegreeMap() $= v \dot{\longmapsto} c$, and

(iii) G.degreeMap() $= v \dot{\longmapsto} 2 \cdot c$.

Let G be a trivial graph. Let us note that G.degreeMap() is trivial and G.inDegreeMap() is trivial and G.outDegreeMap() is trivial. Now we state the propositions:

(62) Let us consider a graph G_2, a set V, and a supergraph G_1 of G_2 extended by the vertices from V. Then

(i) G_1.degreeMap() $= G_2$.degreeMap() $+\cdot (V \setminus$ (the vertices of G_2) $\longmapsto 0)$, and

(ii) G_1.inDegreeMap() $= G_2$.inDegreeMap() $+\cdot (V \setminus$ (the vertices of G_2) $\longmapsto 0)$, and

(iii) G_1.outDegreeMap() $= G_2$.outDegreeMap() $+\cdot (V \setminus$ (the vertices of G_2) $\longmapsto 0)$.

(63) Let us consider a graph G, and a component C of G. Then

(i) C.degreeMap() $= G$.degreeMap() $\restriction$(the vertices of C), and

(ii) C.inDegreeMap() $= G$.inDegreeMap() $\restriction$(the vertices of C), and

(iii) C.outDegreeMap() $= G$.outDegreeMap() $\restriction$(the vertices of C).

Let G be a graph and v be a denumeration of the vertices of G. Let us observe that (G.degreeMap()) $\cdot\, v$ is transfinite sequence-like and (G.order())-elements and (G.inDegreeMap()) $\cdot\, v$ is transfinite sequence-like and (G.order())-elements and (G.outDegreeMap())$\cdot v$ is transfinite sequence-like and (G.order())-elements.

Let us consider a finite graph G and a denumeration v of the vertices of G. Now we state the propositions:

(64) (G.degreeMap()) $\cdot\, v =$ (G.inDegreeMap()) $\cdot\, v +$ (G.outDegreeMap()) $\cdot\, v$. The theorem is a consequence of (60).

(65) (i) G.size() $= \sum$(G.inDegreeMap()) $\cdot\, v$, and

(ii) G.size() $= \sum$(G.outDegreeMap()) $\cdot\, v$.

(66) $2 \cdot$ (G.size()) $= \sum$(G.degreeMap()) $\cdot\, v$. The theorem is a consequence of (65) and (64).

(67) Handshaking Lemma:
Let us consider a finite graph G, and a natural number k. Suppose $k = \overline{\overline{\{w, \text{ where } w \text{ is a vertex of } G : w.\text{degree() is not even}\}}}$. Then k is even.
Proof: Set $v =$ the denumeration of the vertices of G. Define $\mathcal{M}$(natural number) $=$ ((G.degreeMap()) $\cdot\, v$)($\$_1$) mod 2. Consider m being a finite 0-sequence of $\mathbb{N}$ such that $\operatorname{len} m = \operatorname{len}(G.\text{degreeMap}()) \cdot v$ and for every natural number k such that $k \in \operatorname{len}(G.\text{degreeMap}()) \cdot v$ holds $m(k) = \mathcal{M}(k)$. $\square$

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