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# Isosceles Triangular and Isosceles Trapezoidal Membership Functions Using Centroid Method

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**Summary.** Since isosceles triangular and trapezoidal membership functions [4] are easy to manage, they were applied to various fuzzy approximate reasoning [10], [13], [14]. The centroids of isosceles triangular and trapezoidal membership functions are mentioned in this article [16], [9] and formalized in [11] and [12]. Some propositions of the composition mapping ($f +\cdot\, g$, or `f +* g` using Mizar formalism, where $f$, $g$ are affine mappings), are proved following [3], [15]. Then different notations for the same isosceles triangular and trapezoidal membership function are formalized.

We proved the agreement of the same function expressed with different parameters and formalized those centroids with parameters. In addition, various properties of membership functions on intervals where the endpoints of the domain are fixed and on general intervals are formalized in Mizar [1], [2]. Our formal development contains also some numerical results which can be potentially useful to encode either fuzzy numbers [7], or even fuzzy implications [5], [6] and extends the possibility of building hybrid rough-fuzzy approach in the future [8].

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## 1. Preliminaries

Let us consider real numbers $a$, $b$, $c$, $d$. Now we state the propositions:

(1) $[a,d] \setminus [b,c] \subseteq [a,b[ \cup ]c,d]$.

(2) If $a < b < c < d$, then $[a,d] \setminus [b,c] \subseteq [a,b] \cup [c,d]$.

(3) Let us consider real numbers $p$, $q$, $r$, $s$. If $p < r \leqslant s < q$, then $[r, s] \subset [p, q]$.

## 2. Continuous Functions

Let us consider functions $f$, $g$ from $\mathbb{R}$ into $\mathbb{R}$. Now we state the propositions:

(4) If $f$ is continuous and $g$ is continuous, then $\max(f, g)$ is continuous.

(5) If $f$ is continuous and $g$ is continuous, then $\min(f, g)$ is continuous.

Let us consider non empty, closed interval subsets $A$, $B$ of $\mathbb{R}$. Now we state the propositions:

(6) If $B \subset A$, then $\inf A < \inf B$ or $\sup B < \sup A$.

(7) If $B \subseteq A$, then $\inf A \leqslant \inf B$ and $\sup B \leqslant \sup A$.

(8) Let us consider a real number $r$, and functions $f$, $g$ from $\mathbb{R}$ into $\mathbb{R}$. Then $r \cdot (f {+\cdot} g) = r \cdot f {+\cdot} r \cdot g$.
PROOF: Set $F_1 = r \cdot (f {+\cdot} g)$. Set $F_2 = r \cdot f {+\cdot} r \cdot g$. For every object $x$ such that $x \in \operatorname{dom} F_1$ holds $F_1(x) = F_2(x)$. $\square$

From now on $A$ denotes a non empty subset of $\mathbb{R}$. Now we state the propositions:

(9) Let us consider a real number $r$, and a function $f$ from $\mathbb{R}$ into $\mathbb{R}$. Then $(r \cdot f){\restriction}A = r \cdot (f{\restriction}A)$.
PROOF: Set $F = (r \cdot f){\restriction}A$. Set $g = r \cdot (f{\restriction}A)$. For every object $x$ such that $x \in \operatorname{dom} F$ holds $F(x) = g(x)$. $\square$

(10) Let us consider a real number $r$, and a partial function $f$ from $\mathbb{R}$ to $\mathbb{R}$. Suppose $A \subseteq \operatorname{dom} f$. Then $(r \cdot f){\restriction}A = r \cdot (f{\restriction}A)$.
PROOF: Set $F = (r \cdot f){\restriction}A$. Set $g = r \cdot (f{\restriction}A)$. For every object $x$ such that $x \in \operatorname{dom} F$ holds $F(x) = g(x)$. $\square$

(11) Let us consider a real number $s$, and functions $f$, $g$ from $\mathbb{R}$ into $\mathbb{R}$. Then $f{\restriction}]-\infty, s] {+\cdot} g{\restriction}[s, +\infty[$ is a function from $\mathbb{R}$ into $\mathbb{R}$.

(12) Let us consider real numbers $a$, $b$, $r$.
Then $r \cdot (\mathrm{AffineMap}(a, b)) = \mathrm{AffineMap}(r \cdot a, r \cdot b)$.

(13) Let us consider a real number $s$, and functions $f$, $g$ from $\mathbb{R}$ into $\mathbb{R}$. Then

(i) $\operatorname{dom}(f{\restriction}]-\infty, s] {+\cdot} g{\restriction}[s, +\infty[) = \mathbb{R}$, and

(ii) $\operatorname{dom}(f{\restriction}]-\infty, s[ {+\cdot} g{\restriction}[s, +\infty[) = \mathbb{R}$.

(14) Let us consider real numbers $a$, $b$, $c$. Suppose $b > 0$ and $c > 0$. Let us consider a real number $x$. Then $((\mathrm{AffineMap}(\frac{b}{c}, b - \frac{a \cdot b}{c})){\restriction}]-\infty, a] {+\cdot} (\mathrm{AffineMap}(-\frac{b}{c}, b + \frac{a \cdot b}{c})){\restriction}[a, +\infty[)(x) = b - |\frac{b \cdot (x-a)}{c}|$.
PROOF: For every real number $x$, $((\mathrm{AffineMap}(\frac{b}{c}, b - \frac{a \cdot b}{c})){\restriction}]-\infty, a] {+\cdot} (\mathrm{AffineMap}(-\frac{b}{c}, b + \frac{a \cdot b}{c})){\restriction}[a, +\infty[)(x) = b - |\frac{b \cdot (x-a)}{c}|$. $\square$

(15) Let us consider real numbers $a$, $b$, $c$, and a function $f$ from $\mathbb{R}$ into $\mathbb{R}$. Suppose $b > 0$ and $c > 0$ and for every real number $x$, $f(x) = b - |\frac{b\cdot(x-a)}{c}|$. Then $f = (\mathrm{AffineMap}(\frac{b}{c}, b - \frac{a\cdot b}{c}))\restriction ]-\infty, a] +\cdot (\mathrm{AffineMap}(-\frac{b}{c}, b + \frac{a\cdot b}{c}))\restriction [a, +\infty[$. The theorem is a consequence of (14).

Let us consider real numbers $a$, $b$. Now we state the propositions:

(16) Suppose $a > 0$. Then $|\,\mathrm{AffineMap}(a, b)| = -(\mathrm{AffineMap}(a, b))\restriction ]-\infty, \frac{-b}{a}[ +\cdot (\mathrm{AffineMap}(a, b))\restriction [\frac{-b}{a}, +\infty[$.

Proof: For every object $x$ such that $x \in \mathrm{dom}\,|\,\mathrm{AffineMap}(a, b)|$ holds $|\,\mathrm{AffineMap}(a, b)|(x) = (-(\mathrm{AffineMap}(a, b))\restriction ]-\infty, \frac{-b}{a}[ +\cdot (\mathrm{AffineMap}(a, b)) \restriction [\frac{-b}{a}, +\infty[)(x)$. $\square$

(17) Suppose $a < 0$. Then $|\,\mathrm{AffineMap}(a, b)| = (\mathrm{AffineMap}(a, b))\restriction ]-\infty, \frac{-b}{a}[ +\cdot -(\mathrm{AffineMap}(a, b))\restriction [\frac{-b}{a}, +\infty[$.

Proof: Set $f = (\mathrm{AffineMap}(a, b))\restriction ]-\infty, \frac{-b}{a}[ +\cdot -(\mathrm{AffineMap}(a, b))\restriction [\frac{-b}{a}, +\infty[$. For every object $x$ such that $x \in \mathrm{dom}((-(\mathrm{AffineMap}(a, b)))\restriction [\frac{-b}{a}, +\infty[)$ holds $(-(\mathrm{AffineMap}(a, b))\restriction [\frac{-b}{a}, +\infty[)(x) = ((-(\mathrm{AffineMap}(a, b))) \restriction [\frac{-b}{a}, +\infty[)(x)$. For every element $x$ of $\mathbb{R}$, $f(x) = |\,\mathrm{AffineMap}(a, b)|(x)$. $\square$

(18) Let us consider real numbers $a$, $b$, $c$, and a function $f$ from $\mathbb{R}$ into $\mathbb{R}$. Suppose $b > 0$ and $c > 0$ and for every real number $x$, $f(x) = \max(0, b - |\frac{b\cdot(x-a)}{c}|)$. Let us consider a real number $x$. If $x \notin [a - c, a + c]$, then $f(x) = 0$.

(19) Let us consider real numbers $a$, $b$, $c$, and functions $f$, $g$ from $\mathbb{R}$ into $\mathbb{R}$. Suppose $a < b < c$. Then $(f\restriction ]-\infty, b] +\cdot g\restriction [b, +\infty[)\restriction [a, c] = f\restriction [a, b] +\cdot g\restriction [b, c]$.

Proof: For every object $x$ such that $x \in \mathrm{dom}((f\restriction ]-\infty, b] +\cdot g\restriction [b, +\infty[)\restriction [a, c])$ holds $((f\restriction ]-\infty, b] +\cdot g\restriction [b, +\infty[)\restriction [a, c])(x) = (f\restriction [a, b] +\cdot g\restriction [b, c])(x)$. $\square$

Let us consider real numbers $a$, $b$, $c$ and a function $f$ from $\mathbb{R}$ into $\mathbb{R}$. Now we state the propositions:

(20) Suppose $b > 0$ and $c > 0$. Then $((\mathrm{AffineMap}(\frac{b}{c}, b - \frac{a\cdot b}{c}))\restriction ]-\infty, a] +\cdot (\mathrm{AffineMap}(-\frac{b}{c}, b + \frac{a\cdot b}{c}))\restriction [a, +\infty[)\restriction [a - c, a + c] = (\mathrm{AffineMap}(\frac{b}{c}, b - \frac{a\cdot b}{c}))\restriction [a - c, a] +\cdot (\mathrm{AffineMap}(-\frac{b}{c}, b + \frac{a\cdot b}{c}))\restriction [a, a + c]$. The theorem is a consequence of (19).

(21) Suppose $a < b < c$ and $f$ is integrable on $[a, c]$ and $f\restriction [a, c]$ is bounded. Then

(i) $f$ is integrable on $[a, b]$, and

(ii) $f$ is integrable on $[b, c]$, and

(iii) $f\restriction [a, b]$ is bounded, and

(iv) $[a, b] \subseteq \mathrm{dom}\, f$, and

(v) $\int\limits_{a}^{c} f(x)dx = \int\limits_{a}^{b} f(x)dx + \int\limits_{b}^{c} f(x)dx$.

(22) Let us consider real numbers $a$, $b$, $c$, $d$, and a function $f$ from $\mathbb{R}$ into $\mathbb{R}$. Suppose $a < b < c < d$ and $f$ is integrable on $[a, d]$ and $f{\restriction}[a, d]$ is bounded and for every real number $x$ such that $x \in [a, b] \cup [c, d]$ holds $f(x) = 0$. Then $\mathrm{centroid}(f, [a, d]) = \mathrm{centroid}(f, [b, c])$.

(23) Let us consider non empty, closed interval subsets $A$, $B$ of $\mathbb{R}$, and a function $f$ from $\mathbb{R}$ into $\mathbb{R}$. Suppose $\inf B \neq \sup B$ and $B \subseteq A$ and $f$ is integrable on $A$ and $f{\restriction}A$ is bounded and for every real number $x$ such that $x \in A \setminus B$ holds $f(x) = 0$ and $f(\inf B) = 0$ and $f(\sup B) = 0$. Then $\mathrm{centroid}(f, A) = \mathrm{centroid}(f, B)$.
PROOF: $\inf A \leqslant \inf B$ and $\sup B \leqslant \sup A$. For every real number $x$ such that $x \in [\inf A, \inf B] \cup [\sup B, \sup A]$ holds $f(x) = 0$. $\square$

## 3. Triangular and Trapezoidal Membership Functions

Now we state the proposition:

(24) Let us consider real numbers $a$, $c$, and a function $f$ from $\mathbb{R}$ into $\mathbb{R}$. Suppose $c > 0$ and for every real number $x$, $f(x) = \max(0, 1 - |\frac{x-a}{c}|)$. Then $f$ is a triangular fuzzy set of $\mathbb{R}$.
PROOF: Define $\mathcal{H}(\text{element of } \mathbb{R}) = (1 - |\frac{\$_1 - a}{c}|)(\in \mathbb{R})$. Consider $h$ being a function from $\mathbb{R}$ into $\mathbb{R}$ such that for every element $x$ of $\mathbb{R}$, $h(x) = \mathcal{H}(x)$. For every real number $x$, $f(x) = \max(0, \min(1, h(x)))$. $\square$

Let us consider real numbers $a$, $b$, $c$ and a function $f$ from $\mathbb{R}$ into $\mathbb{R}$. Now we state the propositions:

(25) Suppose $b > 1$ and $c > 0$ and for every real number $x$, $f(x) = \min(1, \max(0, b - |\frac{b\cdot(x-a)}{c}|))$. Then $f$ is trapezoidal fuzzy set of $\mathbb{R}$ and normalized fuzzy set of $\mathbb{R}$.

(26) If $b > 0$ and $c > 0$ and for every real number $x$, $f(x) = \max(0, b - |\frac{b\cdot(x-a)}{c}|)$, then $f = b \cdot \mathrm{TriangularFS}((a - c), a, (a + c))$.
PROOF: Set $g = b \cdot \mathrm{TriangularFS}((a - c), a, (a + c))$. For every object $x$ such that $x \in \mathrm{dom}\, f$ holds $f(x) = g(x)$. $\square$

(27) If $b > 0$ and $c > 0$ and for every real number $x$, $f(x) = \max(0, b - |\frac{b\cdot(x-a)}{c}|)$, then $f$ is Lipschitzian.
PROOF: For every real number $x$, $f(x) = \max(0, \min(b, b \cdot (1 - |\frac{x-a}{c}|)))$. $\square$

(28) Suppose $b > 0$ and $c > 0$ and $f{\restriction}[a - c, a + c] = (\mathrm{AffineMap}(\frac{b}{c}, b - \frac{a\cdot b}{c})){\restriction}[\inf[a-c, a+c], \frac{b + \frac{a\cdot b}{c} - (b - \frac{a\cdot b}{c})}{\frac{b}{c} - -\frac{b}{c}}] + \cdot (\mathrm{AffineMap}(-\frac{b}{c}, b + \frac{a\cdot b}{c})){\restriction}[\frac{b + \frac{a\cdot b}{c} - (b - \frac{a\cdot b}{c})}{\frac{b}{c} - -\frac{b}{c}},

$\sup[a-c, a+c]]$. Then $\mathrm{centroid}(f, [a-c, a+c]) = a$.

(29) Suppose $b > 0$ and $c > 0$ and for every real number $x$, $f(x) = \max(0, b - |\frac{b\cdot(x-a)}{c}|)$. Then $f \restriction [a-c, a+c] = (\mathrm{AffineMap}(\frac{b}{c}, b - \frac{a\cdot b}{c})) \restriction [\inf[a-c, a+c], \frac{b+\frac{a\cdot b}{c}-(b-\frac{a\cdot b}{c})}{\frac{b}{c}--\frac{b}{c}}] +\cdot (\mathrm{AffineMap}(-\frac{b}{c}, b + \frac{a\cdot b}{c})) \restriction [\frac{b+\frac{a\cdot b}{c}-(b-\frac{a\cdot b}{c})}{\frac{b}{c}--\frac{b}{c}}, \sup[a-c, a+c]]$.

PROOF: Set $g = (\mathrm{AffineMap}(\frac{b}{c}, b - \frac{a\cdot b}{c})) \restriction [\inf[a-c, a+c], \frac{b+\frac{a\cdot b}{c}-(b-\frac{a\cdot b}{c})}{\frac{b}{c}--\frac{b}{c}}] +\cdot$ $(\mathrm{AffineMap}(-\frac{b}{c}, b + \frac{a\cdot b}{c})) \restriction [\frac{b+\frac{a\cdot b}{c}-(b-\frac{a\cdot b}{c})}{\frac{b}{c}--\frac{b}{c}}, \sup[a-c, a+c]]$. For every object $x$ such that $x \in \mathrm{dom}(f \restriction [a-c, a+c])$ holds $(f \restriction [a-c, a+c])(x) = g(x)$. □

(30) If $b > 0$ and $c > 0$ and for every real number $x$, $f(x) = \max(0, b - |\frac{b\cdot(x-a)}{c}|)$, then $\mathrm{centroid}(f, [a-c, a+c]) = a$. The theorem is a consequence of (29) and (28).

In the sequel $A$ denotes a non empty, closed interval subset of $\mathbb{R}$. Let us consider real numbers $a$, $b$, $c$ and a function $f$ from $\mathbb{R}$ into $\mathbb{R}$. Now we state the propositions:

(31) If $b > 0$ and $c > 0$ and for every real number $x$, $f(x) = \max(0, b - |\frac{b\cdot(x-a)}{c}|)$, then $f$ is integrable on $A$ and $f \restriction A$ is bounded. The theorem is a consequence of (27).

(32) Suppose $b > 0$ and $c > 0$ and for every real number $x$, $f(x) = \max(0, b - |\frac{b\cdot(x-a)}{c}|)$. Then

(i) $f(\inf[a-c, a+c]) = 0$, and

(ii) $f(a-c) = 0$, and

(iii) $f(\sup[a-c, a+c]) = 0$, and

(iv) $f(a+c) = 0$.

(33) If $b > 0$ and $c > 0$ and $[a-c, a+c] \subseteq A$ and for every real number $x$, $f(x) = \max(0, b - |\frac{b\cdot(x-a)}{c}|)$, then $\mathrm{centroid}(f, A) = a$. The theorem is a consequence of (18), (32), (31), (23), and (30).

Let us consider real numbers $a$, $b$, $c$. Now we state the propositions:

(34) If $a < b < c$ and $b-a = c-b$, then $\mathrm{centroid}(\mathrm{TriangularFS}(a, b, c), [a, c]) = b$.

PROOF: For every real number $x$, $(\mathrm{TriangularFS}(a, b, c))(x) = \max(0, 1 - |\frac{1\cdot(x-b)}{b-a}|)$. $\mathrm{centroid}(\mathrm{TriangularFS}(a, b, c), [b-(b-a), b+(b-a)]) = b$. □

(35) If $a < b < c$, then $\mathrm{TriangularFS}(a, b, c)$ is integrable on $A$ and $\mathrm{TriangularFS}(a, b, c) \restriction A$ is bounded.

Let us consider real numbers $a$, $b$, $c$, $d$. Now we state the propositions:

(36) If $a < b < c$ and $b-a = c-b$ and $d \neq 0$, then $\mathrm{centroid}(d \cdot \mathrm{TriangularFS}(a, b, c), [a, c]) = b$. The theorem is a consequence of (35) and (34).

(37) If $a < b < c < d$, then $\mathrm{TrapezoidalFS}(a, b, c, d)$ is integrable on $A$ and $\mathrm{TrapezoidalFS}(a, b, c, d) \restriction A$ is bounded.

(38) Let us consider real numbers $a$, $b$, $c$, $d$, $r$. If $a < b < c < d$, then $r \cdot \mathrm{TrapezoidalFS}(a, b, c, d)$ is integrable on $A$. The theorem is a consequence of (37).

(39) Let us consider real numbers $a_1$, $c$, $a_2$, $d$, and a function $f$ from $\mathbb{R}$ into $\mathbb{R}$. Suppose $c > 0$ and $d > 0$ and $a_1 < a_2$ and $f = (d \cdot \mathrm{TrapezoidalFS}((a_1 - c), a_1, a_2, (a_2 + c))) \restriction [a_1 - c, a_2 + c]$. Then $f$ is integrable on $[a_1 - c, a_2 + c]$. The theorem is a consequence of (38).

(40) Let us consider real numbers $a$, $b$, $c$, functions $f$, $g$ from $\mathbb{R}$ into $\mathbb{R}$, and a partial function $h$ from $\mathbb{R}$ to $\mathbb{R}$. Suppose $a \leqslant b \leqslant c$ and $f$ is continuous and $g$ is continuous and $h \restriction [a, c] = f \restriction [a, b] {+\cdot} g \restriction [b, c]$ and $f(b) = g(b)$ and $[a, c] \subseteq \operatorname{dom} h$. Then $h \restriction [a, c]$ is continuous.

PROOF: For every real numbers $x_0$, $r$ such that $x_0 \in [a, c]$ and $0 < r$ there exists a real number $s$ such that $0 < s$ and for every real number $x_1$ such that $x_1 \in [a, c]$ and $|x_1 - x_0| < s$ holds $|h(x_1) - h(x_0)| < r$. □

(41) Let us consider real numbers $a$, $b$, $p$, $q$, and a function $f$ from $\mathbb{R}$ into $\mathbb{R}$. Suppose $a \neq p$ and $f = (\mathrm{AffineMap}(a, b)) \restriction ]-\infty, \frac{q-b}{a-p}] {+\cdot} (\mathrm{AffineMap}(p, q)) \restriction [\frac{q-b}{a-p}, +\infty[$. Then $f$ is Lipschitzian.

(42) Let us consider real numbers $a$, $b$, $c$, and functions $f$, $g$, $h$ from $\mathbb{R}$ into $\mathbb{R}$. Suppose $a \leqslant b \leqslant c$ and $f$ is continuous and $g$ is continuous and $h \restriction [a, c] = f \restriction [a, b] {+\cdot} g \restriction [b, c]$ and $f(b) = g(b)$. Then $\int\limits_{[a,c]} (\mathrm{id}_{\mathbb{R}} \cdot h)(x) dx = \int\limits_{[a,b]} (\mathrm{id}_{\mathbb{R}} \cdot f)(x) dx + \int\limits_{[b,c]} (\mathrm{id}_{\mathbb{R}} \cdot g)(x) dx$.

PROOF: Set $G = (\mathrm{id}_{\mathbb{R}} \cdot f) \restriction [a, b] {+\cdot} (\mathrm{id}_{\mathbb{R}} \cdot g) \restriction [b, c]$. $[a, c] = \mathbb{R} \cap [a, c]$. For every object $x$ such that $x \in \operatorname{dom}((\mathrm{id}_{\mathbb{R}} \cdot h) \restriction [a, c])$ holds $(\mathrm{id}_{\mathbb{R}} \cdot (h \restriction [a, c]))(x) = ((\mathrm{id}_{\mathbb{R}} \cdot h) \restriction [a, c])(x)$. For every object $x$ such that $x \in \operatorname{dom} G$ holds $G(x) = (\mathrm{id}_{\mathbb{R}} \cdot (h \restriction [a, c]))(x)$. Reconsider $h_1 = h$ as a partial function from $\mathbb{R}$ to $\mathbb{R}$. $h_1 \restriction [a, c]$ is continuous. □

Let us consider real numbers $a$, $b$, $c$, $d$, $r$. Now we state the propositions:

(43) Suppose $a < b < c < d$. Then $((\mathrm{AffineMap}(\frac{1}{b-a}, -\frac{a}{b-a})) \restriction [a, b] {+\cdot} (\mathrm{AffineMap}(0, 1)) \restriction [b, c]) {+\cdot} (\mathrm{AffineMap}(-\frac{1}{d-c}, \frac{d}{d-c})) \restriction [c, d] = \mathrm{TrapezoidalFS}(a, b, c, d) \restriction [a, d]$.

PROOF: For every object $x$ such that $x \in \operatorname{dom}(\mathrm{TrapezoidalFS}(a, b, c, d) \restriction [a, d])$ holds $(((\mathrm{AffineMap}(\frac{1}{b-a}, -\frac{a}{b-a})) \restriction [a, b] {+\cdot} (\mathrm{AffineMap}(0, 1)) \restriction [b, c]) {+\cdot} (\mathrm{AffineMap}(-\frac{1}{d-c}, \frac{d}{d-c})) \restriction [c, d])(x) = (\mathrm{TrapezoidalFS}(a, b, c, d) \restriction [a, d])(x)$. □

(44) Suppose $a < b < c < d$. Then $\mathrm{TrapezoidalFS}(a, b, c, d) = (\mathrm{AffineMap}(0,$

$0)) \restriction \mathbb{R} \setminus ]a,d[+\cdot$ TrapezoidalFS$(a,b,c,d) \restriction [a,d]$. The theorem is a consequence of (43).

(45) Suppose $a < b < c < d$. Then $((r \cdot (\text{AffineMap}(\frac{1}{b-a}, -\frac{a}{b-a}))) \restriction [a,b] +\cdot (r \cdot (\text{AffineMap}(0,1))) \restriction [b,c]) +\cdot (r \cdot (\text{AffineMap}(-\frac{1}{d-c}, \frac{d}{d-c}))) \restriction [c,d] = (r \cdot \text{TrapezoidalFS}(a,b,c,d)) \restriction [a,d]$.

PROOF: Set $f_1 = (\text{AffineMap}(\frac{1}{b-a}, -\frac{a}{b-a})) \restriction [a,b]$. Set $f_2 = (\text{AffineMap}(0, 1)) \restriction [b,c]$. Set $f_3 = (\text{AffineMap}(-\frac{1}{d-c}, \frac{d}{d-c})) \restriction [c,d]$. Set $F_1 = \text{AffineMap}(\frac{1}{b-a}, -\frac{a}{b-a})$. Set $F_2 = \text{AffineMap}(0,1)$. Set $F_3 = \text{AffineMap}(-\frac{1}{d-c}, \frac{d}{d-c})$. For every object $x$ such that $x \in \text{dom}(r \cdot ((f_1 +\cdot f_2) +\cdot f_3))$ holds $(((r \cdot F_1) \restriction [a,b] +\cdot (r \cdot F_2) \restriction [b,c]) +\cdot (r \cdot F_3) \restriction [c,d])(x) = (r \cdot ((f_1 +\cdot f_2) +\cdot f_3))(x)$. $\square$

Let us consider real numbers $a_1$, $c$, $a_2$, $d$. Now we state the propositions:

(46) Suppose $c > 0$ and $d > 0$ and $a_1 < a_2$.
Then $((\text{AffineMap}(\frac{d}{c}, -\frac{d}{c} \cdot (a_1 - c))) \restriction [a_1 - c, a_1] +\cdot (\text{AffineMap}(0,d)) \restriction [a_1, a_2]) +\cdot (\text{AffineMap}(-\frac{d}{c}, \frac{d}{c} \cdot (a_2 + c))) \restriction [a_2, a_2 + c] = (d \cdot \text{TrapezoidalFS}((a_1 - c), a_1, a_2, (a_2 + c))) \restriction [a_1 - c, a_2 + c]$. The theorem is a consequence of (12) and (45).

(47) Suppose $c > 0$ and $d > 0$ and $a_1 < a_2$. Then $\int\limits_{[a_1-c,a_1]} (\text{AffineMap}(\frac{d}{c}, -\frac{d}{c} \cdot (a_1 - c)))(x)dx + \int\limits_{[a_1,a_2]} (\text{AffineMap}(0,d))(x)dx + \int\limits_{[a_2,a_2+c]} (\text{AffineMap}(-\frac{d}{c}, \frac{d}{c} \cdot (a_2 + c)))(x)dx = d \cdot (a_2 - a_1 + c)$.

(48) Let us consider real numbers $a_1$, $c$, $a_2$, $d$, and a function $f$ from $\mathbb{R}$ into $\mathbb{R}$. Suppose $c > 0$ and $d > 0$ and $a_1 < a_2$ and $f \restriction [a_1 - c, a_2 + c] = ((\text{AffineMap}(\frac{d}{c}, -\frac{d}{c} \cdot (a_1 - c))) \restriction [a_1 - c, a_1] +\cdot (\text{AffineMap}(0,d)) \restriction [a_1, a_2]) +\cdot (\text{AffineMap}(-\frac{d}{c}, \frac{d}{c} \cdot (a_2 + c))) \restriction [a_2, a_2 + c]$. Then $\int\limits_{[a_1-c,a_2+c]} f(x)dx = \int\limits_{[a_1-c,a_1]} (\text{AffineMap}(\frac{d}{c}, -\frac{d}{c} \cdot (a_1 - c)))(x)dx + \int\limits_{[a_1,a_2]} (\text{AffineMap}(0,d))(x)dx + \int\limits_{[a_2,a_2+c]} (\text{AffineMap}(-\frac{d}{c}, \frac{d}{c} \cdot (a_2 + c)))(x)dx$. The theorem is a consequence of (46).

(49) Let us consider real numbers $a_1$, $c$, $a_2$, $d$. Suppose $c > 0$ and $d > 0$ and $a_1 < a_2$. Then $\text{centroid}(d \cdot \text{TrapezoidalFS}((a_1 - c), a_1, a_2, (a_2 + c)), [a_1 - c, a_2 + c]) = \frac{a_1 + a_2}{2}$. The theorem is a consequence of (46), (48), and (47).

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