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Differentiation on Interval

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Summary. This article generalizes the differential method on intervals, using the Mizar system [2], [3], [12]. Differentiation of real one-variable functions is introduced in Mizar [13], along standard lines (for interesting survey of formalizations of real analysis in various proof-assistants like ACL2 [11], Isabelle/HOL [10], Coq [4], see [5]), but the differentiable interval is restricted to open intervals. However, when considering the relationship with integration [9], since integration is an operation on a closed interval, it would be convenient for differentiation to be able to handle derivates on a closed interval as well. Regarding differentiability on a closed interval, the right and left differentiability have already been formalized [6], but they are the derivatives at the endpoints of an interval and not demonstrated as a differentiation over intervals.

Therefore, in this paper, based on these results, although it is limited to real one-variable functions, we formalize the differentiation on arbitrary intervals and summarize them as various basic propositions. In particular, the chain rule [1] is an important formula in relation to differentiation and integration, extending recent formalized results [7], [8] in the latter field of research.

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1. Preliminaries

Now we state the propositions:

(1) Let us consider open subsets A, B of $\mathbb{R}$, and partial functions f, g from $\mathbb{R}$ to $\mathbb{R}$. Suppose f is differentiable on A and $\operatorname{rng}(f{\restriction}A) \subseteq B$ and g is differentiable on B. Then

(i) $g \cdot f$ is differentiable on A, and

(ii) $(g \cdot f)'_{\restriction A} = g'_{\restriction B} \cdot f \cdot f'_{\restriction A}$.

(2) Let us consider an interval I. Then

(i) $]\inf I, \sup I[$ is an open subset of $\mathbb{R}$, and

(ii) $]\inf I, \sup I[\subseteq I$.

(3) Let us consider an interval I, and a real number x. Suppose $x \in I$ and $x \neq \inf I$ and $x \neq \sup I$. Then $x \in]\inf I, \sup I[$.

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, an interval I, and a real number x. Now we state the propositions:

(4) If f is right differentiable in x and $x \in I$ and $x \neq \sup I$, then $f \restriction I$ is right differentiable in x.

PROOF: Consider r being a real number such that $r > 0$ and $[x, x + r] \subseteq \operatorname{dom} f$. For every 0-convergent, non-zero sequence h of real numbers and for every constant sequence c of real numbers such that $\operatorname{rng} c = \{x\}$ and $\operatorname{rng}(h + c) \subseteq \operatorname{dom}(f \restriction I)$ and for every natural number n, $h(n) > 0$ holds $h^{-1} \cdot ((f \restriction I_*(h + c)) - (f \restriction I_* c))$ is convergent. □

(5) If f is left differentiable in x and $x \in I$ and $x \neq \inf I$, then $f \restriction I$ is left differentiable in x.

PROOF: Consider r being a real number such that $r > 0$ and $[x - r, x] \subseteq \operatorname{dom} f$. For every 0-convergent, non-zero sequence h of real numbers and for every constant sequence c of real numbers such that $\operatorname{rng} c = \{x\}$ and $\operatorname{rng}(h + c) \subseteq \operatorname{dom}(f \restriction I)$ and for every natural number n, $h(n) < 0$ holds $h^{-1} \cdot ((f \restriction I_*(h + c)) - (f \restriction I_* c))$ is convergent. □

(6) Let us consider a set X, and partial functions f_1, f_2 from X to $\mathbb{R}$. Suppose $\operatorname{dom} f_1 = \operatorname{dom} f_2$. Then

(i) $f_1 + f_2 - f_2 = f_1$, and

(ii) $f_1 - f_2 + f_2 = f_1$.

2. Differentiation on Intervals

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and I be a non empty interval. We say that f is differentiable on interval I if and only if

(Def. 1) $I \subseteq \operatorname{dom} f$ and $\inf I < \sup I$ and if $\inf I \in I$, then f is right differentiable in $\inf I$ and if $\sup I \in I$, then f is left differentiable in $\sup I$ and f is differentiable on $]\inf I, \sup I[$.

Let I be an interval, non empty subset of $\mathbb{R}$. Assume f is differentiable on interval I. The functor f'_I yielding a partial function from $\mathbb{R}$ to $\mathbb{R}$ is defined by

(Def. 2) dom $it = I$ and for every real number x such that $x \in I$ holds if $x = \inf I$, then $it(x) = f'_{+}(x)$ and if $x = \sup I$, then $it(x) = f'_{-}(x)$ and if $x \neq \inf I$ and $x \neq \sup I$, then $it(x) = f'(x)$.

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$ and real numbers a, b. Now we state the propositions:

(7) If $a < b$ and f is differentiable on interval $[a, b]$, then f is differentiable on $]a, b[$.

(8) Suppose $a \leqslant b$ and f is differentiable on interval $[a, b]$. Then

(i) $f'_{[a,b]}(a) = f'_{+}(a)$, and

(ii) $f'_{[a,b]}(b) = f'_{-}(b)$, and

(iii) for every real number x such that $x \in]a, b[$ holds $f'_{[a,b]}(x) = f'(x)$.

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, an interval I, and a real number x. Now we state the propositions:

(9) If $f{\restriction}I$ is right differentiable in x, then f is right differentiable in x and $(f{\restriction}I)'_{+}(x) = f'_{+}(x)$.

Proof: Consider r being a real number such that $r > 0$ and $[x, x + r] \subseteq \mathrm{dom}(f{\restriction}I)$. For every 0-convergent, non-zero sequence h of real numbers and for every constant sequence c of real numbers such that $\mathrm{rng}\, c = \{x\}$ and $\mathrm{rng}(h + c) \subseteq \mathrm{dom}\, f$ and for every natural number n, $h(n) > 0$ holds $h^{-1} \cdot ((f_*(h+c)) - (f_*c))$ is convergent and $\lim(h^{-1} \cdot ((f_*(h+c)) - (f_*c))) = (f{\restriction}I)'_{+}(x)$. □

(10) If $f{\restriction}I$ is left differentiable in x, then f is left differentiable in x and $(f{\restriction}I)'_{-}(x) = f'_{-}(x)$.

Proof: Consider r being a real number such that $r > 0$ and $[x - r, x] \subseteq \mathrm{dom}(f{\restriction}I)$. For every 0-convergent, non-zero sequence h of real numbers and for every constant sequence c of real numbers such that $\mathrm{rng}\, c = \{x\}$ and $\mathrm{rng}(h + c) \subseteq \mathrm{dom}\, f$ and for every natural number n, $h(n) < 0$ holds $h^{-1} \cdot ((f_*(h+c)) - (f_*c))$ is convergent and $\lim(h^{-1} \cdot ((f_*(h+c)) - (f_*c))) = (f{\restriction}I)'_{-}(x)$. □

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$ and a non empty interval I. Now we state the propositions:

(11) f is differentiable on interval I if and only if $I \subseteq \mathrm{dom}\, f$ and for every real number x such that $x \in I$ holds if $x = \inf I$, then $f{\restriction}I$ is right differentiable in x and if $x = \sup I$, then $f{\restriction}I$ is left differentiable in x and if $x \in]\inf I, \sup I[$, then f is differentiable in x.

Proof: If $\inf I \in I$, then f is right differentiable in $\inf I$. If $\sup I \in I$, then f is left differentiable in $\sup I$. $]\inf I, \sup I[\subseteq I$. For every real number x such that $x \in]\inf I, \sup I[$ holds $f{\restriction}]\inf I, \sup I[$ is differentiable in x. □

(12) If I is open interval, then f is differentiable on I iff f is differentiable on interval I.

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$ and real numbers x_0, r. Now we state the propositions:

(13) If f is right differentiable in x_0 and $\operatorname{rng} f = \{r\}$, then $f'_+(x_0) = 0$.
PROOF: For every non-zero, 0-convergent sequence h of real numbers and for every constant sequence c of real numbers such that $\operatorname{rng} c = \{x_0\}$ and $\operatorname{rng}(h + c) \subseteq \operatorname{dom} f$ and for every natural number n, $h(n) > 0$ holds $h^{-1} \cdot ((f_*(h+c)) - (f_*c))$ is convergent and $\lim(h^{-1} \cdot ((f_*(h+c)) - (f_*c))) = 0$. □

(14) If f is left differentiable in x_0 and $\operatorname{rng} f = \{r\}$, then $f'_-(x_0) = 0$.
PROOF: For every non-zero, 0-convergent sequence h of real numbers and for every constant sequence c of real numbers such that $\operatorname{rng} c = \{x_0\}$ and $\operatorname{rng}(h + c) \subseteq \operatorname{dom} f$ and for every natural number n, $h(n) < 0$ holds $h^{-1} \cdot ((f_*(h+c)) - (f_*c))$ is convergent and $\lim(h^{-1} \cdot ((f_*(h+c)) - (f_*c))) = 0$. □

(15) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and a non empty interval I. Suppose $I \subseteq \operatorname{dom} f$ and $\inf I < \sup I$ and there exists a real number r such that $\operatorname{rng} f = \{r\}$. Then

(i) f is differentiable on interval I, and

(ii) for every real number x such that $x \in I$ holds $f'_I(x) = 0$.

PROOF: Consider r being a real number such that $\operatorname{rng} f = \{r\}$. Set $J =]\inf I, \sup I[$. For every real number x such that $x \in J$ holds $f{\restriction}J$ is differentiable in x. For every real number x such that $x \in I$ holds $f'_I(x) = 0$. □

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$ and a real number x_0. Now we state the propositions:

(16) If $\operatorname{dom} f \subseteq]-\infty, x_0[$ and f is left continuous in x_0, then f is continuous in x_0.

(17) If $\operatorname{dom} f \subseteq]x_0, +\infty[$ and f is right continuous in x_0, then f is continuous in x_0.

3. Fundamental Properties

Now we state the proposition:

(18) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and a non empty interval I. Suppose $I \subseteq \operatorname{dom} f$ and $\inf I < \sup I$ and $f{\restriction}I = \operatorname{id}_I$. Then

(i) f is differentiable on interval I, and

(ii) for every real number x such that $x \in I$ holds $f'_I(x) = 1$.

PROOF: For every set x such that $x \in I$ holds $f(x) = x$. Set $J =]\inf I, \sup I[$. For every set x such that $x \in J$ holds $(f \restriction J)(x) = x$. For every real number x such that $x \in J$ holds $f \restriction J$ is differentiable in x. For every real number x such that $x \in I$ holds $f'_I(x) = 1$. □

Let us consider partial functions f, g from $\mathbb{R}$ to $\mathbb{R}$ and a non empty interval I. Now we state the propositions:

(19) Suppose $I \subseteq \mathrm{dom}(f + g)$ and f is differentiable on interval I and g is differentiable on interval I. Then

(i) $f + g$ is differentiable on interval I, and

(ii) $(f + g)'_I = f'_I + g'_I$, and

(iii) for every real number x such that $x \in I$ holds $(f + g)'_I(x) = f'_I(x) + g'_I(x)$.

PROOF: Set $J =]\inf I, \sup I[$. For every real number x such that $x \in J$ holds $(f + g) \restriction J$ is differentiable in x. For every element x of $\mathbb{R}$ such that $x \in \mathrm{dom}(f + g)'_I$ holds $(f + g)'_I(x) = (f'_I + g'_I)(x)$. □

(20) Suppose $I \subseteq \mathrm{dom}(f - g)$ and f is differentiable on interval I and g is differentiable on interval I. Then

(i) $f - g$ is differentiable on interval I, and

(ii) $(f - g)'_I = f'_I - g'_I$, and

(iii) for every real number x such that $x \in I$ holds $(f - g)'_I(x) = f'_I(x) - g'_I(x)$.

PROOF: Reconsider $J =]\inf I, \sup I[$ as an open subset of $\mathbb{R}$. $J \subseteq I$. For every real number x such that $x \in J$ holds $(f - g) \restriction J$ is differentiable in x. For every element x of $\mathbb{R}$ such that $x \in \mathrm{dom}(f - g)'_I$ holds $(f - g)'_I(x) = (f'_I - g'_I)(x)$. □

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$ and real numbers x_0, r. Now we state the propositions:

(21) If f is right differentiable in x_0, then $r \cdot f$ is right differentiable in x_0 and $(r \cdot f)'_+(x_0) = r \cdot f'_+(x_0)$.

(22) If f is left differentiable in x_0, then $r \cdot f$ is left differentiable in x_0 and $(r \cdot f)'_-(x_0) = r \cdot f'_-(x_0)$.

(23) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, a non empty interval I, and a real number r. Suppose f is differentiable on interval I. Then

(i) $r \cdot f$ is differentiable on interval I, and

(ii) $(r \cdot f)'_I = r \cdot f'_I$, and

(iii) for every real number x such that $x \in I$ holds $(r \cdot f)'_I(x) = r \cdot f'_I(x)$.

Proof: For every real number x such that $x \in]\inf I, \sup I[$ holds $(r \cdot f) \restriction]\inf I, \sup I[$ is differentiable in x. For every element x of $\mathbb{R}$ such that $x \in \mathrm{dom}(r \cdot f)'_I$ holds $(r \cdot f)'_I(x) = (r \cdot f'_I)(x)$. □

Let us consider partial functions f, g from $\mathbb{R}$ to $\mathbb{R}$ and a non empty interval I. Now we state the propositions:

(24) Suppose f is differentiable on interval I and g is differentiable on interval I. Then

(i) $f \cdot g$ is differentiable on interval I, and

(ii) $(f \cdot g)'_I = g \cdot f'_I + f \cdot g'_I$, and

(iii) for every real number x such that $x \in I$ holds $(f \cdot g)'_I(x) = g(x) \cdot f'_I(x) + f(x) \cdot g'_I(x)$.

Proof: Reconsider $J =]\inf I, \sup I[$ as an open subset of $\mathbb{R}$. $J \subseteq I$. For every element x of $\mathbb{R}$ such that $x \in \mathrm{dom}(f \cdot g)'_I$ holds $(f \cdot g)'_I(x) = (g \cdot f'_I + f \cdot g'_I)(x)$. □

(25) Suppose $I \subseteq \mathrm{dom}(\frac{f}{g})$ and f is differentiable on interval I and g is differentiable on interval I. Then

(i) $\frac{f}{g}$ is differentiable on interval I, and

(ii) $(\frac{f}{g})'_I = \frac{f'_I \cdot g - g'_I \cdot f}{g^2}$, and

(iii) for every real number x such that $x \in I$ holds $(\frac{f}{g})'_I(x) = \frac{f'_I(x) \cdot g(x) - g'_I(x) \cdot f(x)}{g(x)^2}$.

Proof: Reconsider $J =]\inf I, \sup I[$ as an open subset of $\mathbb{R}$. $J \subseteq I$. For every set x such that $x \in I$ holds $g(x) \neq 0$. For every element x of $\mathbb{R}$ such that $x \in \mathrm{dom}(\frac{f}{g})'_I$ holds $(\frac{f}{g})'_I(x) = (\frac{f'_I \cdot g - g'_I \cdot f}{g^2})(x)$. □

4. One-Sided Continuity

Now we state the proposition:

(26) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and a real number x_0. Suppose $x_0 \in \mathrm{dom}\, f$ and f is continuous in x_0. Then f is left continuous in x_0 and right continuous in x_0.

Let us consider a real number x_0 and a partial function f from $\mathbb{R}$ to $\mathbb{R}$. Now we state the propositions:

(27) f is left continuous in x_0 if and only if $x_0 \in \operatorname{dom} f$ and for every real number e such that $0 < e$ there exists a real number d such that $0 < d$ and for every real number x such that $x \in \operatorname{dom} f$ and $x_0 - d < x < x_0$ holds $|f(x) - f(x_0)| < e$.

(28) f is right continuous in x_0 if and only if $x_0 \in \operatorname{dom} f$ and for every real number e such that $0 < e$ there exists a real number d such that $0 < d$ and for every real number x such that $x \in \operatorname{dom} f$ and $x_0 < x < x_0 + d$ holds $|f(x) - f(x_0)| < e$.

(29) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and a real number x_0. Suppose f is left continuous in x_0 and right continuous in x_0. Then f is continuous in x_0.

Proof: For every real number e such that $0 < e$ there exists a real number d such that $0 < d$ and for every real number x such that $x \in \operatorname{dom} f$ and $|x - x_0| < d$ holds $|f(x) - f(x_0)| < e$. □

Let us consider a real number x_0 and a partial function f from $\mathbb{R}$ to $\mathbb{R}$. Now we state the propositions:

(30) Suppose f is left continuous in x_0 and for every real number r such that $r < x_0$ there exists a real number g such that $r < g < x_0$ and $g \in \operatorname{dom} f$. Then

(i) f is left convergent in x_0, and

(ii) $\lim_{x_0^-} f = f(x_0)$.

(31) Suppose f is right continuous in x_0 and for every real number r such that $x_0 < r$ there exists a real number g such that $g < r$ and $x_0 < g$ and $g \in \operatorname{dom} f$. Then

(i) f is right convergent in x_0, and

(ii) $\lim_{x_0^+} f = f(x_0)$.

(32) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and a real number x_0. Suppose $x_0 \in \operatorname{dom} f$ and f is right convergent in x_0 and $\lim_{x_0^+} f = f(x_0)$. Then f is right continuous in x_0.

(33) Let us consider a real number x_0, and a partial function f from $\mathbb{R}$ to $\mathbb{R}$. Suppose $x_0 \in \operatorname{dom} f$ and f is left convergent in x_0 and $\lim_{x_0^-} f = f(x_0)$. Then f is left continuous in x_0.

(34) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and a real number x_0. Suppose f is convergent in x_0 and $\lim_{x_0} f = f(x_0)$. Then f is continuous in x_0.

Proof: For every real number e such that $0 < e$ there exists a real number d such that $0 < d$ and for every real number x such that $x \in \operatorname{dom} f$ and $|x - x_0| < d$ holds $|f(x) - f(x_0)| < e$. □

From now on h denotes a non-zero, 0-convergent sequence of real numbers and c denotes a constant sequence of real numbers.

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$ and a real number x_0. Now we state the propositions:

(35) If f is right continuous in x_0, then $f\restriction[x_0, +\infty[$ is continuous in x_0.
PROOF: $x_0 \in \mathrm{dom}\, f$ and for every real number e such that $0 < e$ there exists a real number d such that $0 < d$ and for every real number x such that $x \in \mathrm{dom}\, f$ and $x_0 < x < x_0 + d$ holds $|f(x) - f(x_0)| < e$. Set $f_1 = f\restriction[x_0, +\infty[$. For every real number e such that $0 < e$ there exists a real number d such that $0 < d$ and for every real number x such that $x \in \mathrm{dom}\, f_1$ and $|x - x_0| < d$ holds $|f_1(x) - f_1(x_0)| < e$. □

(36) If f is left continuous in x_0, then $f\restriction]-\infty, x_0]$ is continuous in x_0.
PROOF: $x_0 \in \mathrm{dom}\, f$ and for every real number e such that $0 < e$ there exists a real number d such that $0 < d$ and for every real number x such that $x \in \mathrm{dom}\, f$ and $x_0 - d < x < x_0$ holds $|f(x) - f(x_0)| < e$. Set $f_1 = f\restriction]-\infty, x_0]$. For every real number e such that $0 < e$ there exists a real number d such that $0 < d$ and for every real number x such that $x \in \mathrm{dom}\, f_1$ and $|x - x_0| < d$ holds $|f_1(x) - f_1(x_0)| < e$. □

(37) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and a non empty interval I. If f is differentiable on interval I, then $f\restriction I$ is continuous.
PROOF: For every real number x such that $x \in \mathrm{dom}(f\restriction I)$ holds $f\restriction I$ is continuous in x. □

(38) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and non empty intervals I, J. Suppose f is differentiable on interval I and $J \subseteq I$ and $\inf J < \sup J$. Then

(i) f is differentiable on interval J, and

(ii) for every real number x such that $x \in J$ holds $f'_I(x) = f'_J(x)$.

PROOF: For every real number x such that $x \in J$ holds if $x = \inf J$, then $f\restriction J$ is right differentiable in x and if $x = \sup J$, then $f\restriction J$ is left differentiable in x and if $x \in]\inf J, \sup J[$, then f is differentiable in x. For every real number x such that $x \in J$ holds $f'_I(x) = f'_J(x)$. □

(39) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, an open subset Z of $\mathbb{R}$, and a non empty interval I. Suppose $I \subseteq Z$ and $\inf I < \sup I$ and f is differentiable on Z. Then f is differentiable on interval I.
PROOF: For every real number x such that $x \in I$ holds if $x = \inf I$, then $f\restriction I$ is right differentiable in x and if $x = \sup I$, then $f\restriction I$ is left differentiable in x and if $x \in]\inf I, \sup I[$, then f is differentiable in x. □

5. Chain Rule

From now on R, R_1, R_2 denote rests and L, L_1, L_2 denote linear functions.

Let us consider a real number x_0 and partial functions f, g from $\mathbb{R}$ to $\mathbb{R}$. Now we state the propositions:

(40) Suppose f is right differentiable in x_0 and g is differentiable in $f(x_0)$. Then

(i) $g \cdot f$ is right differentiable in x_0, and

(ii) $(g \cdot f)'_+(x_0) = g'(f(x_0)) \cdot f'_+(x_0)$.

Proof: Consider r being a real number such that $r > 0$ and $[x_0, x_0 + r] \subseteq \mathrm{dom}(g \cdot f)$. For every h and c such that $\mathrm{rng}\, c = \{x_0\}$ and $\mathrm{rng}(h + c) \subseteq \mathrm{dom}(g \cdot f)$ and for every natural number n, $h(n) > 0$ holds $h^{-1} \cdot ((g \cdot f_*(h + c)) - (g \cdot f_* c))$ is convergent and $\lim(h^{-1} \cdot ((g \cdot f_*(h + c)) - (g \cdot f_* c))) = g'(f(x_0)) \cdot f'_+(x_0)$. □

(41) Suppose f is left differentiable in x_0 and g is differentiable in $f(x_0)$. Then

(i) $g \cdot f$ is left differentiable in x_0, and

(ii) $(g \cdot f)'_-(x_0) = g'(f(x_0)) \cdot f'_-(x_0)$.

Proof: Consider r being a real number such that $r > 0$ and $[x_0 - r, x_0] \subseteq \mathrm{dom}(g \cdot f)$. For every h and c such that $\mathrm{rng}\, c = \{x_0\}$ and $\mathrm{rng}(h + c) \subseteq \mathrm{dom}(g \cdot f)$ and for every natural number n, $h(n) < 0$ holds $h^{-1} \cdot ((g \cdot f_*(h + c)) - (g \cdot f_* c))$ is convergent and $\lim(h^{-1} \cdot ((g \cdot f_*(h + c)) - (g \cdot f_* c))) = g'(f(x_0)) \cdot f'_-(x_0)$. □

(42) Suppose f is right differentiable in x_0 and g is right differentiable in $f(x_0)$ and for every real number r_1 such that $r_1 > 0$ there exists a real number r_0 such that $r_0 > 0$ and $[x_0, x_0 + r_0] \subseteq \mathrm{dom}([f(x_0), f(x_0) + r_1] \upharpoonright f)$. Then

(i) $g \cdot f$ is right differentiable in x_0, and

(ii) $(g \cdot f)'_+(x_0) = g'_+(f(x_0)) \cdot f'_+(x_0)$.

Proof: Consider r_1 being a real number such that $r_1 > 0$ and $[f(x_0), f(x_0) + r_1] \subseteq \mathrm{dom}\, g$. Consider r_0 being a real number such that $r_0 > 0$ and $[x_0, x_0 + r_0] \subseteq \mathrm{dom}([f(x_0), f(x_0) + r_1] \upharpoonright f)$. For every h and c such that $\mathrm{rng}\, c = \{x_0\}$ and $\mathrm{rng}(h + c) \subseteq \mathrm{dom}(g \cdot f)$ and for every natural number n, $h(n) > 0$ holds $h^{-1} \cdot ((g \cdot f_*(h + c)) - (g \cdot f_* c))$ is convergent and $\lim(h^{-1} \cdot ((g \cdot f_*(h + c)) - (g \cdot f_* c))) = g'_+(f(x_0)) \cdot f'_+(x_0)$. □

(43) Suppose f is left differentiable in x_0 and g is right differentiable in $f(x_0)$ and for every real number r_1 such that $r_1 > 0$ there exists a real number r_0 such that $r_0 > 0$ and $[x_0 - r_0, x_0] \subseteq \mathrm{dom}([f(x_0), f(x_0) + r_1] \upharpoonright f)$. Then

(i) $g \cdot f$ is left differentiable in x_0, and

(ii) $(g \cdot f)'_-(x_0) = g'_+(f(x_0)) \cdot f'_-(x_0)$.

PROOF: Consider r_1 being a real number such that $r_1 > 0$ and $[f(x_0), f(x_0) + r_1] \subseteq \operatorname{dom} g$. Consider r_0 being a real number such that $r_0 > 0$ and $[x_0 - r_0, x_0] \subseteq \operatorname{dom}([f(x_0), f(x_0) + r_1] \upharpoonright f)$. For every h and c such that $\operatorname{rng} c = \{x_0\}$ and $\operatorname{rng}(h + c) \subseteq \operatorname{dom}(g \cdot f)$ and for every natural number n, $h(n) < 0$ holds $h^{-1} \cdot ((g \cdot f_*(h + c)) - (g \cdot f_*c))$ is convergent and $\lim(h^{-1} \cdot ((g \cdot f_*(h + c)) - (g \cdot f_*c))) = g'_+(f(x_0)) \cdot f'_-(x_0)$. □

(44) Suppose f is differentiable in x_0 and g is right differentiable in $f(x_0)$ and for every real number r_1 such that $r_1 > 0$ there exists a real number r_0 such that $r_0 > 0$ and $[x_0 - r_0, x_0 + r_0] \subseteq \operatorname{dom}([f(x_0), f(x_0) + r_1] \upharpoonright f)$. Then

(i) $g \cdot f$ is differentiable in x_0, and

(ii) $(g \cdot f)'(x_0) = g'_+(f(x_0)) \cdot f'(x_0)$.

The theorem is a consequence of (42) and (43).

(45) Suppose f is right differentiable in x_0 and g is left differentiable in $f(x_0)$ and for every real number r_1 such that $r_1 > 0$ there exists a real number r_0 such that $r_0 > 0$ and $[x_0, x_0 + r_0] \subseteq \operatorname{dom}([f(x_0) - r_1, f(x_0)] \upharpoonright f)$. Then

(i) $g \cdot f$ is right differentiable in x_0, and

(ii) $(g \cdot f)'_+(x_0) = g'_-(f(x_0)) \cdot f'_+(x_0)$.

PROOF: Consider r_1 being a real number such that $r_1 > 0$ and $[f(x_0) - r_1, f(x_0)] \subseteq \operatorname{dom} g$. Consider r_0 being a real number such that $r_0 > 0$ and $[x_0, x_0 + r_0] \subseteq \operatorname{dom}([f(x_0) - r_1, f(x_0)] \upharpoonright f)$. For every h and c such that $\operatorname{rng} c = \{x_0\}$ and $\operatorname{rng}(h + c) \subseteq \operatorname{dom}(g \cdot f)$ and for every natural number n, $h(n) > 0$ holds $h^{-1} \cdot ((g \cdot f_*(h + c)) - (g \cdot f_*c))$ is convergent and $\lim(h^{-1} \cdot ((g \cdot f_*(h + c)) - (g \cdot f_*c))) = g'_-(f(x_0)) \cdot f'_+(x_0)$. □

(46) Suppose f is left differentiable in x_0 and g is left differentiable in $f(x_0)$ and for every real number r_1 such that $r_1 > 0$ there exists a real number r_0 such that $r_0 > 0$ and $[x_0 - r_0, x_0] \subseteq \operatorname{dom}([f(x_0) - r_1, f(x_0)] \upharpoonright f)$. Then

(i) $g \cdot f$ is left differentiable in x_0, and

(ii) $(g \cdot f)'_-(x_0) = g'_-(f(x_0)) \cdot f'_-(x_0)$.

PROOF: Consider r_1 being a real number such that $r_1 > 0$ and $[f(x_0) - r_1, f(x_0)] \subseteq \operatorname{dom} g$. Consider r_0 being a real number such that $r_0 > 0$ and $[x_0 - r_0, x_0] \subseteq \operatorname{dom}([f(x_0) - r_1, f(x_0)] \upharpoonright f)$. For every h and c such that $\operatorname{rng} c = \{x_0\}$ and $\operatorname{rng}(h + c) \subseteq \operatorname{dom}(g \cdot f)$ and for every natural number n, $h(n) < 0$ holds $h^{-1} \cdot ((g \cdot f_*(h + c)) - (g \cdot f_*c))$ is convergent and $\lim(h^{-1} \cdot ((g \cdot f_*(h + c)) - (g \cdot f_*c))) = g'_-(f(x_0)) \cdot f'_-(x_0)$. □

(47) Suppose f is differentiable in x_0 and g is left differentiable in $f(x_0)$ and for every real number r_1 such that $r_1 > 0$ there exists a real number r_0 such that $r_0 > 0$ and $[x_0 - r_0, x_0 + r_0] \subseteq \mathrm{dom}([f(x_0) - r_1, f(x_0)] \upharpoonleft f)$. Then

(i) $g \cdot f$ is differentiable in x_0, and

(ii) $(g \cdot f)'(x_0) = g'_{-}(f(x_0)) \cdot f'(x_0)$.

The theorem is a consequence of (45) and (46).

(48) Suppose f is right differentiable in x_0 and g is right differentiable in $f(x_0)$ and there exists a real number r such that $r > 0$ and $f \restriction [x_0, x_0 + r]$ is non-decreasing. Then

(i) $g \cdot f$ is right differentiable in x_0, and

(ii) $(g \cdot f)'_{+}(x_0) = g'_{+}(f(x_0)) \cdot f'_{+}(x_0)$.

Proof: Consider R being a real number such that $R > 0$ and $f \restriction [x_0, x_0 + R]$ is non-decreasing. $x_0 \in \mathrm{dom}\, f$. For every real number r_1 such that $r_1 > 0$ there exists a real number r_0 such that $r_0 > 0$ and $[x_0, x_0 + r_0] \subseteq \mathrm{dom}([f(x_0), f(x_0) + r_1] \upharpoonleft f)$. □

(49) Suppose f is left differentiable in x_0 and g is right differentiable in $f(x_0)$ and there exists a real number r such that $r > 0$ and $f \restriction [x_0 - r, x_0]$ is non-increasing. Then

(i) $g \cdot f$ is left differentiable in x_0, and

(ii) $(g \cdot f)'_{-}(x_0) = g'_{+}(f(x_0)) \cdot f'_{-}(x_0)$.

Proof: Consider R being a real number such that $R > 0$ and $f \restriction [x_0 - R, x_0]$ is non-increasing. $x_0 \in \mathrm{dom}\, f$. For every real number r_1 such that $r_1 > 0$ there exists a real number r_0 such that $r_0 > 0$ and $[x_0 - r_0, x_0] \subseteq \mathrm{dom}([f(x_0), f(x_0) + r_1] \upharpoonleft f)$. □

(50) Suppose f is right differentiable in x_0 and g is left differentiable in $f(x_0)$ and there exists a real number r such that $r > 0$ and $f \restriction [x_0, x_0 + r]$ is non-increasing. Then

(i) $g \cdot f$ is right differentiable in x_0, and

(ii) $(g \cdot f)'_{+}(x_0) = g'_{-}(f(x_0)) \cdot f'_{+}(x_0)$.

Proof: Consider R being a real number such that $R > 0$ and $f \restriction [x_0, x_0 + R]$ is non-increasing. $x_0 \in \mathrm{dom}\, f$. For every real number r_1 such that $r_1 > 0$ there exists a real number r_0 such that $r_0 > 0$ and $[x_0, x_0 + r_0] \subseteq \mathrm{dom}([f(x_0) - r_1, f(x_0)] \upharpoonleft f)$. □

(51) Suppose f is left differentiable in x_0 and g is left differentiable in $f(x_0)$ and there exists a real number r such that $r > 0$ and $f \restriction [x_0 - r, x_0]$ is non-decreasing. Then

(i) $g \cdot f$ is left differentiable in x_0, and

(ii) $(g \cdot f)'_{-}(x_0) = g'_{-}(f(x_0)) \cdot f'_{-}(x_0)$.

Proof: Consider R being a real number such that $R > 0$ and $f \restriction [x_0 - R, x_0]$ is non-decreasing. $x_0 \in \operatorname{dom} f$. For every real number r_1 such that $r_1 > 0$ there exists a real number r_0 such that $r_0 > 0$ and $[x_0 - r_0, x_0] \subseteq \operatorname{dom}([f(x_0) - r_1, f(x_0)] \upharpoonleft f)$. □

(52) Chain rule:

Let us consider partial functions f, g from $\mathbb{R}$ to $\mathbb{R}$, and non empty intervals I, J. Suppose f is differentiable on interval I and g is differentiable on interval J and $f^{\circ}I \subseteq J$. Then

(i) $g \cdot f$ is differentiable on interval I, and

(ii) $(g \cdot f)'_I = g'_J \cdot f \cdot f'_I$.

The theorem is a consequence of (4), (5), (11), and (3).

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