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# Formalization of Orthogonal Decomposition for Hilbert Spaces

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**Summary.** In this article, we formalize the theorems about orthogonal decomposition of Hilbert spaces, using the Mizar system [1], [2]. For any subspace $S$ of a Hilbert space $H$, any vector can be represented by the sum of a vector in $S$ and a vector orthogonal to $S$. The formalization of orthogonal complements of Hilbert spaces has been stored in the Mizar Mathematical Library [4]. We referred to [5] and [6] in the formalization.

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## 1. Preliminaries

From now on $X$ denotes a real unitary space and $x$, $y$, $y_1$, $y_2$ denote points of $X$. Now we state the proposition:

(1) Let us consider a real unitary space $X$, points $x$, $y$ of $X$, and points $z$, $t$ of MetricSpaceNorm(the real normed space of $X$). If $x = z$ and $y = t$, then $\|x - y\| = \rho(z, t)$.

Let us consider a real unitary space $X$, an element $z$ of MetricSpaceNorm(the real normed space of $X$), and a real number $r$. Now we state the propositions:

(2) There exists a point $x$ of $X$ such that

(i) $x = z$, and

(ii) $\mathrm{Ball}(z, r) = \{y$, where $y$ is a point of $X : \|x - y\| < r\}$.

The theorem is a consequence of (1).

(3) There exists a point $x$ of $X$ such that

(i) $x = z$, and

(ii) $\overline{\mathrm{Ball}}(z, r) = \{y$, where $y$ is a point of $X : \|x - y\| \leqslant r\}$.

The theorem is a consequence of (1).

(4) Let us consider a real unitary space $X$, a sequence $S$ of $X$, a sequence $S_1$ of MetricSpaceNorm(the real normed space of $X$), a point $x$ of $X$, and a point $x_2$ of MetricSpaceNorm(the real normed space of $X$). Suppose $S = S_1$ and $x = x_2$. Then $S_1$ is convergent to $x_2$ if and only if for every real number $r$ such that $0 < r$ there exists a natural number $m$ such that for every natural number $n$ such that $m \leqslant n$ holds $\|S(n) - x\| < r$. The theorem is a consequence of (1).

Let us consider a real unitary space $X$, a sequence $S$ of $X$, and a sequence $S_1$ of MetricSpaceNorm(the real normed space of $X$). Now we state the propositions:

(5) If $S = S_1$, then $S_1$ is convergent iff $S$ is convergent. The theorem is a consequence of (4).

(6) If $S = S_1$ and $S_1$ is convergent, then $\lim S_1 = \lim S$. The theorem is a consequence of (5) and (4).

## 2. Topological Space Generated from Real Unitary Space

Now we state the proposition:

(7) Let us consider a real unitary space $X$, and a subset $V$ of TopSpaceNorm (the real normed space of $X$). Then $V$ is open if and only if for every point $x$ of $X$ such that $x \in V$ there exists a real number $r$ such that $r > 0$ and $\{y$, where $y$ is a point of $X : \|x - y\| < r\} \subseteq V$. The theorem is a consequence of (2).

Let us consider a real unitary space $X$, a point $x$ of $X$, and a real number $r$. Now we state the propositions:

(8) $\{y$, where $y$ is a point of $X : \|x - y\| < r\}$ is an open subset of TopSpaceNorm(the real normed space of $X$). The theorem is a consequence of (2).

(9) $\{y$, where $y$ is a point of $X : \|x - y\| \leqslant r\}$ is a closed subset of TopSpaceNorm(the real normed space of $X$). The theorem is a consequence of (3).

(10) Let us consider a real unitary space $M$, a subset $X$ of TopSpaceNorm(the real normed space of $M$), and an object $x$. Then $x \in \overline{X}$ if and only if there exists a sequence $S$ of $M$ such that for every natural number $n$, $S(n) \in X$ and $S$ is convergent and $\lim S = x$. The theorem is a consequence of (5) and (6).

(11) Let us consider a real unitary space $M$, and a subset $X$ of TopSpaceNorm (the real normed space of $M$). Then $X$ is closed if and only if for every sequence $S$ of $M$ such that for every natural number $n$, $S(n) \in X$ and $S$ is convergent holds $\lim S \in X$. The theorem is a consequence of (5) and (6).

(12) Let us consider a real unitary space $S$, and a subset $X$ of $S$. Then $X$ is a closed subset of TopSpaceNorm(the real normed space of $S$) if and only if for every sequence $s_1$ of $S$ such that $\operatorname{rng} s_1 \subseteq X$ and $s_1$ is convergent holds $\lim s_1 \in X$. The theorem is a consequence of (11).

(13) Let us consider a real unitary space $S$, a point $x$ of $S$, a point $y$ of MetricSpaceNorm(the real normed space of $S$), and a real number $r$. If $x = y$, then $\mathrm{Ball}(x, r) = \mathrm{Ball}(y, r)$. The theorem is a consequence of (1).

(14) Let us consider a real unitary space $S$. Then TopSpaceNorm(the real normed space of $S$) = TopUnitSpace $S$. The theorem is a consequence of (13).

Let us consider a real unitary space $S$, a subset $U$ of $S$, and a subset $V$ of TopSpaceNorm(the real normed space of $S$). Now we state the propositions:

(15) If $U = V$, then $U$ is closed iff $V$ is closed.

(16) If $U = V$, then $U$ is open iff $V$ is open.

(17) Let us consider a real unitary space $X$, a subspace $M$ of $X$, and points $x$, $m_0$ of $X$. Suppose $m_0 \in M$. Then for every point $m$ of $X$ such that $m \in M$ holds $\|x - m_0\| \leqslant \|x - m\|$ if and only if for every point $m$ of $X$ such that $m \in M$ holds $((x - m_0)|m) = 0$.

(18) Let us consider a real unitary space $X$, a subspace $M$ of $X$, and points $x$, $m_1$, $m_2$ of $X$. Suppose $m_1, m_2 \in M$ and for every point $m$ of $X$ such that $m \in M$ holds $\|x - m_1\| \leqslant \|x - m\|$ and for every point $m$ of $X$ such that $m \in M$ holds $\|x - m_2\| \leqslant \|x - m\|$. Then $m_1 = m_2$.

(19) Let us consider a real Hilbert space of $X$, a subspace $M$ of $X$, and a point $x$ of $X$. Suppose the carrier of $M$ is a closed subset of TopSpaceNorm(the real normed space of $X$). Then there exists a point $m_0$ of $X$ such that

   (i) $m_0 \in M$, and

   (ii) for every point $m$ of $X$ such that $m \in M$ holds $\|x - m_0\| \leqslant \|x - m\|$.

The theorem is a consequence of (12).

Let $X$ be a real unitary space and $M$ be a subset of $X$. The functor OrtCompSet($M$) yielding a non empty subset of $X$ is defined by

(Def. 1) for every point $x$ of $X$, $x \in it$ iff for every point $y$ of $X$ such that $y \in M$ holds $(y|x) = 0$.

Now we state the propositions:

(20) Let us consider a real unitary space $X$, and a subset $M$ of $X$. Then OrtCompSet($M$) is linearly closed.

PROOF: For every vectors $v$, $u$ of $X$ such that $v$, $u \in$ OrtCompSet($M$) holds $v+u \in$ OrtCompSet($M$). For every real number $a$ and for every vector $v$ of $X$ such that $v \in$ OrtCompSet($M$) holds $a \cdot v \in$ OrtCompSet($M$). □

(21) Let us consider a real unitary space $X$, a non empty subset $M$ of $X$, and a sequence $s_2$ of $X$. Suppose $\operatorname{rng} s_2 \subseteq$ the carrier of OrtComp($M$) and $s_2$ is convergent. Then $\lim s_2 \in$ the carrier of OrtComp($M$).

(22) Let us consider a real unitary space $S$, a non empty subset $M$ of $S$, and a subset $L$ of $S$. Suppose $L =$ the carrier of OrtComp($M$). Then $L$ is a closed subset of TopSpaceNorm(the real normed space of $S$). The theorem is a consequence of (21) and (12).

(23) Let us consider a real unitary space $X$. Then every non empty subset of $X$ is a subset of OrtComp(OrtComp($M$)).

(24) Let us consider a real unitary space $X$, and non empty subsets $S$, $T$ of $X$. Suppose $S \subseteq T$. Then OrtComp($T$) is a subspace of OrtComp($S$).

(25) Let us consider a real Hilbert space of $X$, and a subspace $M$ of $X$. Suppose $X$ is strict and the carrier of $M$ is a closed subset of TopSpaceNorm(the real normed space of $X$). Then $X$ is the direct sum of $M$ and OrtComp($M$).

PROOF: For every object $z$, $z \in$ the carrier of $M +$ OrtComp($M$) iff $z \in$ the carrier of $X$. For every object $z$, $z \in$ the carrier of $M \cap$ OrtComp($M$) iff $z \in \{0_X\}$. □

(26) Let us consider a real Hilbert space of $X$, and a strict subspace $M$ of $X$. Suppose $X$ is strict and the carrier of $M$ is a closed subset of TopSpaceNorm(the real normed space of $X$).
Then $M =$ OrtComp(OrtComp($M$)).

PROOF: Reconsider $N =$ the carrier of $M$ as a subset of $X$. $N$ is a subset of OrtComp(OrtComp($N$)). The carrier of OrtComp(OrtComp($M$)) $\subseteq N$. □

(27) Let us consider a real unitary space $X$, a subspace $M$ of $X$, a subset $K$ of $X$, and a subset $L$ of TopSpaceNorm(the real normed space of $X$). Suppose the carrier of $M = L$ and $K = \overline{L}$. Then $K$ is linearly closed.

Proof: For every vectors $v$, $u$ of $X$ such that $v, u \in K$ holds $v + u \in K$. For every real number $a$ and for every vector $v$ of $X$ such that $v \in K$ holds $a \cdot v \in K$ by (10), [3, (15)]. □

(28) Let us consider a real Hilbert space of $X$, and a non empty subset $M$ of $X$. Suppose $X$ is strict. Then

- (i) the carrier of OrtComp(OrtComp($M$)) is a closed subset of TopSpaceNorm(the real normed space of $X$), and
- (ii) there exists a subset $L$ of TopSpaceNorm(the real normed space of $X$) such that $L =$ the carrier of Lin($M$) and the carrier of OrtComp(OrtComp($M$)) $= \overline{L}$, and
- (iii) Lin($M$) is a subspace of OrtComp(OrtComp($M$)).

(29) Let us consider a real Hilbert space of $X$, a strict subspace $K$ of $X$, and a non empty subset $M$ of $X$. Suppose $X$ is strict and the carrier of $K$ is a closed subset of TopSpaceNorm(the real normed space of $X$) and Lin($M$) is a subspace of $K$. Then OrtComp(OrtComp($M$)) is a subspace of $K$.

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