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Existence and Uniqueness of Algebraic Closures

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Summary. This is the second part of a two-part article formalizing existence and uniqueness of algebraic closures, using the Mizar [2], [1] formalism. Our proof follows Artin's classical one as presented by Lang in [3]. In the first part we proved that for a given field F there exists a field extension E such that every non-constant polynomial $p \in F[X]$ has a root in E. Artin's proof applies Kronecker's construction to each polynomial $p \in F[X]\backslash F$ simultaneously. To do so we needed the polynomial ring $F[X_1, X_2, ...]$ with infinitely many variables, one for each polynomal $p \in F[X]\backslash F$. The desired field extension E then is $F[X_1, X_2, ...]\backslash I$, where I is a maximal ideal generated by all non-constant polynomials $p \in F[X]$. Note, that to show that I is maximal Zorn's lemma has to be applied.

In this second part this construction is iterated giving an infinite sequence of fields, whose union establishes a field extension A of F, in which every non-constant polynomial $p \in A[X]$ has a root. The field of algebraic elements of A then is an algebraic closure of F. To prove uniqueness of algebraic closures, e.g. that two algebraic closures of F are isomorphic over F, the technique of extending monomorphisms is applied: a monomorphism $F \longrightarrow A$, where A is an algebraic closure of F can be extended to a monomorphism $E \longrightarrow A$, where E is any algebraic extension of F. In case that E is algebraically closed this monomorphism is an isomorphism. Note that the existence of the extended monomorphism again relies on Zorn's lemma.

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1. Preliminaries

Let L be a non empty double loop structure. One can verify that the double loop structure of L is non empty. Let L be a non trivial double loop structure. One can verify that the double loop structure of L is non trivial. Let L be a non degenerated double loop structure. One can verify that the double loop structure of L is non degenerated. Let L be an add-associative double loop structure. One can check that the double loop structure of L is add-associative.

Let L be a right zeroed double loop structure. Let us note that the double loop structure of L is right zeroed. Let L be a right complementable double loop structure. Observe that the double loop structure of L is right complementable. Let L be an Abelian double loop structure. Let us observe that the double loop structure of L is Abelian. Let L be an associative double loop structure. One can check that the double loop structure of L is associative.

Let L be a well unital, non empty double loop structure. Observe that the double loop structure of L is well unital. Let L be a left distributive, non empty double loop structure. One can check that the double loop structure of L is left distributive. Let L be a right distributive, non empty double loop structure. Observe that the double loop structure of L is right distributive. Let L be a commutative double loop structure. One can verify that the double loop structure of L is commutative.

Let L be an integral domain-like, non empty double loop structure. Let us note that the double loop structure of L is integral domain-like. Let L be an almost left invertible double loop structure. Observe that the double loop structure of L is almost left invertible. Now we state the proposition:

(1) Let us consider a field F. Then the double loop structure of $F \approx F$.

Let F be a field. Let us note that there exists an extension of F which is strict. Let L be an F-monomorphic field. Let us note that there exists an extension of L which is F-homomorphic and F-monomorphic and there exists an element of the carrier of PolyRing(F) which is monic and irreducible. Let F be a non algebraic closed field. Observe that there exists an element of the carrier of PolyRing(F) which is monic and non constant and has not roots. Now we state the propositions:

(2) Let us consider a field F_1, an F_1-monomorphic, F_1-homomorphic field F_2, a monomorphism h of F_1 and F_2, and an element p of the carrier of PolyRing(F_1). Then $(\mathrm{PolyHom}(h))(-p) = -(\mathrm{PolyHom}(h))(p)$.

(3) Let us consider a field F_1, an F_1-monomorphic, F_1-homomorphic field F_2, a monomorphism h of F_1 and F_2, and elements p, q of the carrier of PolyRing(F_1). If $p \mid q$, then $(\mathrm{PolyHom}(h))(p) \mid (\mathrm{PolyHom}(h))(q)$.

Let F_1 be a field, F_2 be an F_1-monomorphic, F_1-homomorphic field, h be a monomorphism of F_1 and F_2, and p be a non constant element of the carrier of PolyRing(F_1). Let us observe that (PolyHom(h))(p) is non constant as an element of the carrier of PolyRing(F_2).

Let R be a GCD domain and a, b be elements of R. We say that a and b are relatively prime if and only if

(Def. 1) 1_R is a GCD of a and b.

Let us consider a field F and elements p, q of the carrier of PolyRing(F). Now we state the propositions:

(4) p and q are relatively prime if and only if $\gcd(p, q) = \mathbf{1}.F$.

(5) If p and q are relatively prime, then p and q have no common roots.

(6) Let us consider a field F, and an element p of the carrier of PolyRing(F). Then there exists an extension E of F and there exists an F-algebraic element a of E such that $p = \text{MinPoly}(a, F)$ if and only if p is monic and irreducible.

(7) Let us consider a field F, and an irreducible element p of the carrier of PolyRing(F). Then there exists an F-finite extension E of F such that

(i) $\deg(E, F) = \deg(p)$, and

(ii) p has a root in E.

The theorem is a consequence of (6).

(8) Let us consider a field F, and a non constant element p of the carrier of PolyRing(F). Then there exists an F-finite extension E of F such that

(i) p has a root in E, and

(ii) $\deg(E, F) \leqslant \deg(p)$.

The theorem is a consequence of (7).

(9) Let us consider a field F, an F-algebraic extension E of F, an E-extending extension K of F, and an element a of K. If a is E-algebraic, then a is F-algebraic.

(10) Let us consider fields F_1, F_2, L, an extension E_1 of F_1, a E_1-extending extension K_1 of F_1, a function h_1 from F_1 into L, a function h_2 from E_1 into L, and a function h_3 from K_1 into L. Suppose h_2 is h_1-extending and h_3 is h_2-extending. Then h_3 is h_1-extending.

Let F be a field. Let us observe that every extension of F is F-monomorphic and F-homomorphic.

Let E be an extension of F. Let us note that there exists a field which is E-homomorphic, E-monomorphic, F-homomorphic, and F-monomorphic.

2. Sequences of Fields

A sequence is a function defined by

(Def. 2) dom $it = \mathbb{N}$.

Let us observe that every sequence is $\mathbb{N}$-defined.

Let f be a binary relation. We say that f is field-yielding if and only if

(Def. 3) for every object x such that $x \in \operatorname{rng} f$ holds x is a field.

Observe that there exists a sequence which is field-yielding and every function which is field-yielding is also 1-sorted yielding.

Let f be a field-yielding sequence and i be an element of $\mathbb{N}$. One can check that the functor $f(i)$ yields a field. Let i be a natural number. Observe that the functor $f(i)$ yields a field.

The scheme *RecExField* deals with a field $\mathcal{A}$ and a ternary predicate $\mathcal{P}$ and states that

(Sch. 1) There exists a field-yielding sequence f such that $f(0) = \mathcal{A}$ and for every natural number n, $\mathcal{P}[n, f(n), f(n+1)]$

provided

- for every natural number n and for every field x, there exists a field y such that $\mathcal{P}[n, x, y]$.

Let f be a field-yielding sequence. We say that f is ascending if and only if

(Def. 4) for every element i of $\mathbb{N}$, $f(i+1)$ is an extension of $f(i)$.

Note that there exists a field-yielding sequence which is ascending.

Let f be a field-yielding sequence. The support of f yielding a non empty set is defined by the term

(Def. 5) $\bigcup$the set of all the carrier of $f(i)$ where i is an element of $\mathbb{N}$.

Now we state the propositions:

(11) Let us consider an ascending, field-yielding sequence f, elements i, j of $\mathbb{N}$, and an element a of $f(i)$. If $i \leqslant j$, then $a \in$ the carrier of $f(j)$.
Proof: Define $\mathcal{P}[\text{natural number}] \equiv$ there exists an element k of $\mathbb{N}$ such that $k = i + \$_1$ and $a \in$ the carrier of $f(k)$. For every natural number k, $\mathcal{P}[k]$. Consider n being a natural number such that $i + n = j$. □

(12) Let us consider an ascending, field-yielding sequence f, and elements i, j of $\mathbb{N}$. If $i \leqslant j$, then $f(j)$ is an extension of $f(i)$.
Proof: Define $\mathcal{P}[\text{natural number}] \equiv$ there exists an element k of $\mathbb{N}$ such that $k = i + \$_1$ and $f(k)$ is an extension of $f(i)$. $\mathcal{P}[0]$. For every natural number k, $\mathcal{P}[k]$. Consider n being a natural number such that $i + n = j$. □

(13) Let us consider an ascending, field-yielding sequence f, elements i, j of $\mathbb{N}$, elements x_2, y_2 of $f(i)$, and elements x_3, y_3 of $f(j)$. Suppose $x_2 = x_3$ and $y_2 = y_3$. Then

(i) $x_2 + y_2 = x_3 + y_3$, and

(ii) $x_2 \cdot y_2 = x_3 \cdot y_3$.

The theorem is a consequence of (12).

Let f be an ascending, field-yielding sequence. The functor addseq(f) yielding a binary operation on the support of f is defined by

(Def. 6) for every elements a, b of the support of f, there exists an element i of $\mathbb{N}$ and there exist elements x, y of $f(i)$ such that $x = a$ and $y = b$ and $it(a, b) = x + y$.

The functor multseq(f) yielding a binary operation on the support of f is defined by

(Def. 7) for every elements a, b of the support of f, there exists an element i of $\mathbb{N}$ and there exist elements x, y of $f(i)$ such that $x = a$ and $y = b$ and $it(a, b) = x \cdot y$.

The functor SeqField(f) yielding a strict double loop structure is defined by

(Def. 8) the carrier of it = the support of f and the addition of it = addseq(f) and the multiplication of it = multseq(f) and the one of $it = 1_{f(0)}$ and the zero of $it = 0_{f(0)}$.

Now we state the propositions:

(14) Let us consider an ascending, field-yielding sequence f, and an element i of $\mathbb{N}$. Then

(i) $1_{\mathrm{SeqField}(f)} = 1_{f(i)}$, and

(ii) $0_{\mathrm{SeqField}(f)} = 0_{f(i)}$.

Proof: Define $\mathcal{P}$[natural number] $\equiv$ there exists an element k of $\mathbb{N}$ such that $k = \$_1$ and $1_{f(k)} = 1_{f(0)}$ and $0_{f(k)} = 0_{f(0)}$. For every natural number k, $\mathcal{P}[k]$. $\square$

(15) Let us consider an ascending, field-yielding sequence f, elements a, b of SeqField(f), an element i of $\mathbb{N}$, and elements x, y of $f(i)$. If $x = a$ and $y = b$, then $a + b = x + y$ and $a \cdot b = x \cdot y$. The theorem is a consequence of (13).

Let f be an ascending, field-yielding sequence. Observe that SeqField(f) is non degenerated and SeqField(f) is Abelian, add-associative, right zeroed, and right complementable and SeqField(f) is commutative, associative, well unital, distributive, and almost left invertible. Now we state the propositions:

(16) Let us consider an ascending, field-yielding sequence f, and an element i of $\mathbb{N}$. Then $f(i)$ is a subfield of SeqField(f).
PROOF: Set $F = f(i)$. Set $K =$ SeqField(f). The addition of $F =$ (the addition of K) $\restriction$ (the carrier of F). The multiplication of $F =$ (the multiplication of K) $\restriction$ (the carrier of F). $1_F = 1_K$ and $0_F = 0_K$. $\square$

(17) Let us consider a field E, and an ascending, field-yielding sequence f. Suppose for every element i of $\mathbb{N}$, $f(i)$ is a subfield of E. Then SeqField(f) is a subfield of E.
PROOF: Set $F =$ SeqField(f). The carrier of $F \subseteq$ the carrier of K. The addition of $F =$ (the addition of K) $\restriction$ (the carrier of F). The multiplication of $F =$ (the multiplication of K) $\restriction$ (the carrier of F). $\square$

(18) Let us consider an ascending, field-yielding sequence f, and a finite subset X of SeqField(f). Then there exists an element i of $\mathbb{N}$ such that $X \subseteq$ the carrier of $f(i)$.
PROOF: Define $\mathcal{P}$[natural number] $\equiv$ for every finite subset X of SeqField(f) such that $\overline{\overline{X}} = \$_1$ there exists an element i of $\mathbb{N}$ such that $X \subseteq$ the carrier of $f(i)$. $\mathcal{P}[0]$. $\mathcal{P}[1]$. For every natural number k, $\mathcal{P}[k]$. Consider n being a natural number such that $\overline{\overline{X}} = n$. Consider i being an element of $\mathbb{N}$ such that $X \subseteq$ the carrier of $f(i)$. $\square$

3. Maximal Algebraic and Algebraic Closed Fields

Let F be a field. We say that F is maximal algebraic if and only if

(Def. 9) for every F-algebraic extension E of F, $E \approx F$.

Let us consider a field F. Now we state the propositions:

(19) F is maximal algebraic if and only if F is algebraic closed. The theorem is a consequence of (7).

(20) F is algebraic closed if and only if every non constant polynomial over F has roots.

(21) F is algebraic closed if and only if for every irreducible element p of the carrier of PolyRing(F), $\deg(p) = 1$.

(22) F is algebraic closed if and only if for every non constant polynomial p over F, p splits in F.

(23) F is algebraic closed if and only if every non constant, monic polynomial over F is a product of linear polynomials of F.

(24) F is algebraic closed if and only if for every elements p, q of the carrier of PolyRing(F), p and q are relatively prime iff p and q have no common roots. The theorem is a consequence of (4) and (5).

(25) F is algebraic closed if and only if for every F-algebraic extension E of F, $E \approx F$. The theorem is a consequence of (19).

(26) F is algebraic closed if and only if for every F-finite extension E of F, $E \approx F$. The theorem is a consequence of (19).

Let us note that every field which is algebraic closed is also infinite.

4. Existence of Algebraic Closures

Let F be a field. A closure sequence of F is an ascending, field-yielding sequence defined by

(Def. 10) $it(0) = F$ and for every element i of $\mathbb{N}$ and for every field K and for every extension E of K such that $K = it(i)$ and $E = it(i+1)$ for every non constant element p of the carrier of PolyRing(K), p has a root in E.

Now we state the proposition:

(27) Let us consider an ascending, field-yielding sequence f, and a polynomial p over SeqField(f). Then there exists an element i of $\mathbb{N}$ such that p is a polynomial over $f(i)$. The theorem is a consequence of (18) and (16).

Let F be a field and f be a closure sequence of F. Let us observe that SeqField(f) is F-extending and SeqField(f) is algebraic closed.

Now we state the proposition:

(28) Let us consider a field F. Then there exists an extension E of F such that E is algebraic closed.

Let F be a field. An algebraic closure of F is an extension of F defined by

(Def. 11) *it* is F-algebraic and algebraic closed.

Note that every algebraic closure of F is F-algebraic and algebraic closed and there exists an algebraic closed field which is F-homomorphic and F-monomorphic. Now we state the propositions:

(29) Let us consider a field F. Then there exists a field E such that E is an algebraic closure of F.

(30) Let us consider a field F, and an F-algebraic extension E of F. Then there exists an algebraic closure A of F such that E is a subfield of A.

Let F be a field and E be an F-algebraic extension of F. Let us observe that there exists an algebraic closure of F which is E-extending.

Now we state the propositions:

(31) Let us consider a field F, and an F-algebraic extension E of F. Then every algebraic closure of E is an algebraic closure of F.

(32) Let us consider a field F, an extension E of F, and an algebraic closure A of F. If A is E-extending, then A is an algebraic closure of E.

(33) Let us consider a field F, and algebraic closures A_1, A_2 of F. If A_1 is A_2-extending, then $A_2 \approx A_1$. The theorem is a consequence of (25).

5. Some More Preliminaries

Let R be a ring and S be an R-homomorphic ring. Observe that there exists a ring which is S-homomorphic and R-homomorphic.

Let T be an S-homomorphic ring, f be an additive function from R into S, and g be an additive function from S into T. Let us note that $g \cdot f$ is additive as a function from R into T.

Let f be a multiplicative function from R into S and g be a multiplicative function from S into T. Let us note that $g \cdot f$ is multiplicative as a function from R into T.

Let f be a unity-preserving function from R into S and g be a unity-preserving function from S into T. Let us note that $g \cdot f$ is unity-preserving as a function from R into T. Now we state the propositions:

(34) Let us consider a field F, and an extension E of F. Then id_F is a monomorphism of F and E.

PROOF: Reconsider $f = \mathrm{id}_F$ as a function from F into E. f is additive, multiplicative, unity-preserving, and monomorphic. □

(35) Let us consider a ring R, an R-homomorphic ring S, an S-homomorphic, R-homomorphic ring T, an additive function f from R into S, and an additive function g from S into T. Then $\mathrm{PolyHom}(g \cdot f) = \mathrm{PolyHom}(g) \cdot \mathrm{PolyHom}(f)$.

(36) Let us consider a ring R, an R-homomorphic ring S, an R-homomorphic, S-homomorphic ring T, an additive function f from R into S, and an additive function g from S into T. Suppose $g \cdot f = \mathrm{id}_R$. Then $\mathrm{PolyHom}(g \cdot f) = \mathrm{id}_{\mathrm{PolyRing}(R)}$. The theorem is a consequence of (35).

(37) Let us consider fields F_1, F_2, and an extension E of F_1. If $F_1 \approx F_2$, then E is an extension of F_2.

(38) Let us consider fields F_1, F_2. Suppose $F_1 \approx F_2$. Then

(i) $\mathbf{0}.F_1 = \mathbf{0}.F_2$, and

(ii) $\mathbf{1}.F_1 = \mathbf{1}.F_2$.

(39) Let us consider fields F_1, F_2, and a polynomial p over F_1. If $F_1 \approx F_2$, then p is a polynomial over F_2.

(40) Let us consider fields F_1, F_2, and a non zero polynomial p over F_1. If $F_1 \approx F_2$, then p is a non zero polynomial over F_2. The theorem is a consequence of (39) and (38).

(41) Let us consider fields F_1, F_2, a polynomial p over F_1, a polynomial q over F_2, an element a of F_1, and an element b of F_2. Suppose $F_1 \approx F_2$ and $p = q$ and $a = b$. Then $\mathrm{eval}(p, a) = \mathrm{eval}(q, b)$.

(42) Let us consider fields F_1, F_2, an extension E_1 of F_1, an extension E_2 of F_2, a polynomial p over F_1, a polynomial q over F_2, an element a of E_1, and an element b of E_2. Suppose $F_1 \approx F_2$ and $E_1 \approx E_2$ and $p = q$ and $a = b$. Then $\mathrm{ExtEval}(p, a) = \mathrm{ExtEval}(q, b)$. The theorem is a consequence of (41).

(43) Let us consider fields F_1, F_2, and an F_1-algebraic extension E of F_1. If $F_1 \approx F_2$, then E is an F_2-algebraic extension of F_2. The theorem is a consequence of (37), (40), and (42).

(44) Let us consider fields F_1, F_2, and an algebraic closure E of F_1. If $F_1 \approx F_2$, then E is an algebraic closure of F_2. The theorem is a consequence of (43).

Let X be a set. We say that X is field-membered if and only if

(Def. 12) for every object x such that $x \in X$ holds x is a field.

Observe that there exists a set which is field-membered and non empty.

Let X be a non empty, field-membered set.

One can check that an element of X is a field. Let F be a field. The functor $\mathrm{SubFields}(F)$ yielding a set is defined by

(Def. 13) for every object o, $o \in \mathit{it}$ iff there exists a strict field K such that $o = K$ and K is a subfield of F.

One can check that $\mathrm{SubFields}(F)$ is non empty and field-membered. Now we state the proposition:

(45) Let us consider fields F, K. Then $K \in \mathrm{SubFields}(F)$ if and only if K is a strict subfield of F.

6. Uniqueness of Algebraic Closures

Let F be a field, E be an extension of F, L be an F-monomorphic field, and f be a monomorphism of F and L. The functor $\mathrm{ExtSet}(f, E)$ yielding a non empty set is defined by the term

(Def. 14) $\{\langle K, g\rangle$, where K is an element of $\mathrm{SubFields}(E)$, g is a function from K into L : there exists an extension K_1 of F and there exists a function g_1 from K_1 into L such that $K_1 = K$ and $g_1 = g$ and g_1 is monomorphic and f-extending$\}$.

Note that every element of $\mathrm{ExtSet}(f, E)$ is pair.

Let p be an element of $\mathrm{ExtSet}(f, E)$. One can verify that the functor $(p)_{\mathbf{1}}$ yields a strict extension of F. One can verify that the functor $(p)_{\mathbf{2}}$ yields a function from $(p)_{\mathbf{1}}$ into L. Now we state the proposition:

(46) Let us consider a field F, an extension E of F, an F-monomorphic field L, a monomorphism f of F and L, a strict extension K of F, and a function g from K into L. Suppose g is monomorphic. Then $\langle K, g\rangle \in \mathrm{ExtSet}(f, E)$ if and only if E is an extension of K and F is a subfield of K and g is f-extending. The theorem is a consequence of (45).

Let F be a field, E be an extension of F, L be an F-monomorphic field, f be a monomorphism of F and L, and p, q be elements of $\mathrm{ExtSet}(f, E)$. We say that $p \leqslant q$ if and only if

(Def. 15) $(q)_{\mathbf{1}}$ is an extension of $(p)_{\mathbf{1}}$ and for every extension K of $(p)_{\mathbf{1}}$ and for every function g from K into L such that $K = (q)_{\mathbf{1}}$ and $g = (q)_{\mathbf{2}}$ holds g is $(p)_{\mathbf{2}}$-extending.

Let S be a non empty subset of $\mathrm{ExtSet}(f, E)$. We say that S is ascending if and only if

(Def. 16) for every elements p, q of S, $p \leqslant q$ or $q \leqslant p$.

One can check that there exists a non empty subset of $\mathrm{ExtSet}(f, E)$ which is ascending. Now we state the propositions:

(47) Let us consider a field F, an extension E of F, an F-monomorphic field L, a monomorphism f of F and L, and an element p of $\mathrm{ExtSet}(f, E)$. Then $p \leqslant p$.

(48) Let us consider a field F, an extension E of F, an F-monomorphic field L, a monomorphism f of F and L, and elements p, q of $\mathrm{ExtSet}(f, E)$. If $p \leqslant q \leqslant p$, then $p = q$.

(49) Let us consider a field F, an extension E of F, an F-monomorphic field L, a monomorphism f of F and L, and elements p, q, r of $\mathrm{ExtSet}(f, E)$. If $p \leqslant q \leqslant r$, then $p \leqslant r$.

Let F be a field, E be an extension of F, L be an F-monomorphic field, f be a monomorphism of F and L, and S be a non empty subset of $\mathrm{ExtSet}(f, E)$. The functor $\mathrm{unionCarrier}(S, f, E)$ yielding a non empty set is defined by the term

(Def. 17) $\bigcup$ the set of all the carrier of $(p)_{\mathbf{1}}$ where p is an element of S.

Let S be an ascending, non empty subset of $\mathrm{ExtSet}(f, E)$. The functors: $\mathrm{unionAdd}(S, f, E)$ and $\mathrm{unionMult}(S, f, E)$ yielding binary operations on union $\mathrm{Carrier}(S, f, E)$ are defined by conditions

(Def. 18) for every elements a, b of $\mathrm{unionCarrier}(S, f, E)$, there exists an element p of S and there exist elements x, y of $(p)_{\mathbf{1}}$ such that $x = a$ and $y = b$ and unionAdd$(S, f, E)(a, b) = x + y$,

(Def. 19) for every elements a, b of unionCarrier(S, f, E), there exists an element p of S and there exist elements x, y of $(p)_{\mathbf{1}}$ such that $x = a$ and $y = b$ and unionMult$(S, f, E)(a, b) = x \cdot y$,

respectively. The functors: unionOne(S, f, E) and unionZero(S, f, E) yielding elements of unionCarrier(S, f, E) are defined by conditions

(Def. 20) there exists an element p of S such that unionOne$(S, f, E) = 1_{(p)_{\mathbf{1}}}$,

(Def. 21) there exists an element p of S such that unionZero$(S, f, E) = 0_{(p)_{\mathbf{1}}}$,

respectively. The functor unionField(S, f, E) yielding a strict double loop structure is defined by

(Def. 22) the carrier of it = unionCarrier(S, f, E) and the addition of it = union Add(S, f, E) and the multiplication of it = unionMult(S, f, E) and the one of it = unionOne(S, f, E) and the zero of it = unionZero(S, f, E).

Now we state the propositions:

(50) Let us consider a field F, an extension E of F, an F-monomorphic field L, a monomorphism f of F and L, a non empty subset S of ExtSet(f, E), elements p, q of S, and an element a of $(p)_{\mathbf{1}}$. If $p \leqslant q$, then $a \in$ the carrier of $(q)_{\mathbf{1}}$.

(51) Let us consider a field F, an extension E of F, an F-monomorphic field L, a monomorphism f of F and L, an ascending, non empty subset S of ExtSet(f, E), and an element p of S. Then

(i) $1_{\mathrm{unionField}(S,f,E)} = 1_{(p)_{\mathbf{1}}}$, and

(ii) $0_{\mathrm{unionField}(S,f,E)} = 0_{(p)_{\mathbf{1}}}$.

(52) Let us consider a field F, an extension E of F, an F-monomorphic field L, a monomorphism f of F and L, an ascending, non empty subset S of ExtSet(f, E), elements a, b of unionField(S, f, E), an element p of S, and elements x, y of $(p)_{\mathbf{1}}$. If $x = a$ and $y = b$, then $a + b = x + y$ and $a \cdot b = x \cdot y$.

Let F be a field, E be an extension of F, L be an F-monomorphic field, f be a monomorphism of F and L, and S be an ascending, non empty subset of ExtSet(f, E). Let us observe that unionField(S, f, E) is non degenerated and unionField(S, f, E) is Abelian, add-associative, right zeroed, and right complementable and unionField(S, f, E) is commutative, associative, well unital, distributive, and almost left invertible. Now we state the proposition:

(53) Let us consider a field F, an extension E of F, an F-monomorphic field L, a monomorphism f of F and L, an ascending, non empty subset S of ExtSet(f, E), and an element p of S. Then $(p)_{\mathbf{1}}$ is a subfield of unionField(S, f, E).

PROOF: Set $K = \text{unionField}(S, f, E)$. The addition of $(p)_{\mathbf{1}}$ = (the addition of K) $\restriction$ (the carrier of $(p)_{\mathbf{1}}$). The multiplication of $(p)_{\mathbf{1}}$ = (the multiplication of K) $\restriction$ (the carrier of $(p)_{\mathbf{1}}$). $1_{(p)_{\mathbf{1}}} = 1_K$ and $0_K = 0_{(p)_{\mathbf{1}}}$. □

Let us consider a field F, an extension E of F, an F-monomorphic field L, a monomorphism f of F and L, and an ascending, non empty subset S of $\text{ExtSet}(f, E)$. Now we state the propositions:

(54) F is a subfield of $\text{unionField}(S, f, E)$. The theorem is a consequence of (53).

(55) $\text{unionField}(S, f, E)$ is a subfield of E.
PROOF: Set $K = \text{unionField}(S, f, E)$. The carrier of $K \subseteq$ the carrier of E. The addition of K = (the addition of E) $\restriction$ (the carrier of K). The multiplication of K = (the multiplication of E) $\restriction$ (the carrier of K). Set p = the element of S. Consider U being an element of $\text{SubFields}(E)$, g being a function from U into L such that $p = \langle U, g \rangle$ and there exists an extension K_1 of F and there exists a function g_1 from K_1 into L such that $K_1 = U$ and $g_1 = g$ and g_1 is monomorphic and f-extending. $(p)_{\mathbf{1}}$ is a subfield of E. $1_K = 1_{(p)_{\mathbf{1}}}$. $0_K = 0_{(p)_{\mathbf{1}}}$. □

Let F be a field, E be an extension of F, L be an F-monomorphic field, f be a monomorphism of F and L, and S be an ascending, non empty subset of $\text{ExtSet}(f, E)$. Note that $\text{unionField}(S, f, E)$ is F-extending.

The functor $\text{unionExt}(S, f, E)$ yielding a function from $\text{unionField}(S, f, E)$ into L is defined by

(Def. 23) for every element p of S, $it \restriction$(the carrier of $(p)_{\mathbf{1}}$) $= (p)_{\mathbf{2}}$.

Now we state the proposition:

(56) Let us consider a field F, an extension E of F, an F-monomorphic field L, a monomorphism f of F and L, and an ascending, non empty subset S of $\text{ExtSet}(f, E)$. Then $\text{unionExt}(S, f, E)$ is monomorphic and f-extending. The theorem is a consequence of (51) and (53).

Let F be a field, E be an extension of F, L be an F-monomorphic field, f be a monomorphism of F and L, and S be an ascending, non empty subset of $\text{ExtSet}(f, E)$. The functor $\sup S$ yielding an element of $\text{ExtSet}(f, E)$ is defined by the term

(Def. 24) $\langle \text{unionField}(S, f, E), \text{unionExt}(S, f, E) \rangle$.

Now we state the propositions:

(57) Let us consider a field F, an extension E of F, an F-monomorphic field L, a monomorphism f of F and L, an ascending, non empty subset S of $\text{ExtSet}(f, E)$, and an element p of S. Then $p \leqslant \sup S$. The theorem is a consequence of (53).

(58) Let us consider a field F, an extension E of F, an F-algebraic element a of E, an F-monomorphic, algebraic closed field L, and a monomorphism f of F and L. Then there exists a function g from $\mathrm{FAdj}(F, \{a\})$ into L such that g is monomorphic and f-extending. The theorem is a consequence of (3) and (2).

(59) Let us consider a field F, an F-algebraic extension E of F, an F-monomorphic, algebraic closed field L, and a monomorphism f of F and L. Then there exists a function g from E into L such that g is monomorphic and f-extending. The theorem is a consequence of (47), (49), (48), (57), (45), (58), (10), and (1).

(60) Let us consider a field F, an extension E of F, an F-homomorphic, E-homomorphic field L, a homomorphism f from F to L, and a homomorphism g from E to L. Suppose g is f-extending. Then $\mathrm{Im}\, f$ is a subfield of $\mathrm{Im}\, g$.

(61) Let us consider a field F, an algebraic closure A of F, an A-monomorphic, A-homomorphic field L, and a monomorphism g of A and L. Then $\mathrm{Im}\, g$ is algebraic closed.

Proof: Reconsider $f = g^{-1}$ as a function from $\mathrm{Im}\, g$ into A. f is additive, multiplicative, unity-preserving, and monomorphic. □

(62) Let us consider a field F, an F-monomorphic, F-homomorphic field L, an algebraic closure A of F, and a monomorphism f of F and L. Suppose L is an algebraic closure of $\mathrm{Im}\, f$. Let us consider a function g from A into L. If g is monomorphic and f-extending, then g is isomorphism. The theorem is a consequence of (61), (60), and (33).

(63) Let us consider a field F, and algebraic closures A_1, A_2 of F. Then A_1 and A_2 are isomorphic over F.

Proof: Reconsider $L = A_2$ as an F-monomorphic, F-homomorphic, algebraic closed field. Reconsider $f = \mathrm{id}_F$ as a monomorphism of F and L. Consider g being a function from A_1 into L such that g is monomorphic and f-extending. The double loop structure of $F \approx F$. $\mathrm{Im}\, f =$ the double loop structure of F by [4, (7)]. L is an algebraic closure of $\mathrm{Im}\, f$. g is isomorphism. □

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