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Elementary Number Theory Problems. Part VI

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Summary. This paper reports on the formalization in Mizar system [1], [2] of ten selected problems from W. Sierpinski's book "250 Problems in Elementary Number Theory" [7] (see [6] for details of this concrete dataset). This article is devoted mainly to arithmetic progressions: problems 52, 54, 55, 56, 60, 64, 70, 71, and 73 belong to the chapter "Arithmetic Progressions", and problem 50 is from "Relatively Prime Numbers".

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1. Preliminaries

Now we state the proposition:

(1) Let us consider a prime number p. If $3 \mid p$, then $p = 3$.

Note that there exists a prime number which is even.

Now we state the propositions:

(2) Let us consider an even prime number p. Then $p = 2$.

(3) Let us consider prime numbers p, q. If $p \neq q$, then p and q are relatively prime.

Let f be an integer-valued function. We say that f is with all coprime terms if and only if

(Def. 1) for every natural numbers i, j such that $i, j \in \operatorname{dom} f$ and $i \neq j$ holds $f(i)$ and $f(j)$ are relatively prime.

Now we state the proposition:

(4) Let us consider a sequence f of $\mathbb{R}$, and a natural number n. Then $f{\upharpoonright}n$ is a finite 0-sequence.

2. Arithmetic Progressions

Let f be a real-valued function. We say that f is AP-like if and only if

(Def. 2) for every natural numbers i, k such that $i, i+1, k, k+1 \in \operatorname{dom} f$ holds $f(i+1) - f(i) = f(k+1) - f(k)$.

Let f be a real-valued finite sequence. We say that f is finite arithmetic progression-like if and only if

(Def. 3) for every natural number i such that $i, i+1, i+2 \in \operatorname{dom} f$ holds $f(i+2) - f(i+1) = f(i+1) - f(i)$.

One can check that every real-valued finite sequence which is constant is also finite arithmetic progression-like and every sequence of $\mathbb{R}$ which is constant is also AP-like and $\mathrm{id}_{\mathbb{N}}$ is AP-like and $\mathrm{id}_{\mathbb{R}}$ is AP-like and there exists a sequence of $\mathbb{R}$ which is AP-like and there exists a real-valued function which is AP-like and there exists an integer-valued, real-valued finite 0-sequence which is AP-like.

Let f be an AP-like, real-valued function and n be a natural number. Let us note that $f{\upharpoonright}n$ is AP-like.

An arithmetic progression is an AP-like sequence of $\mathbb{R}$. Let a, r be real numbers. The functor $\mathrm{ArProg}(a, r)$ yielding a sequence of $\mathbb{R}$ is defined by

(Def. 4) $it(0) = a$ and for every natural number i, $it(i+1) = it(i) + r$.

Let us observe that $\mathrm{ArProg}(a, r)$ is AP-like. Now we state the proposition:

(5) Let us consider an arithmetic progression f, and a natural number i. Then $f(i+1) - f(i) = f(1) - f(0)$.

Let f be an arithmetic progression. The functor $\mathrm{difference}(f)$ yielding a real number is defined by the term

(Def. 5) $f(1) - f(0)$.

Now we state the propositions:

(6) Let us consider an arithmetic progression f.
Then $f = \mathrm{ArProg}(f(0), \mathrm{difference}(f))$.
Proof: Set $a = f(0)$. Set $r = f(1) - f(0)$. Define $\mathcal{P}[\text{natural number}] \equiv f(\$_1) = (\mathrm{ArProg}(a, r))(\$_1)$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number n, $\mathcal{P}[n]$. $\square$

(7) Let us consider real numbers a, r, and a natural number i. Then $(\mathrm{ArProg}(a,r))(i) = a + i \cdot r$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv (\mathrm{ArProg}(a,r))(\$_1) = a + \$_1 \cdot r$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number n, $\mathcal{P}[n]$. □

Let a, r be integers. Let us note that $\mathrm{ArProg}(a,r)$ is integer-valued and there exists an arithmetic progression which is integer-valued.

Let a be an integer and r be a non zero integer. Let us observe that $\mathrm{ArProg}(a,r)$ is non constant.

Let a be a real number and r be a positive real number. Let us observe that $\mathrm{ArProg}(a,r)$ is increasing.

Let r be a non positive real number. One can verify that $\mathrm{ArProg}(a,r)$ is non-increasing.

Let r be a negative real number. Note that $\mathrm{ArProg}(a,r)$ is decreasing.

Let r be a non negative real number. Let us note that $\mathrm{ArProg}(a,r)$ is non-decreasing and $\mathrm{ArProg}(a,0)$ is constant and there exists an arithmetic progression which is constant and there exists an arithmetic progression which is increasing and non-decreasing and there exists an arithmetic progression which is decreasing and non-increasing.

Let f be an increasing arithmetic progression. One can verify that difference(f) is positive.

Let f be a decreasing arithmetic progression. Note that difference(f) is negative.

Let f be a non-increasing arithmetic progression. Observe that difference(f) is non positive.

Let f be a non-decreasing arithmetic progression. Let us observe that difference(f) is non negative.

Let f be a constant arithmetic progression. One can verify that difference(f) is zero. Now we state the proposition:

(8) Let us consider an arithmetic progression f. Suppose there exists a natural number i such that $f(i)$ is an integer and difference(f) is an integer. Then f is integer-valued.

PROOF: Consider i being a natural number such that $f(i)$ is an integer and difference(f) is an integer. Define $\mathcal{P}[\text{natural number}] \equiv f(\$_1)$ is integer. For every natural number k such that $k \neq 0$ and $\mathcal{P}[k]$ there exists a natural number n such that $n < k$ and $\mathcal{P}[n]$. $\mathcal{P}[0]$. For every object n such that $n \in \mathrm{dom}\, f$ holds $f(n)$ is integer. □

3. Problem 50

Let n be a natural number. We say that n is Fibonacci if and only if

(Def. 6) there exists a natural number k such that $n = \mathrm{Fib}(k)$.

Let us note that there exists a natural number which is Fibonacci.

Now we state the propositions:

(9) Let us consider a natural number n. If $\mathrm{Fib}(n) > 1$, then $n > 2$.

(10) Let us consider a natural number k. If $k > 0$, then $\mathrm{Fib}(k) > 0$.

(11) Let us consider natural numbers k, m. Suppose $\mathrm{Fib}(k) < \mathrm{Fib}(m+1)$ and $1 < k$. Then $\mathrm{Fib}(k) \leqslant \mathrm{Fib}(m)$.

(12) Let us consider natural numbers k, n. Suppose $n \neq 1$ and $k \neq 0$ and $k \neq 1$. If $\mathrm{Fib}(k) = \mathrm{Fib}(n)$, then $k = n$. The theorem is a consequence of (10).

Let us consider a natural number n. Now we state the propositions:

(13) If $n > 2$, then $\mathrm{Fib}(n) \geqslant 2$.

(14) If $n > 3$, then $\mathrm{Fib}(n) \geqslant 3$.

Let us consider natural numbers m, n. Now we state the propositions:

(15) If $m < n$ and $m > 3$, then $\mathrm{Fib}(n) - \mathrm{Fib}(m) > 1$. The theorem is a consequence of (13).

(16) If $m < n$ and $m > 4$, then $\mathrm{Fib}(n) - \mathrm{Fib}(m) > 2$. The theorem is a consequence of (14).

Let f be a sequence of $\mathbb{R}$. We say that f is Fibonacci-valued if and only if

(Def. 7) for every natural number n, there exists a natural number f_4 such that $f_4 = f(n)$ and f_4 is Fibonacci.

Let us observe that every sequence of $\mathbb{R}$ which is Fibonacci-valued is also integer-valued and there exists a sequence of $\mathbb{R}$ which is Fibonacci-valued.

Let n be a natural number. One can verify that $\mathrm{Fib}(n)$ is Fibonacci.

Now we state the proposition:

(17) There exists a Fibonacci-valued sequence f of $\mathbb{R}$ such that f is increasing and with all coprime terms.

Proof: Define $\mathcal{F}(\text{natural number}) = \mathrm{Fib}(\mathrm{pr}(\$_1))$. Consider f being a sequence of $\mathbb{R}$ such that for every natural number n, $f(n) = \mathcal{F}(n)$. For every natural number n, $f(n) < f(n+1)$ by [5, (46)]. For every natural number n, there exists a natural number f_4 such that $f_4 = f(n)$ and f_4 is Fibonacci. For every natural numbers i, j such that $i, j \in \mathrm{dom}\, f$ and $i \neq j$ holds $f(i)$ and $f(j)$ are relatively prime by [3, (21)], (3), [8, (5)]. □

Let us observe that there exists an integer-valued sequence of $\mathbb{R}$ which is Fibonacci-valued, increasing, and with all coprime terms.

4. Triangular Numbers

Let us consider a natural number n. Now we state the propositions:

(18) (i) $3 \mid n$, or

(ii) $3 \mid n+1$, or

(iii) $3 \mid n+2$.

PROOF: $3 \mid n-1$ iff $3 \mid n+2$. □

(19) (i) $4 \mid n$, or

(ii) $4 \mid n+1$, or

(iii) $4 \mid n+2$, or

(iv) $4 \mid n+3$.

(20) Let us consider natural numbers n, k, l. Then $3 \mid n+l$ if and only if $3 \mid n+l+3 \cdot k$.

Let f be a function. We say that f is triangular-valued if and only if

(Def. 8) for every object n, $f(n)$ is triangular.

One can check that every number which is triangular is also integer and every sequence of $\mathbb{R}$ which is triangular-valued is also integer-valued and there exists an integer-valued sequence of $\mathbb{R}$ which is triangular-valued and $\langle 0 \rangle$ is triangular-valued as a finite sequence.

5. Problem 52

Now we state the propositions:

(21) Let us consider natural numbers m, k, l. Suppose $k \neq l$ and $1 \leqslant k \leqslant m$ and $1 \leqslant l \leqslant m$. Then $m! \cdot k+1$ and $m! \cdot l+1$ are relatively prime.

(22) Let us consider a natural number n. Then there exists an AP-like, integer-valued finite 0-sequence f such that

(i) $\operatorname{dom} f \geqslant n$, and

(ii) f is with all coprime terms.

PROOF: Set $f = \text{ArProg}(n!+1, n!)$. Reconsider $f_3 = f \restriction n$ as an integer-valued finite 0-sequence. For every natural number k, $f(k) = n! \cdot (k+1)+1$. For every natural number k such that $k+1 \leqslant n$ holds $f_3(k) = n! \cdot (k+1)+1$. For every natural numbers i, j such that $i, j \in \operatorname{dom} f_3$ and $i \neq j$ holds $f_3(i)$ and $f_3(j)$ are relatively prime. □

6. Problem 54

Let x, y, z be real numbers. We say that x, y and z form an arithmetic progression if and only if

(Def. 9) $y - x = z - y$.

Now we state the propositions:

(23) Let us consider natural numbers x, y, z. Suppose $y = 5 \cdot x + 2$ and $z = 7 \cdot x + 3$. Then

(i) $x \cdot (x + 1)$, $y \cdot (y + 1)$ and $z \cdot (z + 1)$ form an arithmetic progression, and

(ii) $x < y < z$.

(24) $\{\langle x, y, z\rangle$, where x is a real number, y is a real number, z is a real number $: x \cdot (x + 1)$, $y \cdot (y + 1)$ and $z \cdot (z + 1)$ form an arithmetic progression$\}$ is infinite.

Proof: Set $A_1 = \{\langle x, y, z\rangle$, where x is a real number, y is a real number, z is a real number $: x \cdot (x+1)$, $y \cdot (y+1)$ and $z \cdot (z+1)$ form an arithmetic progression$\}$. Reconsider $x = 1$ as a natural number. Reconsider $y = 5 \cdot x + 2$ as a natural number. Define $\mathcal{P}[\text{element of } \mathbb{R}, \text{element of } A_1] \equiv \$_2 = \langle \$_1, 5 \cdot \$_1 + 2, 7 \cdot \$_1 + 3\rangle$. For every element x of $\mathbb{R}$, there exists an element y of A_1 such that $\mathcal{P}[x, y]$. Consider f being a function from $\mathbb{R}$ into A_1 such that for every element x of $\mathbb{R}$, $\mathcal{P}[x, f(x)]$. For every objects x_1, x_2 such that x_1, $x_2 \in \mathbb{R}$ and $f(x_1) = f(x_2)$ holds $x_1 = x_2$. $\square$

7. Problem 55

Now we state the proposition:

(25) Let us consider natural numbers a, b, c. Suppose $a^{\mathbf{2}} + b^{\mathbf{2}} = c^{\mathbf{2}}$ and a, b and c form an arithmetic progression. Then there exists an integer i such that

(i) $a = 3 \cdot i$, and

(ii) $b = 4 \cdot i$, and

(iii) $c = 5 \cdot i$.

8. Problem 56

Let k be a natural number. Observe that Triangle$(4 \cdot k + 1)$ is odd and Triangle $4 \cdot k$ is even.

Let us consider a natural number n. Now we state the propositions:

(26) $3 \mid$ Triangle$(3 \cdot n + 2)$.

(27) $3 \mid$ Triangle $3 \cdot n$.

(28) $3 \mid$ Triangle$(3 \cdot n + 1) - 1$.

(29) Let us consider a natural number i. Then $3 \nmid$ (ArProg$(2, 3))(i)$. The theorem is a consequence of (7).

(30) $\{i$, where i is a natural number : (ArProg$(0, 1))(i)$ is triangular$\}$ is infinite.

PROOF: Set $X = \{i$, where i is a natural number : (ArProg$(0, 1))(i)$ is triangular$\}$. For every natural number m, there exists a natural number n such that $n \geqslant m$ and $n \in X$ by [4, (19)], (7). □

(31) $\{i$, where i is a natural number : (ArProg$(0, 2))(i)$ is triangular$\}$ is infinite.

PROOF: Set $X = \{i$, where i is a natural number : (ArProg$(0, 2))(i)$ is triangular$\}$. For every natural number m, there exists a natural number n such that $n \geqslant m$ and $n \in X$. □

(32) $\{i$, where i is a natural number : (ArProg$(1, 2))(i)$ is triangular$\}$ is infinite.

PROOF: Set $X = \{i$, where i is a natural number : (ArProg$(1, 2))(i)$ is triangular$\}$. For every natural number m, there exists a natural number n such that $n \geqslant m$ and $n \in X$. □

(33) Let us consider a natural number i. Then $3 \nmid$ (ArProg$(2, 3))(i) - 1$. The theorem is a consequence of (7).

(34) Let us consider a natural number i. Then (ArProg$(2, 3))(i)$ is not triangular. The theorem is a consequence of (28), (33), (29), (26), and (27).

9. Problem 60

Let n be a natural number. We say that n is perfect power if and only if

(Def. 10) there exists a natural number x and there exists a natural number k such that $k > 1$ and $n = x^k$.

Now we state the proposition:

(35) There exists a natural number n such that

(i) n is perfect power, and

(ii) $n + 1$ is perfect power.

Let us note that there exists a natural number which is even and perfect power. Now we state the propositions:

(36) Let us consider an even natural number n, and a natural number k. If $k > 1$, then $4 \mid n^k$.

(37) Let us consider an even, perfect power natural number n. Then $4 \mid n$. The theorem is a consequence of (36).

(38) Let us consider a natural number k. Then $4 \cdot k + 2$ is not perfect power. The theorem is a consequence of (37).

(39) Let us consider a prime number p. Then p is not perfect power.

One can verify that every natural number which is prime is also non perfect power and every natural number which is a square is also perfect power.

Now we state the proposition:

(40) There exists no natural number n such that n is perfect power and $n+1$ is perfect power and $n+2$ is perfect power and $n+3$ is perfect power. The theorem is a consequence of (38).

10. Problem 64

Now we state the propositions:

(41) Let us consider natural numbers k, l, m. Suppose $0 < k < l < m$ and it is not true that $k = 2$ and $l = 3$ and $m = 4$ and it is not true that $k = 1$ and $l = 4$ and $m = 5$ and $\mathrm{Fib}(m) - \mathrm{Fib}(l) = \mathrm{Fib}(l) - \mathrm{Fib}(k)$ and $\mathrm{Fib}(l) - \mathrm{Fib}(k) > 0$. Then

(i) $l > 2$, and

(ii) $k = l - 2$, and

(iii) $m = l + 1$.

Proof: Set $u_2 = \mathrm{Fib}(l)$. Set $u_3 = \mathrm{Fib}(m)$. $\mathrm{Fib}(l) > 1$. $l > 2$. $u_3 < u_2 + u_2$. $\mathrm{Fib}(m) \leqslant \mathrm{Fib}(l + 1)$. □

(42) $\mathrm{Fib}(1) - \mathrm{Fib}(0) \neq \mathrm{Fib}(2) - \mathrm{Fib}(1)$.

(43) $\mathrm{Fib}(1) - \mathrm{Fib}(0) = \mathrm{Fib}(3) - \mathrm{Fib}(1)$.

(44) $\mathrm{Fib}(2) - \mathrm{Fib}(0) = \mathrm{Fib}(3) - \mathrm{Fib}(2)$.

(45) $\mathrm{Fib}(3) - \mathrm{Fib}(2) = \mathrm{Fib}(4) - \mathrm{Fib}(3)$.

(46) $\mathrm{Fib}(5) = 5$.

(47) $\mathrm{Fib}(5) - \mathrm{Fib}(4) = \mathrm{Fib}(4) - \mathrm{Fib}(1)$.

(48) There exist no natural numbers k, l, m, n such that $0 < k < l < m < n$ and $\mathrm{Fib}(m) - \mathrm{Fib}(l) = \mathrm{Fib}(l) - \mathrm{Fib}(k) = \mathrm{Fib}(n) - \mathrm{Fib}(m)$ and $\mathrm{Fib}(l) - \mathrm{Fib}(k) > 0$. The theorem is a consequence of (41), (15), and (16).

11. PROBLEM 70

Now we state the propositions:

(49) Let us consider an arithmetic progression f, and prime numbers p_1, p_2, p_3. Suppose difference$(f) = 10$ and there exists a natural number i such that $p_1 = f(i)$ and $p_2 = f(i + 1)$ and $p_3 = f(i + 2)$. Then $p_1 = 3$. The theorem is a consequence of (20), (5), and (18).

(50) There exists no arithmetic progression f such that difference$(f) = 10$ and there exist prime numbers p_1, p_2, p_3, p_4 and there exists a natural number i such that p_1, p_2, p_3, p_4 are mutually different and $p_1 = f(i)$ and $p_2 = f(i + 1)$ and $p_3 = f(i + 2)$ and $p_4 = f(i + 3)$. The theorem is a consequence of (8), (5), (20), (18), and (1).

12. PROBLEM 71

Now we state the propositions:

(51) There exists no arithmetic progression f such that difference$(f) = 100$ and there exist prime numbers p_1, p_2, p_3 and there exists a natural number i such that p_1, p_2, p_3 are mutually different and $p_1 = f(i)$ and $p_2 = f(i+1)$ and $p_3 = f(i+2)$. The theorem is a consequence of (8), (5), (20), (1), and (18).

(52) There exists no arithmetic progression f such that difference$(f) = 1000$ and there exist prime numbers p_1, p_2, p_3 and there exists a natural number i such that p_1, p_2, p_3 are mutually different and $p_1 = f(i)$ and $p_2 = f(i+1)$ and $p_3 = f(i+2)$. The theorem is a consequence of (8), (5), (20), (1), and (18).

13. PROBLEM 73

Let k be an integer. We say that k is not representable by a sum or a difference of two primes if and only if

(Def. 11) there exist no prime numbers p_1, p_2 such that $k = p_1 + p_2$ or $k = p_1 - p_2$.

Let f be an integer-valued sequence of $\mathbb{R}$. We say that f is with terms not representable by a sum or a difference of two primes if and only if

(Def. 12) for every natural number i, $f(i)$ is not representable by a sum or a difference of two primes.

Now we state the propositions:

(53) Let us consider an integer k. Then $30 \cdot k + 7$ is odd.

(54) Let us consider a natural number k. Suppose $k \geqslant 1$. Then $30 \cdot k + 7$ is not representable by a sum or a difference of two primes. The theorem is a consequence of (53).

Note that $\mathrm{ArProg}(37, 30)$ is with terms not representable by a sum or a difference of two primes.

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