FORMALIZED MATHEMATICS
Vol. 30, No. 3, Pages 211–221, 2022
ISSN: 1426–2630, e-ISSN: 1898–9934
DOI: 10.2478/forma-2022-0016

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sciendo.com/journal/forma

Ring of Endomorphisms and Modules over a Ring

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Summary. We formalize in the Mizar system [3], [4] some basic properties on left module over a ring such as constructing a module via a ring of endomorphism of an abelian group and the set of all homomorphisms of modules form a module [1] along with Ch. 2 set. 1 of [2].

The formalized items are shown in the below list with notations: M_{ab} for an Abelian group with a suffix "$_{ab}$" and M without a suffix is used for left modules over a ring.

1. The endomorphism ring of an abelian group denoted by $\mathbf{End}(M_{ab})$.
2. A pair of an Abelian group M_{ab} and a ring homomorphism $R \xrightarrow{\rho} \mathbf{End}(M_{ab})$ determines a left R-module, formalized as a function $\mathbf{AbGrLMod}(M_{ab}, \rho)$ in the article.
3. The set of all functions from M to N form R-module and denoted by $\mathbf{Func\_Mod}_R(M, N)$.
4. The set R-module homomorphisms of M to N, denoted by $\mathbf{Hom}_R(M, N)$, forms R-module.
5. A formal proof of $\mathbf{Hom}_R(\bar{R}, M) \cong M$ is given, where the $\bar{R}$ denotes the regular representation of R, i.e. we regard R itself as a left R-module.
6. A formal proof of $\mathbf{AbGrLMod}(M'_{ab}, \rho') \cong M$ where M'_{ab} is an abelian group obtained by removing the scalar multiplication from M, and ρ' is obtained by currying the scalar multiplication of M.

The removal of the multiplication from M has been done by the forgettable functor defined as **AbGr** in the article.

MSC: 13C05 13C60 68V20

Keywords: module; endomorphism ring

MML identifier: `LMOD_XX1`, version: `8.1.12 5.71.1431`

Let M, N be Abelian groups. The functor $\mathrm{ADD}(M,N)$ yielding a binary operation on (the carrier of $N)^{(\text{the carrier of } M)}$ is defined by

(Def. 1) for every elements f, g of (the carrier of $N)^{\alpha}$, $it(f,g) =$ (the addition of $N)^{\circ}(f, g)$, where α is the carrier of M.

Now we state the propositions:

(1) Let us consider Abelian groups M, N, and elements f, g, h of (the carrier of $N)^{\alpha}$. Then $h = (\mathrm{ADD}(M,N))(f,g)$ if and only if for every element x of the carrier of M, $h(x) = f(x) + g(x)$, where α is the carrier of M.

(2) Let us consider Abelian groups M, N, and homomorphisms f, g from M to N. Then $(\mathrm{ADD}(M,N))(f,g)$ is a homomorphism from M to N. The theorem is a consequence of (1).

Let M be an Abelian group. The functor set_End(M) yielding a non empty subset of (the carrier of $M)^{(\text{the carrier of } M)}$ is defined by the term

(Def. 2) $\{f$, where f is a function from M into $M : f$ is an endomorphism of $M\}$.

The functor add_End(M) yielding a binary operation on set_End(M) is defined by the term

(Def. 3) $\mathrm{ADD}(M,M) \restriction (\mathrm{set\_End}(M) \times \mathrm{set\_End}(M))$.

Now we state the proposition:

(3) Let us consider an Abelian group M, and endomorphisms f, g of M. Then

(i) f, $g \in$ (the carrier of $M)^{\alpha}$, and

(ii) $(\mathrm{add\_End}(M))(\langle f, g\rangle) = (\mathrm{ADD}(M,M))(f,g)$, and

(iii) $(\mathrm{ADD}(M,M))(f,g)$ is an endomorphism of M,

where α is the carrier of M. The theorem is a consequence of (2).

From now on M, N denote Abelian groups. Let M be an Abelian group and f, g be elements of (the carrier of $M)^{(\text{the carrier of } M)}$. Let us note that the functor $g \cdot f$ yields an element of (the carrier of $M)^{(\text{the carrier of } M)}$.

We prepare composition of homomorphisms.

Let M be an Abelian group. The functor FuncComp(M) yielding a binary operation on (the carrier of $M)^{(\text{the carrier of } M)}$ is defined by

(Def. 4) for every elements f, g of (the carrier of $M)^{\alpha}$, $it(f,g) = f \cdot g$, where α is the carrier of M.

(4) Let us consider Abelian groups M, N, and elements f, g of (the carrier of $N)^{\alpha}$. Then $(\mathrm{ADD}(M,N))(f,g) = (\mathrm{ADD}(M,N))(g,f)$, where α is the carrier of M. The theorem is a consequence of (1).

(5) Endomorphism of M is closed under Composition:
Let us consider an Abelian group M, and endomorphisms f, g of M. Then $(\mathrm{FuncComp}(M))(f,g)$ is an endomorphism of M.
Proof: Reconsider $F = (\mathrm{FuncComp}(M))(f,g)$ as an element of (the carrier of $M)^{(\text{the carrier of } M)}$. F is additive. □

Let M be an Abelian group. The functor mult_End(M) yielding a binary operation on set_End(M) is defined by the term

(Def. 5) $\mathrm{FuncComp}(M){\restriction}(\mathrm{set\_End}(M) \times \mathrm{set\_End}(M))$.

Now we state the proposition:

(6) Let us consider an Abelian group M, and endomorphisms f, g of M. Then
  (i) f, $g \in$ (the carrier of $M)^{\alpha}$, and
  (ii) $(\mathrm{mult\_End}(M))(\langle f, g \rangle) = (\mathrm{FuncComp}(M))(f,g)$, and
  (iii) $(\mathrm{FuncComp}(M))(f,g)$ is an endomorphism of M,

where α is the carrier of M. The theorem is a consequence of (5).

Let M be an Abelian group. The functors: 0_End(M) and 1_End(M) yielding elements of set_End(M) are defined by terms

(Def. 6) $\mathrm{ZeroMap}(M, M)$,

(Def. 7) id_M,

respectively. Let f be an element of set_End(M). The functor $\mathrm{Inv}\, f$ yielding an element of set_End(M) is defined by

(Def. 8) for every element x of M, $it(x) = f(-x)$.

Now we state the proposition:

(7) Let us consider an Abelian group M, and an element f of set_End(M). Then $(\mathrm{ADD}(M,M))(f, \mathrm{Inv}\, f) = \mathrm{ZeroMap}(M,M)$.
Proof: Consider f_1 being a function from the carrier of M into the carrier of M such that $f_1 = f$ and f_1 is an endomorphism of M. Consider g_1 being a function from the carrier of M into the carrier of M such that $g_1 = \mathrm{Inv}\, f$ and g_1 is an endomorphism of M. For every element x of the carrier of M, $(\mathrm{ADD}(M,M))(f_1, g_1)(x) = (\mathrm{ZeroMap}(M,M))(x)$. □

We define the Ring of Endomorphism of M as a structure.

Let M be an Abelian group. The functor End_Ring(M) yielding a strict, non empty double loop structure is defined by the term

(Def. 9) $\langle \mathrm{set\_End}(M), \mathrm{add\_End}(M), \mathrm{mult\_End}(M), \mathrm{1\_End}(M), \mathrm{0\_End}(M) \rangle$.

Now we state the proposition:

(8) The structure of End-Ring(M) turns to be a Ring:
Let us consider an Abelian group M. Then End_Ring(M) is a ring.

Let M be an Abelian group. One can verify that End_Ring(M) is Abelian, add-associative, right zeroed, right complementable, associative, well unital, and distributive and End_Ring(M) is strict.

In the sequel R denotes a ring and r denotes an element of R.

Let us consider R. Let M, N be left modules over R.

A homomorphism from M to N by R is a function from M into N defined by

(Def. 10) *it* is additive and homogeneous.

Now we state the proposition:

(9) Let us consider left modules M, N over R, and a homomorphism f from M to N by R. Suppose f is one-to-one and onto. Then f^{-1} is a homomorphism from N to M by R.
PROOF: Reconsider $g = f^{-1}$ as a function from N into M. For every elements a, b of the carrier of N, $g(a+b) = g(a) + g(b)$. For every element r of R and for every element a of the carrier of N, $g(r \cdot a) = r \cdot g(a)$. □

Let us consider R. Let M, N be left modules over R. We say that $M \cong N$ if and only if

(Def. 11) there exists a homomorphism f from M to N by R such that f is one-to-one and onto.

Let M be a left module over R.

An endomorphism of R and M is a homomorphism from M to M by R. Now we state the propositions:

(10) Let us consider a left module M over R. Then $M \cong M$.

(11) Let us consider left modules M, N over R. If $M \cong N$, then $N \cong M$. The theorem is a consequence of (9).

Let us consider R. Let M, N be left modules over R. Observe that the predicate $M \cong N$ is reflexive and symmetric. Now we state the propositions:

(12) Let us consider left modules L, M, N over R. If $L \cong M$ and $M \cong N$, then $L \cong N$.
PROOF: Consider f being a homomorphism from L to M by R such that f is one-to-one and onto. Consider g being a homomorphism from M to N by R such that g is one-to-one and onto. Reconsider $G = g \cdot f$ as a function from L into N. For every elements x, y of L, $G(x+y) = G(x) + G(y)$. For every element x of L and for every element a of R, $G(a \cdot x) = a \cdot G(x)$. □

(13) Let us consider left modules M, N over R, and a homomorphism f from M to N by R. Then f is one-to-one if and only if $\ker f = \{0_M\}$.
PROOF: If f is one-to-one, then $\ker f = \{0_M\}$. For every objects x_1, x_2 such that $x_1, x_2 \in \operatorname{dom} f$ and $f(x_1) = f(x_2)$ holds $x_1 = x_2$. □

Let us consider R. Let M be an Abelian group and s be a function from R into End_Ring(M). The functor LModlmult(M, s) yielding a function from (the carrier of R) $\times$ (the carrier of M) into the carrier of M is defined by

(Def. 12) for every element x of the carrier of R and for every element y of the carrier of M, there exists an endomorphism h of M such that $h = s(x)$ and $it(x, y) = h(y)$.

The functor AbGrLMod(M, s) yielding a strict, non empty vector space structure over R is defined by the term

(Def. 13) $\langle$the carrier of M, the addition of M, 0_M, LModlmult(M, s)$\rangle$.

Now we state the proposition:

(14) Let us consider an Abelian group M, and a function s from R into End_Ring(M). Suppose s inherits ring homomorphism.
Then AbGrLMod(M, s) is a left module over R.
Proof: AbGrLMod(M, s) is Abelian. AbGrLMod(M, s) is add-associative. AbGrLMod(M, s) is right zeroed. AbGrLMod(M, s) is right complementable. AbGrLMod(M, s) is scalar unital. □

The set of all functions from R-module M into R-module N form R-module.

In the sequel M, N denote left modules over R.

Let us consider R, M, and N. The functor 0_Funcs(M, N) yielding an element of (the carrier of N)$^{(\text{the carrier of } M)}$ is defined by the term

(Def. 14) ZeroMap(M, N).

The functor ADD(M, N) yielding a binary operation on (the carrier of N)$^{(\text{the carrier of } M)}$ is defined by

(Def. 15) for every elements f, g of (the carrier of N)$^\alpha$, $it(f, g) =$ (the addition of N)$^\circ(f, g)$, where α is the carrier of M.

From now on f, g, h denote elements of (the carrier of N)$^{(\text{the carrier of } M)}$.

Now we state the proposition:

(15) $h = (\text{ADD}(M, N))(f, g)$ if and only if for every element x of the carrier of M, $h(x) = f(x) + g(x)$.

Let us consider R, M, and N. Let F be a function from (the carrier of R) $\times$ (the carrier of N) into the carrier of N, a be an element of the carrier of R, and f be a function from M into N. Observe that the functor $F^\circ(a, f)$ yields an element of (the carrier of N)$^{(\text{the carrier of } M)}$. The functor LMULT($M, N$) yielding a function from (the carrier of R) $\times$ (the carrier of N)$^{(\text{the carrier of } M)}$ into (the carrier of N)$^{(\text{the carrier of } M)}$ is defined by

(Def. 16) for every element a of the carrier of R and for every element f of (the carrier of N)$^\alpha$ and for every element x of the carrier of M, $it(\langle a, f\rangle)(x) = a \cdot f(x)$, where α is the carrier of M.

The functor Func_Mod(R, M, N) yielding a non empty vector space structure over R is defined by the term

(Def. 17) $\langle$(the carrier of $N)^{\alpha}$, ADD(M, N), 0_Funcs(M, N), LMULT$(M, N)\rangle$, where α is the carrier of M.

Now we state the proposition:

(16) Let us consider an element a of the carrier of R, and elements f, h of (the carrier of $N)^{\alpha}$. Then $h =$ (LMULT$(M, N))(\langle a, f\rangle)$ if and only if for every element x of M, $h(x) = a \cdot f(x)$, where α is the carrier of M.

In the sequel a, b denote elements of the carrier of R.

Let us consider R, M, and N. Note that Func_Mod(R, M, N) is Abelian, add-associative, right zeroed, right complementable, vector distributive, scalar distributive, scalar associative, and scalar unital. Now we state the proposition:

(17) Func_Mod(R, M, N) is a left module over R.

From now on R denotes a commutative ring and M, M_1, N, N_1 denote left modules over R. Now we state the proposition:

(18) HOM(M, N) THE SET OF ALL R HOMOMORPHISMS FORM LEFT R-MODULE:

Let us consider homomorphisms f, g from M to N by R.

Then (ADD$(M, N))(f, g)$ is a homomorphism from M to N by R. The theorem is a consequence of (15).

Let us consider R, M_1, M, and N. Let f be an element of (the carrier of $M)^{(\text{the carrier of } M_1)}$ and g be an element of (the carrier of $N)^{(\text{the carrier of } M)}$. Let us observe that the functor $g \cdot f$ yields an element of (the carrier of $N)^{(\text{the carrier of } M_1)}$. Now we state the propositions:

(19) Let us consider left modules M, N, M_1 over R, a homomorphism f from M to N by R, and a homomorphism u from M_1 to M by R. Then $f \cdot u$ is a homomorphism from M_1 to N by R.

PROOF: For every elements x_1, y_1 of the carrier of M_1 and for every element a of R, $(f \cdot u)(x_1 + y_1) = (f \cdot u)(x_1) + (f \cdot u)(y_1)$ and $a \cdot (f \cdot u)(x_1) = a \cdot (f \cdot u)(x_1)$. For every element x_1 of the carrier of M_1 and for every element a of R, $(f \cdot u)(a \cdot x_1) = a \cdot (f \cdot u)(x_1)$. □

(20) Let us consider an element a of the carrier of R, and a homomorphism g from M to N by R. Then (LMULT$(M, N))(\langle a, g\rangle)$ is a homomorphism from M to N by R.

Let us consider R, M, and N. The functor set_Hom(M, N) yielding a non empty subset of (the carrier of $N)^{(\text{the carrier of } M)}$ is defined by the term

(Def. 18) $\{f$, where f is a function from M into N : f is a homomorphism from M to N by $R\}$.

The functor add_Hom(M, N) yielding a binary operation on set_Hom(M, N) is defined by the term

(Def. 19) ADD$(M, N)\restriction$(set_Hom$(M, N) \times$ set_Hom(M, N)).

Let F be a function from (the carrier of R)$\times$(the carrier of N) into the carrier of N, a be an element of the carrier of R, and f be a function from M into N. One can verify that the functor $F^{\circ}(a, f)$ yields an element of (the carrier of N)$^{(\text{the carrier of } M)}$. The functor lmult_Hom$(M, N)$ yielding a function from (the carrier of R) $\times$ set_Hom(M, N) into set_Hom(M, N) is defined by the term

(Def. 20) LMULT$(M, N)\restriction$((the carrier of R) $\times$ set_Hom(M, N)).

The functor 0_Hom(M, N) yielding an element of set_Hom(M, N) is defined by the term

(Def. 21) ZeroMap(M, N).

The functor Hom(R, M, N) yielding a non empty vector space structure over R is defined by the term

(Def. 22) $\langle$set_Hom(M, N), add_Hom(M, N), 0_Hom(M, N), lmult_Hom$(M, N)\rangle$.

Let us note that Hom(R, M, N) is non empty. Now we state the propositions:

(21) Let us consider homomorphisms f, g from M to N by R. Then

(i) $f, g \in$ (the carrier of N)$^{\alpha}$, and

(ii) (add_Hom(M, N))$(\langle f, g\rangle)$ = (ADD(M, N))(f, g), and

(iii) (ADD(M, N))(f, g) is a homomorphism from M to N by R,

where α is the carrier of M. The theorem is a consequence of (18).

(22) Let us consider an element a of the carrier of R, and a homomorphism f from M to N by R. Then

(i) (lmult_Hom(M, N))$(\langle a, f\rangle)$ = (LMULT(M, N))$(\langle a, f\rangle)$, and

(ii) (LMULT(M, N))$(\langle a, f\rangle)$ is a homomorphism from M to N by R.

The theorem is a consequence of (20).

(23) Let us consider elements f_1, g_1 of Func_Mod(R, M, N), and elements f, g of Hom(R, M, N). If $f_1 = f$ and $g_1 = g$, then $f + g = f_1 + g_1$. The theorem is a consequence of (21).

(24) Hom(R, M, N) is a left module over R. The theorem is a consequence of (23).

Let us consider R, M, and N. Note that Hom(R, M, N) is Abelian, add-associative, right zeroed, right complementable, vector distributive, scalar distributive, scalar associative, and scalar unital.

Let us consider M_1. Let u be a homomorphism from M_1 to M by R. The functor $\tau(N, u)$ yielding a function from Hom(R, M, N) into Hom(R, M_1, N) is defined by

(Def. 23) for every element f of $\mathrm{Hom}(R, M, N)$, there exists a homomorphism f_1 from M to N by R such that $f = f_1$ and $it(f) = f_1 \cdot u$.

Let us note that $\tau(N, u)$ is additive and homogeneous. Now we state the proposition:

(25) Let us consider a homomorphism u from M_1 to M by R. Then $\tau(N, u)$ is a homomorphism from $\mathrm{Hom}(R, M, N)$ to $\mathrm{Hom}(R, M_1, N)$ by R.

Let us consider R, M, N, and N_1. Let u be a homomorphism from N to N_1 by R. The functor $\phi(M, u)$ yielding a function from $\mathrm{Hom}(R, M, N)$ into $\mathrm{Hom}(R, M, N_1)$ is defined by

(Def. 24) for every element f of $\mathrm{Hom}(R, M, N)$, there exists a homomorphism f_1 from M to N by R such that $f = f_1$ and $it(f) = u \cdot f_1$.

Let us observe that $\phi(M, u)$ is additive and homogeneous. Now we state the propositions:

(26) Let us consider a homomorphism u from N to N_1 by R. Then $\phi(M, u)$ is a homomorphism from $\mathrm{Hom}(R, M, N)$ to $\mathrm{Hom}(R, M, N_1)$ by R.

(27) $\mathrm{Hom}(R, \mathrm{LeftMod}(R), M) \cong M$.

PROOF: Reconsider $R_1 = \mathrm{LeftMod}(R)$ as a left module over R. Reconsider $m_1 = 1_R$ as an element of R_1. Define $\mathcal{F}$(element of (the carrier of $M)^{(\text{the carrier of } R_1)}) = \$_1(m_1)$. Consider G being a function from (the carrier of $M)^{(\text{the carrier of } R_1)}$ into M such that For every element x of (the carrier of $M)^{\alpha}$, $G(x) = \mathcal{F}(x)$, where α is the carrier of R_1. For every elements f, g of (the carrier of $M)^{\alpha}$, $G((\mathrm{ADD}(R_1, M))(f, g)) = G(f) + G(g)$, where α is the carrier of R_1.

For every element f of (the carrier of $M)^{\alpha}$ and for every element a of R, $G((\mathrm{LMULT}(R_1, M))(\langle a, f \rangle)) = a \cdot G(f)$, where α is the carrier of R_1. Set $c =$ the carrier of $\mathrm{Hom}(R, R_1, M)$. Set $G_1 = G \restriction c$. For every object y such that $y \in \mathrm{rng}\, G_1$ holds $y \in$ the carrier of M. For every elements f, g of $\mathrm{Hom}(R, R_1, M)$, $G_1(f + g) = G_1(f) + G_1(g)$. For every element f of $\mathrm{Hom}(R, R_1, M)$ and for every element a of R, $G_1(a \cdot f) = a \cdot G_1(f)$. $\ker G_1 = \{0_{\mathrm{Hom}(R, R_1, M)}\}$. For every object y such that $y \in$ the carrier of M holds $y \in \mathrm{rng}\, G_1$. $\square$

Correspondence between Abelian Group (AbGr) and left R-module.

Let us consider R and M. The functor $\mathrm{AbGr}(M)$ yielding a non empty, strict Abelian group is defined by the term

(Def. 25) $\langle$the carrier of M, the addition of M, $0_M\rangle$.

Let us consider N. Let f be a homomorphism from M to N by R. The functor $\mathrm{AbGr}(f)$ yielding a function from $\mathrm{AbGr}(M)$ into $\mathrm{AbGr}(N)$ is defined by

(Def. 26) for every object x such that $x \in$ the carrier of AbGr(M) holds $it(x) = f(x)$.

Now we state the proposition:

(28) Let us consider a homomorphism f from M to N by R. Then AbGr(f) is a homomorphism from AbGr(M) to AbGr(N).

Let us consider endomorphisms f, g, h of R and M. Now we state the propositions:

(29) AbGr(h) = (FuncComp(AbGr(M)))(AbGr(f), AbGr(g)) if and only if for every element x of the carrier of AbGr(M), (AbGr(h))(x) = ((AbGr(f))$\cdot$ (AbGr(g)))(x).

(30) If $h = f \cdot g$, then AbGr(h) = (AbGr(f)) $\cdot$ (AbGr(g)).
Proof: For every element x of the carrier of AbGr(M), (AbGr(h))(x) = ((AbGr(f)) $\cdot$ (AbGr(g)))(x). □

(31) AbGr(h) = (ADD(AbGr(M), AbGr(M)))(AbGr(f), AbGr(g)) if and only if for every element x of the carrier of AbGr(M), (AbGr(h))(x) = (AbGr(f))(x) + (AbGr(g))(x).
Proof: If AbGr(h) = (ADD(AbGr(M), AbGr(M)))(AbGr(f), AbGr(g)), then for every element x of the carrier of AbGr(M), (AbGr(h))(x) = (AbGr(f))(x) + (AbGr(g))(x). AbGr(h) = (ADD(AbGr(M), AbGr(M))) (AbGr(f), AbGr(g)). □

(32) If h = (ADD(M, M))(f, g), then AbGr(h) = (ADD(AbGr(M), AbGr(M)))(AbGr(f), AbGr(g)). The theorem is a consequence of (15) and (31).

(33) Let us consider a ring R, a left module M over R, an element a of R, and an element m of M. Then (curry(the left multiplication of M))(a)(m) = $a \cdot m$.

(34) Let us consider a commutative ring R, a left module M over R, and an element a of R. Then (curry(the left multiplication of M))(a) is an endomorphism of R and M.
Proof: Set f = (curry(the left multiplication of M))(a). For every elements m_1, m_2 of M, $f(m_1 + m_2) = f(m_1) + f(m_2)$. For every element b of R and for every element m of M, $f(b \cdot m) = b \cdot f(m)$. □

(35) Let us consider endomorphisms f, g, h of R and M. Suppose $h = f \cdot g$. Then AbGr(h) = (FuncComp(AbGr(M)))(AbGr(f), AbGr(g)). The theorem is a consequence of (30) and (29).

Let R be a commutative ring and M be a left module over R. The canonical homomorphism of M into quotient field yielding a function from R into End_Ring(AbGr(M)) is defined by

(Def. 27) for every object x such that $x \in$ the carrier of R there exists an endomorphism f of R and M such that $f =$ (curry(the left multiplication of M))(x) and $it(x) = \mathrm{AbGr}(f)$.

Observe that the canonical homomorphism of M into quotient field is additive. Now we state the proposition:

(36) Let us consider a commutative ring R, a left module M over R, and an element a of R. Then (the canonical homomorphism of M into quotient field)(a) is a homomorphism from $\mathrm{AbGr}(M)$ to $\mathrm{AbGr}(M)$.

Let R be a commutative ring and M be a left module over R. One can verify that the canonical homomorphism of M into quotient field is linear and AbGrLMod(AbGr(M), the canonical homomorphism of M into quotient field) is non empty, Abelian, add-associative, right zeroed, right complementable, vector distributive, scalar distributive, scalar associative, and scalar unital.

Now we state the propositions:

(37) Let us consider a commutative ring R, and a left module M over R. Then LModlmult(AbGr(M), the canonical homomorphism of M into quotient field) = the left multiplication of M.

PROOF: Set $F =$ LModlmult(AbGr(M), the canonical homomorphism of M into quotient field). For every object z such that $z \in$ (the carrier of R) $\times$ (the carrier of M) holds $F(z) =$ (the left multiplication of M)(z). □

(38) Let us consider a commutative ring R, and a strict left module M over R. Then AbGrLMod(AbGr(M), the canonical homomorphism of M into quotient field) $= M$.

PROOF: AbGrLMod(AbGr(M), the canonical homomorphism of M into quotient field) is a submodule of M. □

Let R be a commutative ring and M be a left module over R. The functor $\rho(M)$ yielding a function from M into AbGrLMod(AbGr(M), the canonical homomorphism of M into quotient field) is defined by the term

(Def. 28) id_M.

Now we state the proposition:

(39) Let us consider a commutative ring R, and a left module M over R. Then $\rho(M)$ is additive and homogeneous.

PROOF: For every element x of the carrier of M and for every element a of R, $\rho(M)(a \cdot x) = a \cdot \rho(M)(x)$ by [5, (7)]. □

Let R be a commutative ring and M be a left module over R. Observe that $\rho(M)$ is additive and homogeneous.

Let us consider a commutative ring R and a left module M over R. Now we state the propositions:

(40) $\rho(M)$ is one-to-one and onto.

(41) $M \cong$ AbGrLMod(AbGr(M), the canonical homomorphism of M into quotient field). The theorem is a consequence of (40).

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Accepted September 30, 2022