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The Divergence of the Sum of Prime Reciprocals[1]

Mario Carneiro
Carnegie Mellon University
Pittsburgh PA, USA

Summary. This is Erdős's proof of the divergence of the sum of prime reciprocals, using the Mizar system [2], [3], as reported in "Proofs from THE BOOK" [1].

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From now on i, j, k, k_0, m, n, N denote natural numbers, x, y denote real numbers, and p denotes a prime number. Now we state the propositions:

(1) k is not zero if and only if $1 \leqslant k$.

(2) If $x^{\mathbf{2}} \leqslant y$, then $x \leqslant \sqrt{y}$.

(3) If $x^{\mathbf{2}} < y$, then $x < \sqrt{y}$.

(4) If $0 \leqslant x$ and $0 \leqslant y$ and $x \leqslant y^{\mathbf{2}}$, then $\sqrt{x} \leqslant y$.

(5) If $0 \leqslant x$ and $0 \leqslant y$ and $x < y^{\mathbf{2}}$, then $\sqrt{x} < y$.

Let x be a non negative real number. Let us note that the functor $\lfloor x \rfloor$ yields a natural number. In the sequel s denotes a sequence of real numbers. Now we state the propositions:

(6) If for every n, $0 \leqslant s(n)$, then $0 \leqslant ((\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}})(n)$.

(7) If s is summable and for every n, $0 \leqslant s(n)$, then $((\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}})(i) \leqslant \sum s$.

(8) If s is summable and for every n, $0 \leqslant s(n)$ and $i \leqslant j$, then $\sum (s \uparrow j) \leqslant \sum (s \uparrow i)$. The theorem is a consequence of (6).

[1]Work performed while visiting the Czech Institute for Informatics, Robotics and Cybernetics.

(9) If s is summable and for every n, $0 \leqslant s(n)$, then $\sum(s \uparrow i) \leqslant \sum s$. The theorem is a consequence of (8).

(10) If $p < n$, then $\overline{\overline{\mathbb{P}(p)}} + 1 \leqslant \overline{\overline{\mathbb{P}(n)}}$.

(11) $n \leqslant \mathrm{pr}(n)$.

(12) If $p < \mathrm{pr}(n+1)$, then $p \leqslant \mathrm{pr}(n)$. The theorem is a consequence of (10).

From now on N denotes a non zero natural number. Now we state the proposition:

(13) **Main Result** The sum of the reciprocals of the primes diverges:

$\mathrm{inv}_{\mathbb{P}}$ is not summable.

Proof: Define $\mathcal{P}$[non zero natural number, natural number, natural number] $\equiv \$_1 \leqslant \$_3$ and for every p such that $p \mid \$_1$ holds $p \leqslant \mathrm{pr}(\$_2)$. Define $\mathcal{M}$(natural number, natural number) $= \{n$, where n is a non zero natural number : $\mathcal{P}[n, \$_1, \$_2]\}$.

For every k and N, $\mathcal{M}(k, N)$ is finite and $\overline{\overline{\mathcal{M}(k, N)}} \subseteq 2^{\mathrm{pr}(k)} \cdot \lfloor\sqrt{N}\rfloor$ by (1), (2), [4, (47)]. For every k and N, $N \cdot ((\sum_{\alpha=0}^{\kappa}(\mathrm{inv}_{\mathbb{P}})(\alpha))_{\kappa\in\mathbb{N}})(k) + \overline{\overline{(\mathrm{Seg}\, N) \setminus \mathcal{M}(k, N)}} \leqslant N \cdot ((\sum_{\alpha=0}^{\kappa}(\mathrm{inv}_{\mathbb{P}})(\alpha))_{\kappa\in\mathbb{N}})(k+N)$. Consider k being an element of $\mathbb{N}$ such that $\sum(\mathrm{inv}_{\mathbb{P}} \uparrow k) < \frac{1}{2}$. Set $p = \mathrm{pr}(k)$. For every N, $\frac{N}{2} < 2^p \cdot \lfloor\sqrt{N}\rfloor$ by (8), (7), [5, (3)]. □

Observe that $\mathrm{inv}_{\mathbb{P}}$ is non summable as a sequence of real numbers.

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