FORMALIZED MATHEMATICS
Vol. 30, No. 3, Pages 199–207, 2022
ISSN: 1426–2630, e-ISSN: 1898–9934
DOI: 10.2478/forma-2022-0014

sciendo.com/journal/forma

Artin's Theorem Towards the Existence of Algebraic Closures

Christoph Schwarzweller
Institute of Informatics
University of Gdańsk
Poland

Summary. This is the first part of a two-part article formalizing existence and uniqueness of algebraic closures using the Mizar system [1], [2]. Our proof follows Artin's classical one as presented by Lang in [3]. In this first part we prove that for a given field F there exists a field extension E such that every non-constant polynomial $p \in F[X]$ has a root in E. Artin's proof applies Kronecker's construction to each polynomial $p \in F[X] \backslash F$ simultaneously. To do so we need the polynomial ring $F[X_1, X_2, ...]$ with infinitely many variables, one for each polynomal $p \in F[X] \backslash F$. The desired field extension E then is $F[X_1, X_2, ...] \backslash I$, where I is a maximal ideal generated by all non-constant polynomials $p \in F[X]$. Note, that to show that I is maximal Zorn's lemma has to be applied.

In the second part this construction is iterated giving an infinite sequence of fields, whose union establishes a field extension A of F, in which every non-constant polynomial $p \in A[X]$ has a root. The field of algebraic elements of A then is an algebraic closure of F. To prove uniqueness of algebraic closures, e.g. that two algebraic closures of F are isomorphic over F, the technique of extending monomorphisms is applied: a monomorphism $F \longrightarrow A$, where A is an algebraic closure of F can be extended to a monomorphism $E \longrightarrow A$, where E is any algebraic extension of F. In case that E is algebraically closed this monomorphism is an isomorphism. Note that the existence of the extended monomorphism again relies on Zorn's lemma.

MSC: 12F05 68V20

Keywords: algebraic closures; polynomial rings with countably infinite number of variables; Emil Artin

MML identifier: `FIELD_11`, version: `8.1.12 5.71.1431`

Let us consider ordinal numbers n, m and bags b_1, b_2 of n. Now we state the propositions:

(1) If $\operatorname{support} b_1 = \{m\}$ and $\operatorname{support} b_2 = \{m\}$, then $b_1 \leqslant b_2$ iff $b_1(m) \leqslant b_2(m)$.

(2) If $\operatorname{support} b_1 = \{m\}$, then $b_2 \mid b_1$ iff $b_2 = \operatorname{EmptyBag} n$ or $\operatorname{support} b_2 = \{m\}$ and $b_2(m) \leqslant b_1(m)$. The theorem is a consequence of (1).

(3) Let us consider a field F, ordinal numbers m, n, and a bag b of n. Suppose $\operatorname{support} b = \{m\}$. Then

   (i) $\operatorname{len} \operatorname{divisors} b = b(m) + 1$, and

   (ii) for every natural number k and for every finite subset S of n such that $S = \{m\}$ and $k \in \operatorname{dom}(\operatorname{divisors} b)$ holds $(\operatorname{divisors} b)(k) = (S, k -' 1)$-bag.

   The theorem is a consequence of (1) and (2).

Let n be an ordinal number and L be a right zeroed, add-associative, right complementable, right unital, distributive, non degenerated double loop structure. Let us note that $\operatorname{PolyRing}(n, L)$ is non degenerated.

Now we state the proposition:

(4) Let us consider a non degenerated commutative ring R, a commutative ring extension S of R, and an ordinal number n. Then $\operatorname{PolyRing}(n, S)$ is a commutative ring extension of $\operatorname{PolyRing}(n, R)$.

   Proof: Every polynomial of n,R is a polynomial of n,S. The carrier of $\operatorname{PolyRing}(n, R) \subseteq$ the carrier of $\operatorname{PolyRing}(n, S)$. For every polynomials p, q of n,R and for every polynomials p_1, q_1 of n,S such that $p = p_1$ and $q = q_1$ holds $p + q = p_1 + q_1$. The addition of $\operatorname{PolyRing}(n, R) =$ (the addition of $\operatorname{PolyRing}(n, S)) \upharpoonright$ (the carrier of $\operatorname{PolyRing}(n, R)$). For every polynomials p, q of n,R and for every polynomials p_1, q_1 of n,S such that $p = p_1$ and $q = q_1$ holds $p * q = p_1 * q_1$. The multiplication of $\operatorname{PolyRing}(n, R) =$ (the multiplication of $\operatorname{PolyRing}(n, S)) \upharpoonright$ (the carrier of $\operatorname{PolyRing}(n, R)$). $\square$

Let R be a non degenerated ring, n be an ordinal number, and p be a polynomial of n,R. The functor Leading-Term(p) yielding a bag of n is defined by the term

(Def. 1) $$\begin{cases} (\operatorname{SgmX}(\operatorname{BagOrder} n, \operatorname{Support} p))(\operatorname{len} \operatorname{SgmX}(\operatorname{BagOrder} n, \operatorname{Support} p)), \\ \quad \textbf{if } p \neq 0_n R, \\ \operatorname{EmptyBag} n, \textbf{otherwise}. \end{cases}$$

The leading coefficient of p yielding an element of R is defined by the term

(Def. 2) $p(\text{Leading-Term}(p))$.

The functor Leading-Monomial p yielding a monomial of n,R is defined by the term

(Def. 3) Monom(the leading coefficient of p, Leading-Term(p)).

We introduce the notation LC p as a synonym of the leading coefficient of p and LT p as a synonym of Leading-Term(p) and LM(p) as a synonym of Leading-Monomial p.

Let us consider a non degenerated ring R, an ordinal number n, and a polynomial p of n,R. Now we state the propositions:

(5) $p = 0_n R$ if and only if Support $p = \emptyset$.

(6) LC $p = 0_R$ if and only if $p = 0_n R$. The theorem is a consequence of (5).

(7) Let us consider a non degenerated ring R, an ordinal number n, a polynomial p of n,R, and a bag b of n. Suppose $b \in$ Support p. Then $b =$ LT p if and only if for every bag b_1 of n such that $b_1 \in$ Support p holds $b_1 \leqslant b$. The theorem is a consequence of (5).

(8) Let us consider a non degenerated ring R, an ordinal number n, and a polynomial p of n,R. Then Support LM(p) $\subseteq$ Support p.

(9) Let us consider a field F, an ordinal number n, and a monomial p of n,F. Then

(i) LC $p =$ coefficient p, and

(ii) LT $p =$ term p.

The theorem is a consequence of (5).

Let us consider a non degenerated ring R, an ordinal number n, and a polynomial p of n,R. Now we state the propositions:

(10) (i) Support LM(p) $= \emptyset$, or

(ii) Support LM(p) $= \{$LT $p\}$.

The theorem is a consequence of (5), (8), and (6).

(11) LM(p) $= 0_n R$ if and only if $p = 0_n R$. The theorem is a consequence of (5), (8), and (6).

(12) (i) (LM(p))(LT p) $=$ LC p, and

(ii) for every bag b of n such that $b \neq$ LT p holds (LM(p))(b) $= 0_R$.

(13) (i) LT LM(p) $=$ LT p, and

(ii) LC LM(p) $=$ LC p.

Let us consider an ordinal number n, a non degenerated ring R, and elements a, b of R. Now we state the propositions:

(14) $(a\restriction(n, R)) + (b\restriction(n, R)) = a + b\restriction(n, R)$.

(15) $(a\restriction(n, R)) * (b\restriction(n, R)) = a \cdot b\restriction(n, R)$.

Let R, S be non degenerated commutative rings, n be an ordinal number, p be a polynomial of n,R, and x be a function from n into S. The functor $\mathrm{ExtEval}(p, x)$ yielding an element of S is defined by

(Def. 4) there exists a finite sequence y of elements of S such that $it = \sum y$ and $\mathrm{len}\, y = \mathrm{len}\, \mathrm{SgmX}(\mathrm{BagOrder}\, n, \mathrm{Support}\, p)$ and for every element i of $\mathbb{N}$ such that $1 \leqslant i \leqslant \mathrm{len}\, y$ holds $y(i) = (p \cdot (\mathrm{SgmX}(\mathrm{BagOrder}\, n, \mathrm{Support}\, p)))(i)(\in S) \cdot (\mathrm{eval}((\mathrm{SgmX}(\mathrm{BagOrder}\, n, \mathrm{Support}\, p))_{/i}, x))$.

Let us consider non degenerated commutative rings R, S, an ordinal number n, and a function x from n into S. Now we state the propositions:

(16) $\mathrm{ExtEval}(0_n R, x) = 0_S$. The theorem is a consequence of (5).

(17) If R is a subring of S, then $\mathrm{ExtEval}(1\_(n, R), x) = 1_S$.

(18) Let us consider non degenerated commutative rings R, S, an ordinal number n, a polynomial p of n,R, and a bag b of n. Suppose $\mathrm{Support}\, p = \{b\}$. Let us consider a function x from n into S. Then $\mathrm{ExtEval}(p, x) = p(b)(\in S) \cdot (\mathrm{eval}(b, x))$.

PROOF: Reconsider $s_2 = \mathrm{Support}\, p$ as a finite subset of $\mathrm{Bags}\, n$. Set $s_1 = \mathrm{SgmX}(\mathrm{BagOrder}\, n, s_2)$. For every object u such that $u \in \mathrm{dom}\, s_1$ holds $u \in \{1\}$. Consider y being a finite sequence of elements of the carrier of S such that $\mathrm{ExtEval}(p, x) = \sum y$ and $\mathrm{len}\, y = \mathrm{len}\, \mathrm{SgmX}(\mathrm{BagOrder}\, n, \mathrm{Support}\, p)$ and for every element i of $\mathbb{N}$ such that $1 \leqslant i \leqslant \mathrm{len}\, y$ holds $y(i) = (p \cdot (\mathrm{SgmX}(\mathrm{BagOrder}\, n, s_2)))(i)(\in S) \cdot (\mathrm{eval}((\mathrm{SgmX}(\mathrm{BagOrder}\, n, s_2))_{/i}, x))$. □

Let us consider non degenerated commutative rings R, S, an ordinal number n, polynomials p, q of n,R, and a function x from n into S. Now we state the propositions:

(19) If R is a subring of S, then $\mathrm{ExtEval}(p+q, x) =$ $\mathrm{ExtEval}(p, x) + \mathrm{ExtEval}(q, x)$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every polynomial p of n,R such that $\overline{\overline{\mathrm{Support}\, p}} = \$_1$ holds $\mathrm{ExtEval}(p+q, x) = \mathrm{ExtEval}(p, x) + \mathrm{ExtEval}(q, x)$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. $\mathcal{P}[0]$. For every natural number k, $\mathcal{P}[k]$. □

(20) If R is a subring of S, then $\mathrm{ExtEval}(p * q, x) =$ $(\mathrm{ExtEval}(p, x)) \cdot (\mathrm{ExtEval}(q, x))$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every polynomial p of n,R such that $\overline{\overline{\mathrm{Support}\, p}} = \$_1$ holds $\mathrm{ExtEval}(p*q, x) = (\mathrm{ExtEval}(p, x)) \cdot (\mathrm{ExtEval}(q, x))$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. $\mathcal{P}[0]$. For every natural number k, $\mathcal{P}[k]$. □

Let F be a field. The functor $\mathrm{nCP}(F)$ yielding a non empty subset of the carrier of $\mathrm{PolyRing}(F)$ is defined by the term

(Def. 5) the set of all p where p is a non constant element of the carrier of PolyRing(F).

One can verify that $\overline{\overline{\mathrm{nCP}(F)}}$ is non empty and there exists a function from nCP(F) into $\overline{\overline{\mathrm{nCP}(F)}}$ which is bijective.

Let g be a function from nCP(F) into $\overline{\overline{\mathrm{nCP}(F)}}$ and p be a non constant element of the carrier of PolyRing(F). Observe that the functor $g(p)$ yields an ordinal number. Let m be an ordinal number and p be a polynomial over F. The functor Poly(m, p) yielding a polynomial of $\overline{\overline{\mathrm{nCP}(F)}}$,$F$ is defined by

(Def. 6) $it(\mathrm{EmptyBag}\,\overline{\overline{\mathrm{nCP}(F)}}) = p(0)$ and for every bag b of $\overline{\overline{\mathrm{nCP}(F)}}$ such that support $b = \{m\}$ holds $it(b) = p(b(m))$ and for every bag b of $\overline{\overline{\mathrm{nCP}(F)}}$ such that support $b \neq \emptyset$ and support $b \neq \{m\}$ holds $it(b) = 0_F$.

Let g be a bijective function from nCP(F) into $\overline{\overline{\mathrm{nCP}(F)}}$. The functor nCP$(g, F)$ yielding a non empty subset of PolyRing$(\overline{\overline{\mathrm{nCP}(F)}}, F)$ is defined by the term

(Def. 7) the set of all Poly$(g(p), p)$ where p is a non constant element of the carrier of PolyRing(F).

Let m be an ordinal number and p be a polynomial over F. Observe that Poly$(m, \mathrm{LM}(p))$ is monomial-like. Now we state the propositions:

(21) Let us consider a field F, and an ordinal number m. Suppose $m \in \overline{\overline{\mathrm{nCP}(F)}}$. Let us consider a polynomial p over F. Then Poly$(m, p) = 0_{\overline{\overline{\mathrm{nCP}(F)}}}F$ if and only if $p = \mathbf{0}.F$. The theorem is a consequence of (5).

(22) Let us consider a field F, and an ordinal number m. Suppose $m \in \overline{\overline{\mathrm{nCP}(F)}}$. Let us consider a polynomial p over F, and an element a of F. Then Poly$(m, p) = a\restriction(\overline{\overline{\mathrm{nCP}(F)}}, F)$ if and only if $p = a\restriction F$.

(23) Let us consider a field F, and an ordinal number m. Suppose $m \in \overline{\overline{\mathrm{nCP}(F)}}$. Let us consider a non zero element p of the carrier of PolyRing(F). Then Support Poly$(m, p) = \{\mathrm{EmptyBag}\,\overline{\overline{\mathrm{nCP}(F)}}\}$ if and only if p is constant. The theorem is a consequence of (22) and (21).

(24) Let us consider a field F, and ordinal numbers m_1, m_2. Suppose m_1, $m_2 \in \overline{\overline{\mathrm{nCP}(F)}}$. Let us consider non constant polynomials p_1, p_2 over F. Suppose Poly$(m_1, p_1) = $ Poly(m_2, p_2). Then

(i) $m_1 = m_2$, and

(ii) $p_1 = p_2$.

The theorem is a consequence of (21), (23), and (5).

(25) Let us consider a field F, and an ordinal number m. Suppose $m \in \overline{\overline{\mathrm{nCP}(F)}}$. Let us consider a constant polynomial p over F. Then

(i) $\mathrm{LT}\,\mathrm{Poly}(m, p) = \mathrm{EmptyBag}\,\overline{\overline{\mathrm{nCP}(F)}}$, and

(ii) $\mathrm{LC}\,\mathrm{Poly}(m, p) = p(0)$.

The theorem is a consequence of (22).

(26) Let us consider a field F, and an ordinal number m. Suppose $m \in \overline{\overline{\mathrm{nCP}(F)}}$. Let us consider a non constant polynomial p over F. Then

(i) $(\mathrm{LT}\,\mathrm{Poly}(m, p))(m) = \deg(p)$, and

(ii) for every ordinal number o such that $o \neq m$ holds $(\mathrm{LT}\,\mathrm{Poly}(m, p))(o) = 0$.

Proof: Set $n = \overline{\overline{\mathrm{nCP}(F)}}$. Set $q = \mathrm{Poly}(m, p)$. Reconsider $S = \{m\}$ as a finite subset of n. Reconsider $d = \deg(p)$ as a non zero element of $\mathbb{N}$. Set $b = (S, d)$-bag. $b \in \mathrm{Support}\,q$. For every bag b_1 of n such that $b_1 \in \mathrm{Support}\,q$ holds $b_1 \leqslant b$ by [4, (7),(6)]. $b = \mathrm{LT}\,q$. □

Let us consider a field F, an ordinal number m, and a polynomial p over F. Now we state the propositions:

(27) Suppose $m \in \overline{\overline{\mathrm{nCP}(F)}}$. Then

(i) $\mathrm{LC}\,\mathrm{Poly}(m, \mathrm{LM}(p)) = \mathrm{LC}\,\mathrm{Poly}(m, p)$, and

(ii) $\mathrm{LT}\,\mathrm{Poly}(m, \mathrm{LM}(p)) = \mathrm{LT}\,\mathrm{Poly}(m, p)$.

The theorem is a consequence of (25) and (26).

(28) Suppose $m \in \overline{\overline{\mathrm{nCP}(F)}}$. Then $\mathrm{Poly}(m, \mathrm{LM}(p)) = \mathrm{Monom}(\mathrm{LC}\,\mathrm{Poly}(m, p), \mathrm{LT}\,\mathrm{Poly}(m, p))$. The theorem is a consequence of (9) and (27).

(29) If $m \in \overline{\overline{\mathrm{nCP}(F)}}$, then $\mathrm{LM}(\mathrm{Poly}(m, p)) = \mathrm{Poly}(m, \mathrm{LM}(p))$.

(30) Let us consider a field F, an ordinal number m, and polynomials p, q over F. Then $\mathrm{Poly}(m, p + q) = \mathrm{Poly}(m, p) + \mathrm{Poly}(m, q)$.

(31) Let us consider a field F, an ordinal number m, and a polynomial p over F. Then $\mathrm{Poly}(m, -p) = -\mathrm{Poly}(m, p)$.

(32) Let us consider a field F, a non zero element a of F, a natural number i, and an ordinal number m. Suppose $m \in \overline{\overline{\mathrm{nCP}(F)}}$. Then $\mathrm{Poly}(m, \mathrm{anpoly}(a, 0)) * \mathrm{Poly}(m, \mathrm{anpoly}(1_F, i)) = \mathrm{Poly}(m, \mathrm{anpoly}(a, i))$. The theorem is a consequence of (22).

(33) Let us consider a field F, an element i of $\mathbb{N}$, and an ordinal number m. Suppose $m \in \overline{\overline{\mathrm{nCP}(F)}}$. Then $\mathrm{Poly}(m, \mathrm{anpoly}(1_F, 1)) * \mathrm{Poly}(m, \mathrm{anpoly}(1_F, i)) = \mathrm{Poly}(m, \mathrm{anpoly}(1_F, i + 1))$. The theorem is a consequence of (22) and (3).

(34) Let us consider a field F, a natural number i, and an ordinal number m. Suppose $m \in \overline{\overline{\mathrm{nCP}(F)}}$. Then $\mathrm{power}_{\mathrm{PolyRing}(\overline{\overline{\mathrm{nCP}(F)}}, F)}(\mathrm{Poly}(m, \mathrm{anpoly}(1_F,$

$1)), i) = \mathrm{Poly}(m, \mathrm{anpoly}(1_F, i))$.

PROOF: Set $f = \mathrm{power}_{\mathrm{PolyRing}(\overline{\overline{\mathrm{nCP}(F)}}, F)}$. Define $\mathcal{P}[\text{natural number}] \equiv f(\mathrm{Poly}(m, \mathrm{anpoly}(1_F, 1)), \$_1) = \mathrm{Poly}(m, \mathrm{anpoly}(1_F, \$_1))$. $\mathcal{P}[0]$ by [5, (7)], (22). For every natural number k, $\mathcal{P}[k]$. $\square$

(35) Let us consider a field F, a non constant element p of the carrier of PolyRing(F), and an ordinal number m. Suppose $m \in \overline{\overline{\mathrm{nCP}(F)}}$. Then $\mathrm{Poly}(m, \mathrm{anpoly}(\mathrm{LC}\, p, \deg(p))) = \mathrm{LM}(\mathrm{Poly}(m, p))$. The theorem is a consequence of (28).

(36) Let us consider a field F, and a finite subset P of the carrier of PolyRing (F). Then there exists an extension E of F such that for every non constant element p of the carrier of PolyRing(F) such that $p \in P$ holds p has a root in E.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every field F for every finite subset P of the carrier of PolyRing(F) such that $\overline{\overline{P}} = \$_1$ there exists an extension E of F such that for every non constant element p of the carrier of PolyRing(F) such that $p \in P$ holds p has a root in E. $\mathcal{P}[0]$ by [6, (6)]. For every natural number k, $\mathcal{P}[k]$. Consider n being a natural number such that $\overline{\overline{P}} = n$. $\square$

(37) Let us consider a field F, an extension E of F, and an ordinal number m. Suppose $m \in \overline{\overline{\mathrm{nCP}(F)}}$. Let us consider a polynomial p over F, and a function x from $\overline{\overline{\mathrm{nCP}(F)}}$ into E. Then $\mathrm{ExtEval}(\mathrm{Poly}(m, p), x) = \mathrm{ExtEval}(p, x_{/m})$.

PROOF: Set $q = \mathrm{Poly}(m, p)$. Set $n = \overline{\overline{\mathrm{nCP}(F)}}$. Define $\mathcal{P}[\text{natural number}] \equiv$ for every polynomial p over F for every function x from n into E such that $\overline{\overline{\mathrm{Support}\,\mathrm{Poly}(m, p)}} = \$_1$ holds $\mathrm{ExtEval}(\mathrm{Poly}(m, p), x) = \mathrm{ExtEval}(p, x_{/m})$. For every natural number k, $\mathcal{P}[k]$. Consider n being a natural number such that $\overline{\overline{\mathrm{Support}\, q}} = n$. $\square$

(38) Let us consider a non degenerated commutative ring R, a non empty subset M of R, and an object o. Then $o \in M$–ideal if and only if there exists a non empty, finite subset P of R and there exists a linear combination L of P such that $P \subseteq M$ and $o = \sum L$.

Let F be a field and g be a bijective function from $\mathrm{nCP}(F)$ into $\overline{\overline{\mathrm{nCP}(F)}}$. Let us observe that $(\mathrm{nCP}(g, F))$–ideal is proper.

Let R be a non degenerated, commutative ring and I be a proper ideal of R.

A maximal ideal of I is an ideal of R defined by

(Def. 8) $I \subseteq it$ and it is maximal.

Observe that every maximal ideal of I is maximal.

Let F be a field, g be a bijective function from $\mathrm{nCP}(F)$ into $\overline{\overline{\mathrm{nCP}(F)}}$, and I be a maximal ideal of $(\mathrm{nCP}(g, F))$–ideal. The functor $\mathrm{KroneckerField}(F, g, I)$ yielding a field is defined by the term

(Def. 9) $\frac{\mathrm{PolyRing}(\overline{\overline{\mathrm{nCP}(F)}}, F)}{I}$.

Let n be an ordinal number and R be a non degenerated ring. The functor $\pi_{n \to n/R}$ yielding a function from R into $\mathrm{PolyRing}(n, R)$ is defined by

(Def. 10) for every element a of R, $it(a) = a \restriction (n, R)$.

Let R be a non degenerated commutative ring. One can check that $\pi_{n \to n/R}$ is additive, multiplicative, and unity-preserving and $\pi_{n \to n/R}$ is monomorphic.

Let F be a field, g be a bijective function from $\mathrm{nCP}(F)$ into $\overline{\overline{\mathrm{nCP}(F)}}$, and I be a maximal ideal of $(\mathrm{nCP}(g, F))$–ideal. The functor $\mathrm{emb}(F, I, g)$ yielding a function from F into $\mathrm{KroneckerField}(F, g, I)$ is defined by the term

(Def. 11) (the canonical homomorphism of I into quotient field)$\cdot$
$(\pi_{\overline{\overline{\mathrm{nCP}(F)}} \to \overline{\overline{\mathrm{nCP}(F)}}/F})$.

Note that $\mathrm{emb}(F, I, g)$ is additive, multiplicative, and unity-preserving and $\mathrm{emb}(F, I, g)$ is monomorphic and $\mathrm{KroneckerField}(F, g, I)$ is F-monomorphic and F-homomorphic.

Let m be an ordinal number. The functor $\mathrm{KrRoot}(I, m)$ yielding an element of $\mathrm{KroneckerField}(F, g, I)$ is defined by the term

(Def. 12) $[\mathrm{Poly}(m, \langle 0_F, 1_F \rangle)]_{\mathrm{EqRel}(\mathrm{PolyRing}(\overline{\overline{\mathrm{nCP}(F)}}, F), I)}$.

Now we state the propositions:

(39) Let us consider a field F, a bijective function g from $\mathrm{nCP}(F)$ into $\overline{\overline{\mathrm{nCP}(F)}}$, a maximal ideal I of $(\mathrm{nCP}(g, F))$–ideal, and an element a of F. Then $(\mathrm{emb}(F, I, g))(a) = [a \restriction (\overline{\overline{\mathrm{nCP}(F)}}, F)]_{\mathrm{EqRel}(\mathrm{PolyRing}(\overline{\overline{\mathrm{nCP}(F)}}, F), I)}$.

(40) Let us consider a field F, a bijective function g from $\mathrm{nCP}(F)$ into $\overline{\overline{\mathrm{nCP}(F)}}$, a maximal ideal I of $(\mathrm{nCP}(g, F))$–ideal, an element p of the carrier of $\mathrm{PolyRing}(F)$, and an element n of $\mathbb{N}$. Then $(\mathrm{PolyHom}(\mathrm{emb}(F, I, g)))(p)(n) = [p(n) \restriction (\overline{\overline{\mathrm{nCP}(F)}}, F)]_{\mathrm{EqRel}(\mathrm{PolyRing}(\overline{\overline{\mathrm{nCP}(F)}}, F), I)}$.
The theorem is a consequence of (39).

(41) Let us consider a field F, a bijective function g from $\mathrm{nCP}(F)$ into $\overline{\overline{\mathrm{nCP}(F)}}$, a maximal ideal I of $(\mathrm{nCP}(g, F))$–ideal, an element p of the carrier of $\mathrm{PolyRing}(F)$, and an ordinal number m. Suppose $m \in \overline{\overline{\mathrm{nCP}(F)}}$. Then $\mathrm{eval}((\mathrm{PolyHom}(\mathrm{emb}(F, I, g)))(p), \mathrm{KrRoot}(I, m)) = [\mathrm{Poly}(m, p)]_{\mathrm{EqRel}(\mathrm{PolyRing}(\overline{\overline{\mathrm{nCP}(F)}}, F), I)}$.

(42) Let us consider a field F, a bijective function g from $\mathrm{nCP}(F)$ into

$\overline{\overline{\mathrm{nCP}(F)}}$, a maximal ideal I of $(\mathrm{nCP}(g, F))$–ideal, and a non constant element p of the carrier of PolyRing(F). Then KrRoot$(I, g(p))$ is a root of (PolyHom(emb(F, I, g)))(p). The theorem is a consequence of (41).

(43) Let us consider a field F. Then there exists an extension E_1 of F such that for every non constant element p of the carrier of PolyRing(F), p has a root in E_1. The theorem is a consequence of (42), (39), and (40).

References

[1] Grzegorz Bancerek, Czesław Byliński, Adam Grabowski, Artur Korniłowicz, Roman Matuszewski, Adam Naumowicz, Karol Pąk, and Josef Urban. Mizar: State-of-the-art and beyond. In Manfred Kerber, Jacques Carette, Cezary Kaliszyk, Florian Rabe, and Volker Sorge, editors, *Intelligent Computer Mathematics*, volume 9150 of *Lecture Notes in Computer Science*, pages 261–279. Springer International Publishing, 2015. ISBN 978-3-319-20614-1. doi:10.1007/978-3-319-20615-8_17.

[2] Grzegorz Bancerek, Czesław Byliński, Adam Grabowski, Artur Korniłowicz, Roman Matuszewski, Adam Naumowicz, and Karol Pąk. The role of the Mizar Mathematical Library for interactive proof development in Mizar. *Journal of Automated Reasoning*, 61(1):9–32, 2018. doi:10.1007/s10817-017-9440-6.

[3] Serge Lang. *Algebra*. Springer Verlag, 2002 (Revised Third Edition).

[4] Piotr Rudnicki. Little Bezout theorem (factor theorem). *Formalized Mathematics*, 12(**1**): 49–58, 2004.

[5] Christoph Schwarzweller. On roots of polynomials over $F[X]/\langle p\rangle$. *Formalized Mathematics*, 27(**2**):93–100, 2019. doi:10.2478/forma-2019-0010.

[6] Christoph Schwarzweller. Field extensions and Kronecker's construction. *Formalized Mathematics*, 27(**3**):229–235, 2019. doi:10.2478/forma-2019-0022.

Accepted September 30, 2022