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Prime Representing Polynomial with 10 Unknowns – Introduction

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Summary. The main purpose of the article is to construct a sophisticated polynomial proposed by Matiyasevich and Robinson [5] that is often used to reduce the number of unknowns in diophantine representations, using the Mizar [1], [2] formalism. The polynomial

$$J_k(a_1,\dots,a_k,x) = \prod_{\epsilon_1,\dots,\epsilon_k\in\{\pm1\}} (x+\epsilon_1\sqrt{a_1}+\epsilon_2\sqrt{a_2}W+\ldots+\epsilon_k\sqrt{a_k}W^{k-1})$$

with $W=\sum_{i=1}^{k} x_i^2$ has integer coefficients and $J_k(a_1,\dots,a_k,x)=0$ for some $a_1,\dots,a_k,x\in\mathbb{Z}$ if and only if $a_1,\dots,a_k$ are all squares. However although it is nontrivial to observe that this expression is a polynomial, i.e., eliminating similar elements in the product of all combinations of signs we obtain an expression where every square root will occur with an even power. This work has been partially presented in [7].

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1. Preliminaries

From now on i, j, n, k, m denote natural numbers, a, b, x, y, z denote objects, F, G denote finite sequence-yielding finite sequences, f, g, p, q denote finite sequences, X, Y denote sets, and D denotes a non empty set.

Let X be a finite set. The functor Ω_X yielding an element of Fin X is defined by the term

(Def. 1) X.

Now we state the propositions:

(1) Let us consider non empty sets X_1, X_2, Y, a binary operation F on Y, an element B_1 of $\operatorname{Fin} X_1$, and an element B_2 of $\operatorname{Fin} X_2$. Suppose $B_1 = B_2$ and ($B_1 \neq \emptyset$ or F is unital) and F is associative and commutative. Let us consider a function f_1 from X_1 into Y, and a function f_2 from X_2 into Y. Suppose $f_1 \restriction B_1 = f_2 \restriction B_2$. Then $F\text{-}\sum_{B_1} f_1 = F\text{-}\sum_{B_2} f_2$.

Proof: Consider G_1 being a function from $\operatorname{Fin} X_1$ into Y such that $F\text{-}\sum_{B_1} f_1 = G_1(B_1)$ and for every element e of Y such that e is a unity w.r.t. F holds $G_1(\emptyset) = e$ and for every element x of X_1, $G_1(\{x\}) = f_1(x)$ and for every element B' of $\operatorname{Fin} X_1$ such that $B' \subseteq B_1$ and $B' \neq \emptyset$ for every element x of X_1 such that $x \in B_1 \setminus B'$ holds $G_1(B' \cup \{x\}) = F(G_1(B'), f_1(x))$.

Consider G_2 being a function from $\operatorname{Fin} X_2$ into Y such that $F\text{-}\sum_{B_2} f_2 = G_2(B_2)$ and for every element e of Y such that e is a unity w.r.t. F holds $G_2(\emptyset) = e$ and for every element x of X_2, $G_2(\{x\}) = f_2(x)$ and for every element B' of $\operatorname{Fin} X_2$ such that $B' \subseteq B_2$ and $B' \neq \emptyset$ for every element x of X_2 such that $x \in B_2 \setminus B'$ holds $G_2(B' \cup \{x\}) = F(G_2(B'), f_2(x))$. Define $\mathcal{P}[\text{set}] \equiv$ if $\$_1 \subseteq B_1$, then $G_1(\$_1) = G_2(\$_1)$ or $\$_1 = \emptyset$. For every element B' of $\operatorname{Fin} X_1$ and for every element b of X_1 such that $\mathcal{P}[B']$ and $b \notin B'$ holds $\mathcal{P}[B' \cup \{b\}]$. For every element B of $\operatorname{Fin} X_1$, $\mathcal{P}[B]$. □

(2) Let us consider a non empty set D, elements d_1, d_2 of D, and a binary operation B on D. Suppose B is unital, associative, and commutative and has inverse operation. Then

  (i) $B(($the inverse operation w.r.t. $B)(d_1), d_2) =($the inverse operation w.r.t. $B)(B(d_1, ($the inverse operation w.r.t. $B)(d_2)))$, and

  (ii) $B(d_1, ($the inverse operation w.r.t. $B)(d_2)) =($the inverse operation w.r.t. $B)(B(($the inverse operation w.r.t. $B)(d_1), d_2))$.

(3) Let us consider a non empty set D, and binary operations A, M on D. Suppose A is commutative, associative, and unital and M is commutative and distributive w.r.t. A and for every element d of D, $M(\mathbf{1}_A, d) = \mathbf{1}_A$. Let us consider non empty, finite sets X, Y, a function f from X into D, a function g from Y into D, an element a of $\operatorname{Fin} X$, and an element b of $\operatorname{Fin} Y$. Then $A\text{-}\sum_{a \times b} M_{f,g} = M(A\text{-}\sum_a f, A\text{-}\sum_b g)$.

Proof: Set $m = M_{f,g}$. Define $\mathcal{P}[\text{set}] \equiv$ for every element a of $\operatorname{Fin} X$ for every element b of $\operatorname{Fin} Y$ such that $a = \$_1$ holds $A\text{-}\sum_{a \times b} m = M(A\text{-}\sum_a f,$ $A\text{-}\sum_b g)$. $\mathcal{P}[\emptyset_X]$. For every element E of $\operatorname{Fin} X$ and for every element e of X such that $\mathcal{P}[E]$ and $e \notin E$ holds $\mathcal{P}[E \cup \{e\}]$. For every element E of $\operatorname{Fin} X$, $\mathcal{P}[E]$. □

(4) Let us consider a non empty set D, binary operations M, A on D, and an element d of D. Suppose M is unital and A is associative and unital and has inverse operation and M is distributive w.r.t. A. Then

(i) if n is even, then $M \odot n \mapsto$ (the inverse operation w.r.t. $A)(d) = M \odot n \mapsto d$, and

(ii) if n is odd, then $M \odot n \mapsto$ (the inverse operation w.r.t. $A)(d) =$ (the inverse operation w.r.t. $A)(M \odot n \mapsto d)$.

Proof: Set $I =$ the inverse operation w.r.t. A. Define $\mathcal{P}$[natural number] $\equiv$ if $\$_1$ is even, then $M \odot \$_1 \mapsto I(d) = M \odot \$_1 \mapsto d$ and if $\$_1$ is not even, then $M \odot \$_1 \mapsto I(d) = I(M \odot \$_1 \mapsto d)$. If $\mathcal{P}[i]$, then $\mathcal{P}[i+1]$. $\mathcal{P}[i]$. □

(5) Let us consider a finite sequence s. Suppose $s^{-1}(\{y\}) \neq \emptyset$. Then there exists a permutation p of Seg len s such that

(i) $(s \cdot p)(\operatorname{len} s) = y$, and

(ii) $p = p^{-1}$.

Let D be a non empty set. Let us note that there exists a finite sequence of elements of D^* which is non empty and non-empty. Let X, Y be non empty sets. Let us note that $X \Cup Y$ is non empty. Let X, Y be finite sets. One can check that $X \Cup Y$ is finite. Now we state the propositions:

(6) Let us consider sets X, Y. Then $2^X \Cup 2^Y = 2^{X \cup Y}$.

(7) Let us consider sets X, Y_1, Y_2. Then $X \Cup (Y_1 \cup Y_2) = (X \Cup Y_1) \cup (X \Cup Y_2)$.

(8) If X misses $\bigcup Y$, then $\overline{\overline{Y \Cup \{X\}}} = \overline{\overline{Y}}$.

Proof: Define $\mathcal{F}(\text{set}) = \$_1 \cup X$. Consider f being a function such that $\operatorname{dom} f = Y$ and for every set A such that $A \in Y$ holds $f(A) = \mathcal{F}(A)$. $\operatorname{rng} f \subseteq Y \Cup \{X\}$. $Y \Cup \{X\} \subseteq \operatorname{rng} f$. f is one-to-one. □

(9) Suppose $m \neq 0$. Then $2 \cdot \overline{\overline{2^{(\operatorname{Seg} m) \setminus \{1\}}}} = \overline{\overline{2^{(\operatorname{Seg}(1+m)) \setminus \{1\}}}}$.

Proof: Set $S = (\operatorname{Seg} m) \setminus \{1\}$. Set $F = 2^S$. $\overline{\overline{F \Cup \{\emptyset\}}} = \overline{\overline{F}}$. $\{m+1\}$ misses $\bigcup F$. $\overline{\overline{F \Cup \{\{m+1\}\}}} = \overline{\overline{F}}$. $F \Cup 2^{\{m+1\}} = (F \Cup \{\emptyset\}) \cup (F \Cup \{\{m+1\}\})$. $F \Cup \{\emptyset\}$ misses $F \Cup \{\{m+1\}\}$. □

2. Selected Operations on Set Families

Let X be a set and a, b be objects. The functor $\operatorname{ext}(X, a, b)$ yielding a set is defined by the term

(Def. 2) $\{A \cup \{b\}$, where A is an element of $X : a \in A\} \cup \{A$, where A is an element of $X : a \notin A$ and $A \in X\}$.

The functor $\operatorname{swap}(X, a, b)$ yielding a set is defined by the term

(Def. 3) $\{A \setminus \{a\} \cup \{b\}$, where A is an element of $X : a \in A\} \cup \{A \cup \{a\}$, where A is an element of $X : a \notin A$ and $A \in X\}$.

Now we state the propositions:

(10) If $y \notin \bigcup Y$, then $\overline{\overline{Y}} = \overline{\overline{\mathrm{ext}(Y, x, y)}}$.

PROOF: Set $P = \{X$, where X is an element of $Y : x \in X\}$. Set $P_5 = \{X \cup \{y\}$, where X is an element of $Y : x \in X\}$. Set $N = \{X$, where X is an element of $Y : x \notin X$ and $X \in Y\}$. Define $\mathcal{F}(\text{set}) = \$_1 \cup \{y\}$. Consider f being a function such that $\mathrm{dom}\, f = P$ and for every set A such that $A \in P$ holds $f(A) = \mathcal{F}(A)$. $\mathrm{rng}\, f \subseteq P_5$. $P_5 \subseteq \mathrm{rng}\, f$. f is one-to-one. $P \subseteq Y$. $N \subseteq Y$. $Y \subseteq N \cup P$. N misses P_5. N misses P. $\square$

(11) If $y \notin \bigcup Y$, then $\overline{\overline{Y}} = \overline{\overline{\mathrm{swap}(Y, x, y)}}$.

PROOF: Set $P = \{X$, where X is an element of $Y : x \in X\}$. Set $P_5 = \{X \setminus \{x\} \cup \{y\}$, where X is an element of $Y : x \in X\}$. Set $N = \{X$, where X is an element of $Y : x \notin X$ and $X \in Y\}$. Set $N_2 = \{X \cup \{x\}$, where X is an element of $Y : x \notin X$ and $X \in Y\}$. Define $\mathcal{F}(\text{set}) = \$_1 \setminus \{x\} \cup \{y\}$.

Consider f being a function such that $\mathrm{dom}\, f = P$ and for every set A such that $A \in P$ holds $f(A) = \mathcal{F}(A)$. $\mathrm{rng}\, f \subseteq P_5$. $P_5 \subseteq \mathrm{rng}\, f$. f is one-to-one. Define $\mathcal{G}(\text{set}) = \$_1 \cup \{x\}$. Consider g being a function such that $\mathrm{dom}\, g = N$ and for every set A such that $A \in N$ holds $g(A) = \mathcal{G}(A)$. $\mathrm{rng}\, g \subseteq N_2$. $N_2 \subseteq \mathrm{rng}\, g$. g is one-to-one. $P \subseteq Y$. $N \subseteq Y$. $Y \subseteq N \cup P$. N_2 misses P_5. N misses P. $\square$

(12) $\mathrm{swap}(\emptyset, x, y) = \emptyset$.

(13) $\mathrm{swap}(X \cup Y, x, y) = \mathrm{swap}(X, x, y) \cup \mathrm{swap}(Y, x, y)$.

(14) If $Y \in \mathrm{swap}(X, x, y)$ and $x \neq y$ and $y \notin \bigcup X$, then $x \in Y$ iff $y \notin Y$.

(15) $\mathrm{ext}(\emptyset, x, y) = \emptyset$.

(16) $\mathrm{ext}(X \cup Y, x, y) = \mathrm{ext}(X, x, y) \cup \mathrm{ext}(Y, x, y)$.

(17) If $Y \in \mathrm{ext}(X, x, y)$ and $y \notin \bigcup X$, then $x \in Y$ iff $y \in Y$.

Let X be a finite set and a, b be objects. Observe that $\mathrm{swap}(X, a, b)$ is finite and $\mathrm{ext}(X, a, b)$ is finite.

Let f be a function. The functor $\mathrm{Swap}(f, a, b)$ yielding a function is defined by

(Def. 4) $\mathrm{dom}\, it = \mathrm{dom}\, f$ and for every x such that $x \in \mathrm{dom}\, f$ holds if $a \in f(x)$, then $it(x) = f(x) \setminus \{a\} \cup \{b\}$ and if $a \notin f(x)$, then $it(x) = f(x) \cup \{a\}$.

The functor $\mathrm{Ext}(f, a, b)$ yielding a function is defined by

(Def. 5) $\mathrm{dom}\, it = \mathrm{dom}\, f$ and for every x such that $x \in \mathrm{dom}\, f$ holds if $a \in f(x)$, then $it(x) = f(x) \cup \{b\}$ and if $a \notin f(x)$, then $it(x) = f(x)$.

Let f be a finite sequence. Observe that $\mathrm{Swap}(f, a, b)$ is $(\mathrm{len}\, f)$-element and finite sequence-like and $\mathrm{Ext}(f, a, b)$ is $(\mathrm{len}\, f)$-element and finite sequence-like.

Let us consider finite sequences f, g. Now we state the propositions:

(18) $\mathrm{Swap}(f \frown g, a, b) = \mathrm{Swap}(f, a, b) \frown \mathrm{Swap}(g, a, b)$.
PROOF: Set $S_9 = \mathrm{Swap}(f, a, b)$. Set $S_{11} = \mathrm{Swap}(g, a, b)$. Set $S_{10} = \mathrm{Swap}(f \frown g, a, b)$. For every k such that $1 \leqslant k \leqslant \mathrm{len}\, S_{10}$ holds $S_{10}(k) = (S_9 \frown S_{11})(k)$. $\square$

(19) $\mathrm{Ext}(f \frown g, a, b) = \mathrm{Ext}(f, a, b) \frown \mathrm{Ext}(g, a, b)$.
PROOF: Set $E_{25} = \mathrm{Ext}(f, a, b)$. Set $E_{27} = \mathrm{Ext}(g, a, b)$. Set $E_{26} = \mathrm{Ext}(f \frown g, a, b)$. For every k such that $1 \leqslant k \leqslant \mathrm{len}\, E_{26}$ holds $E_{26}(k) = (E_{25} \frown E_{27})(k)$. $\square$

Let us consider a function f. Now we state the propositions:

(20) If $b \neq x$ and $b \neq y$, then $b \in (\mathrm{Ext}(f, x, y))(a)$ iff $b \in f(a)$.
PROOF: If $b \in (\mathrm{Ext}(f, x, y))(a)$, then $b \in f(a)$. $\square$

(21) If $b \neq x$ and $b \neq y$, then $b \in (\mathrm{Swap}(f, x, y))(a)$ iff $b \in f(a)$.
PROOF: If $b \in (\mathrm{Swap}(f, x, y))(a)$, then $b \in f(a)$. $\square$

(22) If $x \neq y$ and $y \notin \bigcup X$ and $y \notin \bigcup Y$, then $\mathrm{ext}(X, x, y)$ misses $\mathrm{swap}(Y, x, y)$. The theorem is a consequence of (14) and (17).

(23) Let us consider functions f, g. Then $(\mathrm{Swap}(f, x, y)) \cdot g = \mathrm{Swap}(f \cdot g, x, y)$.
PROOF: Set $S = \mathrm{Swap}(f, x, y)$. Set $S_{11} = \mathrm{Swap}(f \cdot g, x, y)$. $\mathrm{dom}(S \cdot g) \subseteq \mathrm{dom}(f \cdot g)$. $\mathrm{dom}(f \cdot g) \subseteq \mathrm{dom}(S \cdot g)$. For every a such that $a \in \mathrm{dom}\, S_{11}$ holds $S_{11}(a) = (S \cdot g)(a)$. $\square$

(24) Let us consider a function f. Then $\mathrm{Swap}(f, x, y) \restriction X = \mathrm{Swap}(f \restriction X, x, y)$. The theorem is a consequence of (23).

(25) Let us consider functions f, g. Then $(\mathrm{Ext}(f, x, y)) \cdot g = \mathrm{Ext}(f \cdot g, x, y)$.
PROOF: Set $E = \mathrm{Ext}(f, x, y)$. Set $E_{27} = \mathrm{Ext}(f \cdot g, x, y)$. $\mathrm{dom}(E \cdot g) \subseteq \mathrm{dom}(f \cdot g)$. $\mathrm{dom}(f \cdot g) \subseteq \mathrm{dom}(E \cdot g)$. For every a such that $a \in \mathrm{dom}\, E_{27}$ holds $E_{27}(a) = (E \cdot g)(a)$. $\square$

(26) Let us consider a function f. Then $\mathrm{Ext}(f, x, y) \restriction X = \mathrm{Ext}(f \restriction X, x, y)$. The theorem is a consequence of (25).

Let X be a finite set. Let us observe that every enumeration of X is $\overline{\overline{X}}$-element and X-valued. Let us consider a finite set F and an enumeration E of F. Now we state the propositions:

(27) If $y \notin \bigcup F$, then $\mathrm{Swap}(E, x, y)$ is an enumeration of $\mathrm{swap}(F, x, y)$. The theorem is a consequence of (11).

(28) If $y \notin \bigcup F$, then $\mathrm{Ext}(E, x, y)$ is an enumeration of $\mathrm{ext}(F, x, y)$. The theorem is a consequence of (10).

(29) If $x \in X$, then $\mathrm{ext}(\{X\}, x, y) = \{X \cup \{y\}\}$.

(30) If $x \notin X$, then $\mathrm{ext}(\{X\}, x, y) = \{X\}$.

(31) If $x \in X$, then $\mathrm{swap}(\{X\}, x, y) = \{X \setminus \{x\} \cup \{y\}\}$.

(32) If $x \notin X$, then $\mathrm{swap}(\{X\}, x, y) = \{X \cup \{x\}\}$.

Let X be a non empty set and a, b be objects. One can check that $\mathrm{ext}(X, a, b)$ is non empty and $\mathrm{swap}(X, a, b)$ is non empty. Now we state the propositions:

(33) If $y \notin \bigcup X$ and $y \notin \bigcup Y$, then X misses Y iff $\mathrm{ext}(X, x, y)$ misses $\mathrm{ext}(Y, x, y)$.
PROOF: If X misses Y, then $\mathrm{ext}(X, x, y)$ misses $\mathrm{ext}(Y, x, y)$. Consider a being an object such that $a \in X$ and $a \in Y$. □

(34) If $x \neq y$ and $y \notin \bigcup X$ and $y \notin \bigcup Y$, then X misses Y iff $\mathrm{swap}(X, x, y)$ misses $\mathrm{swap}(Y, x, y)$.
PROOF: If X misses Y, then $\mathrm{swap}(X, x, y)$ misses $\mathrm{swap}(Y, x, y)$. Consider a being an object such that $a \in X$ and $a \in Y$. □

Let us consider a function f. Now we state the propositions:

(35) If $z \in \mathrm{dom}\, f$, then $\mathrm{Ext}(\langle f(z) \rangle, x, y) = \langle (\mathrm{Ext}(f, x, y))(z) \rangle$.

(36) If $z \in \mathrm{dom}\, f$, then $\mathrm{Swap}(\langle f(z) \rangle, x, y) = \langle (\mathrm{Swap}(f, x, y))(z) \rangle$.

(37) If $z \in \mathrm{dom}\, f$, then $\mathrm{ext}(\{f(z)\}, x, y) = \{(\mathrm{Ext}(f, x, y))(z)\}$. The theorem is a consequence of (29) and (30).

(38) If $z \in \mathrm{dom}\, f$, then $\mathrm{swap}(\{f(z)\}, x, y) = \{(\mathrm{Swap}(f, x, y))(z)\}$. The theorem is a consequence of (31) and (32).

(39) Suppose $m \neq 0$. Then $2^{(\mathrm{Seg}(m+2)) \setminus \{1\}} = \mathrm{ext}(2^{(\mathrm{Seg}(m+1)) \setminus \{1\}}, 1 + m, 2 + m) \cup \mathrm{swap}(2^{(\mathrm{Seg}(m+1)) \setminus \{1\}}, 1 + m, 2 + m)$. The theorem is a consequence of (10), (11), (9), and (22).

3. Function where Each Value is Repeated an Even Number of Times

Let f be a finite function. We say that f has evenly repeated values if and only if

(Def. 6) $\overline{\overline{f^{-1}(\{y\})}}$ is even.

One can verify that every finite function which is empty has also evenly repeated values.

Let x be an object. Observe that $\langle x, x \rangle$ has evenly repeated values.

Now we state the proposition:

(40) Let us consider finite sequences f, g with evenly repeated values. Then $f \frown g$ has evenly repeated values.

Let F be a set. We say that F is with evenly repeated values-member if and only if

(Def. 7) for every object y such that $y \in F$ holds y is a finite function with evenly repeated values.

One can verify that every set which is empty is also with evenly repeated values-member.

Let X be a finite sequence-membered set. Note that every element of $\mathrm{Fin}\, X$ is finite sequence-membered.

Let Y be a finite sequence-membered set. Note that $X \cup Y$ is finite sequence-membered. Now we state the propositions:

(41) Let us consider finite sequence-membered sets P_1, S_1, S_2. Then $P_1 \frown (S_1 \cup S_2) = P_1 \frown S_1 \cup P_1 \frown S_2$.

(42) Let us consider finite sequence-membered sets P_1, P_2, S_1. Then $(P_1 \cup P_2) \frown S_1 = P_1 \frown S_1 \cup P_2 \frown S_1$.

(43) Let us consider finite sequences f, g. Then $\{f\} \frown \{g\} = \{f \frown g\}$.

Let f be a finite function with evenly repeated values. Observe that $\{f\}$ is with evenly repeated values-member. Let g be a finite function with evenly repeated values. Let us note that $\{f, g\}$ is with evenly repeated values-member. Let F, G be with evenly repeated values-member, finite sequence-membered sets. Let us note that $F \frown G$ is with evenly repeated values-member. Now we state the proposition:

(44) Let us consider a finite function f, and a permutation p of $\mathrm{dom}\, f$. Then f has evenly repeated values if and only if $f \cdot p$ has evenly repeated values.
PROOF: If f has evenly repeated values, then $f \cdot p$ has evenly repeated values. □

4. Cartesian Product of Domains in Finite Sequences

Let F be a finite sequence-yielding finite sequence. The functor $\mathrm{dom}_\kappa\, F(\kappa)$ yielding a finite subset of $\mathbb{N}^*$ is defined by

(Def. 8) for every object x, $x \in \mathit{it}$ iff there exists a finite sequence p such that $p = x$ and $\mathrm{len}\, p = \mathrm{len}\, F$ and for every i such that $i \in \mathrm{dom}\, p$ holds $p(i) \in \mathrm{dom}(F(i))$.

Now we state the propositions:

(45) $\mathrm{dom}_\kappa\, F(\kappa)$ is not empty if and only if F is non-empty.
PROOF: If $\mathrm{dom}_\kappa\, F(\kappa)$ is not empty, then F is non-empty. Set $L = \mathrm{len}\, F \mapsto 1$. For every i such that $i \in \mathrm{dom}\, L$ holds $L(i) \in \mathrm{dom}(F(i))$. □

(46) $\mathrm{dom}_\kappa\, \emptyset(\kappa) = \{\emptyset\}$.

Let F be a finite sequence-yielding finite sequence. Let us observe that $\mathrm{dom}_\kappa\, F(\kappa)$ is finite sequence-membered. Now we state the proposition:

(47) $p \in \mathrm{dom}_\kappa\, F(\kappa)$ if and only if $\mathrm{len}\, p = \mathrm{len}\, F$ and for every i such that $i \in \mathrm{dom}\, p$ holds $p(i) \in \mathrm{dom}(F(i))$.

Let F be a finite sequence-yielding finite sequence. Let us note that every element of $\mathrm{dom}_\kappa\, F(\kappa)$ is $\mathbb{N}$-valued.

Let F be a non-empty, finite sequence-yielding finite sequence. Let us note that $\mathrm{dom}_\kappa\, F(\kappa)$ is non empty. Now we state the propositions:

(48) If $f \in \mathrm{dom}_\kappa\, F(\kappa)$ and $g \in \mathrm{dom}_\kappa\, G(\kappa)$, then $f \frown g \in \mathrm{dom}_\kappa\, F \frown G(\kappa)$.
PROOF: Set $f_{11} = f \frown g$. Set $F_8 = F \frown G$. $\mathrm{len}\, f = \mathrm{len}\, F$ and $\mathrm{len}\, g = \mathrm{len}\, G$. For every i such that $i \in \mathrm{dom}\, f_{11}$ holds $f_{11}(i) \in \mathrm{dom}(F_8(i))$. □

(49) Let us consider finite sequence-membered sets P, S. Suppose $P \subseteq \mathrm{dom}_\kappa\, F(\kappa)$ and $S \subseteq \mathrm{dom}_\kappa\, G(\kappa)$. Then $P \frown S \subseteq \mathrm{dom}_\kappa\, F \frown G(\kappa)$. The theorem is a consequence of (48).

(50) Suppose ($\mathrm{len}\, f = \mathrm{len}\, F$ or $\mathrm{len}\, g = \mathrm{len}\, G$) and $f \frown g \in \mathrm{dom}_\kappa\, F \frown G(\kappa)$. Then

(i) $f \in \mathrm{dom}_\kappa\, F(\kappa)$, and

(ii) $g \in \mathrm{dom}_\kappa\, G(\kappa)$.

PROOF: Set $f_{11} = f \frown g$. Set $F_8 = F \frown G$. $\mathrm{len}\, f_{11} = \mathrm{len}\, f + \mathrm{len}\, g$ and $\mathrm{len}\, F_8 = \mathrm{len}\, F + \mathrm{len}\, G$ and $\mathrm{len}\, F_8 = \mathrm{len}\, f_{11}$. For every i such that $i \in \mathrm{dom}\, f$ holds $f(i) \in \mathrm{dom}(F(i))$. For every i such that $i \in \mathrm{dom}\, g$ holds $g(i) \in \mathrm{dom}(G(i))$. □

(51) $f \in \mathrm{dom}_\kappa \langle g \rangle(\kappa)$ if and only if $\mathrm{len}\, f = 1$ and $f(1) \in \mathrm{dom}\, g$. The theorem is a consequence of (47).

(52) $\mathrm{dom}_\kappa\, F \frown \langle g \frown \langle x \rangle \rangle(\kappa) = \mathrm{dom}_\kappa\, F \frown \langle g \rangle(\kappa) \cup \{f \frown \langle 1 + \mathrm{len}\, g \rangle$, where f is an element of $\mathrm{dom}_\kappa\, F(\kappa) : f \in \mathrm{dom}_\kappa\, F(\kappa)\}$.
PROOF: Set $S = \{f \frown \langle 1 + \mathrm{len}\, g \rangle$, where f is an element of $\mathrm{dom}_\kappa\, F(\kappa) : f \in \mathrm{dom}_\kappa\, F(\kappa)\}$. Set $g_4 = g \frown \langle x \rangle$. $\mathrm{dom}_\kappa\, F \frown \langle g_4 \rangle(\kappa) \subseteq \mathrm{dom}_\kappa\, F \frown \langle g \rangle(\kappa) \cup S$. □

(53) $\mathrm{dom}_\kappa\, F \frown \langle \langle x \rangle \rangle(\kappa) = \{f \frown \langle 1 \rangle$, where f is an element of $\mathrm{dom}_\kappa\, F(\kappa) : f \in \mathrm{dom}_\kappa\, F(\kappa)\}$. The theorem is a consequence of (45) and (52).

(54) Let us consider finite sequence-yielding finite sequences F, G. Then (the concatenation of $\mathbb{N})^\circ((\mathrm{dom}_\kappa\, F(\kappa)) \times (\mathrm{dom}_\kappa\, G(\kappa))) = \mathrm{dom}_\kappa\, F \frown G(\kappa)$.
PROOF: Set $C =$ the concatenation of $\mathbb{N}$. $C^\circ((\mathrm{dom}_\kappa\, F(\kappa)) \times (\mathrm{dom}_\kappa\, G(\kappa))) \subseteq \mathrm{dom}_\kappa\, F \frown G(\kappa)$ by [3, (4)], (48). Reconsider $f_{11} = xy$ as an $\mathbb{N}$-valued finite sequence. $\mathrm{len}\, f_{11} = \mathrm{len}(F \frown G) = \mathrm{len}\, F + \mathrm{len}\, G$. Set $f = f_{11} \restriction \mathrm{len}\, F$. Consider g being a finite sequence such that $f_{11} = f \frown g$. $f \in \mathrm{dom}_\kappa\, F(\kappa)$ and $g \in \mathrm{dom}_\kappa\, G(\kappa)$. □

(55) $\mathrm{dom}_\kappa \langle f \rangle(\kappa) = \{\langle i \rangle$, where i is an element of $\mathbb{N} : i \in \mathrm{dom}\, f\}$.
PROOF: $\mathrm{dom}_\kappa \langle f \rangle(\kappa) \subseteq \{\langle i \rangle$, where i is an element of $\mathbb{N} : i \in \mathrm{dom}\, f\}$. Consider i being an element of $\mathbb{N}$ such that $y = \langle i \rangle$ and $i \in \mathrm{dom}\, f$. □

Let us consider n and F. One can check that $F \restriction n$ is finite sequence-yielding.

Now we state the propositions:

(56) If $f \in \mathrm{dom}_\kappa F(\kappa)$, then $f \restriction n \in \mathrm{dom}_\kappa F \restriction n(\kappa)$. The theorem is a consequence of (47).

(57) $\overline{\overline{\mathrm{dom}_\kappa \langle g \rangle(\kappa)}} = \mathrm{len}\, g$.

PROOF: Set $G = \langle g \rangle$. Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ for every finite sequence f such that $f = \$_1$ holds $f(1) = \$_2$. For every object x such that $x \in \mathrm{dom}_\kappa G(\kappa)$ there exists an object y such that $y \in \mathrm{dom}\, g$ and $\mathcal{P}[x, y]$. Consider F being a function such that $\mathrm{dom}\, F = \mathrm{dom}_\kappa G(\kappa)$ and $\mathrm{rng}\, F \subseteq \mathrm{dom}\, g$ and for every object x such that $x \in \mathrm{dom}_\kappa G(\kappa)$ holds $\mathcal{P}[x, F(x)]$. F is one-to-one. $\mathrm{dom}\, g \subseteq \mathrm{rng}\, F$. □

(58) $\overline{\overline{\mathrm{dom}_\kappa F \frown \langle f \rangle(\kappa)}} = \overline{\overline{\mathrm{dom}_\kappa F(\kappa)}} \cdot (\mathrm{len}\, f)$.

PROOF: Define $\mathcal{D}[\text{natural number}] \equiv$ for every finite sequence f such that $\mathrm{len}\, f = \$_1$ holds $\overline{\overline{\mathrm{dom}_\kappa F \frown \langle f \rangle(\kappa)}} = \overline{\overline{\mathrm{dom}_\kappa F(\kappa)}} \cdot (\mathrm{len}\, f)$. $\mathcal{D}[0]$. If $\mathcal{D}[n]$, then $\mathcal{D}[n+1]$. $\mathcal{D}[n]$. □

5. Some Operations on Finite Sequences

Let F be a finite sequence-yielding finite sequence. The functor $\mathrm{App}(F)$ yielding a finite sequence-yielding function is defined by

(Def. 9) $\mathrm{dom}\, it = \mathrm{dom}_\kappa F(\kappa)$ and for every finite sequence p such that $p \in \mathrm{dom}_\kappa F(\kappa)$ holds $\mathrm{len}\, it(p) = \mathrm{len}\, p$ and for every i such that $i \in \mathrm{dom}\, p$ holds $(it(p))(i) = F(i)(p(i))$.

Let D be a non empty set and F be a (D^*)-valued finite sequence. Let us note that the functor $\mathrm{App}(F)$ yields a function from $\mathrm{dom}_\kappa F(\kappa)$ into D^*. Now we state the propositions:

(59) $(\mathrm{App}(\emptyset))(\emptyset) = \emptyset$. The theorem is a consequence of (46).

(60) If $i \in \mathrm{dom}\, f$, then $(\mathrm{App}(\langle f \rangle))(\langle i \rangle) = \langle f(i) \rangle$. The theorem is a consequence of (51).

(61) Suppose $f \in \mathrm{dom}_\kappa F(\kappa)$ and $g \in \mathrm{dom}_\kappa G(\kappa)$. Then $(\mathrm{App}(F \frown G))(f \frown g) = (\mathrm{App}(F))(f) \frown (\mathrm{App}(G))(g)$.

PROOF: Set $F_8 = F \frown G$. Set $A_1 = \mathrm{App}(F)$. Set $A_3 = \mathrm{App}(G)$. Set $A_2 = \mathrm{App}(F_8)$. $f \frown g \in \mathrm{dom}_\kappa F_8(\kappa)$. $\mathrm{len}\, f = \mathrm{len}\, F$ and $\mathrm{len}\, g = \mathrm{len}\, G$. For every i such that $1 \leqslant i \leqslant \mathrm{len}\, A_2(f \frown g)$ holds $A_2(f \frown g)(i) = (A_1(f) \frown A_3(g))(i)$. □

Let D be a non empty set and F be a non empty, (D^*)-valued finite sequence. One can verify that $\mathrm{App}(F)$ is non-empty.

Let f be a (D^*)-valued function and x be an object. One can check that the functor $f(x)$ yields a finite sequence of elements of D. Let B be a binary operation on D and F be a (D^*)-valued function. The functor $B \odot F$ yielding a function from $\mathrm{dom}\, F$ into D is defined by

(Def. 10) for every x such that $x \in \mathrm{dom}\, F$ holds $it(x) = B \odot F(x)$.

From now on B, A, M denote binary operations on D, F, G denote (D^*)-valued finite sequences, f denotes a finite sequence of elements of D, and d, d_1, d_2 denote elements of D.

Let D be a non empty set, B be a binary operation on D, and F be a (D^*)-valued finite sequence. Let us observe that $B \odot F$ is $(\mathrm{len}\, F)$-element and finite sequence-like.

Let D be a set and f be a finite sequence of elements of D. Observe that the functor $\langle f \rangle$ yields a finite sequence of elements of D^*. Now we state the propositions:

(62) $A \odot \langle f \rangle = \langle A \odot f \rangle$.

(63) $A \odot F \frown G = (A \odot F) \frown (A \odot G)$.
PROOF: Set $F_8 = F \frown G$. For every n such that $1 \leqslant n \leqslant \mathrm{len}\, F + \mathrm{len}\, G$ holds $(A \odot F_8)(n) = ((A \odot F) \frown (A \odot G))(n)$. $\square$

Let f be a non empty finite sequence. Observe that $\langle f \rangle$ is non-empty.

From now on F, G denote non-empty, non empty finite sequences of elements of D^* and f denotes a non empty finite sequence of elements of D.

Now we state the propositions:

(64) Suppose A is commutative and associative. Let us consider non empty finite sequences f, g, a function F from $\mathrm{dom}\, f$ into D, a function G from $\mathrm{dom}\, g$ into D, and a function F_8 from $\mathrm{dom}(f \frown g)$ into D. Suppose $f = F$ and $g = G$ and $f \frown g = F_8$. Then $A\text{-}\sum_{\Omega_{\mathrm{dom}(f \frown g)}} F_8 = A(A\text{-}\sum_{\Omega_{\mathrm{dom}\, f}} F, A\text{-}\sum_{\Omega_{\mathrm{dom}\, g}} G)$.
PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every non empty finite sequences f, g such that $\$_1 = \mathrm{len}\, g$ for every function F from $\mathrm{dom}\, f$ into D for every function G from $\mathrm{dom}\, g$ into D for every function F_8 from $\mathrm{dom}(f \frown g)$ into D such that $f = F$ and $g = G$ and $f \frown g = F_8$ holds $A\text{-}\sum_{\Omega_{\mathrm{dom}(f \frown g)}} F_8 = A(A\text{-}\sum_{\Omega_{\mathrm{dom}\, f}} F, A\text{-}\sum_{\Omega_{\mathrm{dom}\, g}} G)$. $\mathcal{P}[1]$. For every n such that $1 \leqslant n$ holds if $\mathcal{P}[n]$, then $\mathcal{P}[n+1]$. For every n such that $1 \leqslant n$ holds $\mathcal{P}[n]$. $\square$

(65) Suppose M is commutative and associative. Then $M\text{-}\sum_{\Omega_{\mathrm{dom}(F \frown G)}}(A \odot F \frown G) = M(M\text{-}\sum_{\Omega_{\mathrm{dom}\, F}}(A \odot F), M\text{-}\sum_{\Omega_{\mathrm{dom}\, G}}(A \odot G))$. The theorem is a consequence of (63) and (64).

(66) If M is commutative and associative, then $M\text{-}\sum_{\Omega_{\mathrm{dom}\langle f \rangle}}(A \odot \langle f \rangle) = A \odot f$. The theorem is a consequence of (62).

(67) Suppose M is commutative and associative and A is commutative and associative and M is left distributive w.r.t. A. Let us consider a function

f_9 from $\mathrm{dom}\, f$ into D. Suppose for every x such that $x \in \mathrm{dom}\, f$ holds $f_9(x) = M(M\text{-}\sum_{\Omega_{\mathrm{dom}\, F}}(A \odot F), f(x))$. Then $M\text{-}\sum_{\Omega_{\mathrm{dom}(F \frown \langle f \rangle)}}(A \odot F \frown \langle f \rangle) = A\text{-}\sum_{\Omega_{\mathrm{dom}\, f}} f_9$.

PROOF: Define $\mathcal{P}$[natural number] $\equiv$ for every f such that $\mathrm{len}\, f = \$_1$ for every function f_9 from $\mathrm{dom}\, f$ into D such that for every x such that $x \in \mathrm{dom}\, f$ holds $f_9(x) = M(M\text{-}\sum_{\Omega_{\mathrm{dom}\, F}}(A \odot F), f(x))$ holds $M\text{-}\sum_{\Omega_{\mathrm{dom}(F \frown \langle f \rangle)}}(A \odot F \frown \langle f \rangle) = A\text{-}\sum_{\Omega_{\mathrm{dom}\, f}} f_9$. If $\mathcal{P}[n]$, then $\mathcal{P}[n+1]$. $\mathcal{P}[n]$. $\square$

(68) Suppose $\mathrm{len}\, F = 1$ and M is commutative and associative and A is commutative and associative. Then $M\text{-}\sum_{\Omega_{\mathrm{dom}\, F}}(A \odot F) = A\text{-}\sum_{\Omega_{\mathrm{dom}(\mathrm{App}(F))}}(M \odot \mathrm{App}(F))$.

PROOF: Set $F_1 = F(1)$. Set $f = M \odot \mathrm{App}(F)$. Set $X = \mathrm{dom}(\mathrm{App}(F))$. Consider G being a function from $\mathrm{Fin}\, X$ into D such that $A\text{-}\sum_{\Omega_X} f = G(\Omega_X)$ and for every element e of D such that e is a unity w.r.t. A holds $G(\emptyset) = e$ and for every element x of X, $G(\{x\}) = f(x)$ and for every element B' of $\mathrm{Fin}\, X$ such that $B' \subseteq \Omega_X$ and $B' \neq \emptyset$ for every element x of X such that $x \in \Omega_X \setminus B'$ holds $G(B' \cup \{x\}) = A(G(B'), f(x))$.

Consider s being a sequence of D such that $s(1) = F_1(1)$ and for every natural number n such that $0 \neq n$ and $n < \mathrm{len}\, F_1$ holds $s(n+1) = A(s(n), F_1(n+1))$ and $A \odot F_1 = s(\mathrm{len}\, F_1)$. Define $\mathcal{R}$(natural number) $= \{\langle i \rangle$, where i is an element of $\mathbb{N} : i \in \mathrm{Seg}\, \$_1\}$. Define $\mathcal{P}$[natural number] $\equiv$ if $\$_1 \leqslant \mathrm{len}\, F_1$, then for every element B' of $\mathrm{Fin}\, X$ such that $B' = \mathcal{R}(\$_1)$ holds $G(B') = s(\$_1)$. $\mathcal{P}[1]$. For every j such that $1 \leqslant j$ holds if $\mathcal{P}[j]$, then $\mathcal{P}[j+1]$. For every i such that $1 \leqslant i$ holds $\mathcal{P}[i]$. $\mathcal{R}(\mathrm{len}\, F_1) = X$. $\square$

(69) Suppose M is commutative and associative and A is commutative, associative, and unital and M is distributive w.r.t. A. Then $M\text{-}\sum_{\Omega_{\mathrm{dom}\, F}}(A \odot F) = A\text{-}\sum_{\Omega_{\mathrm{dom}(\mathrm{App}(F))}}(M \odot \mathrm{App}(F))$.

PROOF: Define $\mathcal{R}$[natural number] $\equiv$ for every non-empty, non empty finite sequence F of elements of D^* such that $\mathrm{len}\, F = \$_1$ holds $M\text{-}\sum_{\Omega_{\mathrm{dom}\, F}}(A \odot F) = A\text{-}\sum_{\Omega_{\mathrm{dom}(\mathrm{App}(F))}}(M \odot \mathrm{App}(F))$. If $\mathcal{R}[n]$, then $\mathcal{R}[n+1]$. $\mathcal{R}[n]$. $\square$

6. Combination of Sign and Characteristic Functions

Let D be a non empty set, B be a binary operation on D, f be a finite sequence of elements of D, and X be a set. The functor $\mathrm{SignGen}(f, B, X)$ yielding a finite sequence of elements of D is defined by

(Def. 11) $\mathrm{dom}\, it = \mathrm{dom}\, f$ and for every i such that $i \in \mathrm{dom}\, it$ holds if $i \in X$, then $it(i) =$ (the inverse operation w.r.t. B)$(f(i))$ and if $i \notin X$, then $it(i) = f(i)$.

Note that $\mathrm{SignGen}(f, B, X)$ is $(\mathrm{len}\, f)$-element.

From now on f, g denote finite sequences of elements of D, a, b, c denote sets, and F, F_1, F_2 denote finite sets. Now we state the propositions:

(70) If X misses $\operatorname{dom} f$, then $\operatorname{SignGen}(f, B, X) = f$.

(71) $\operatorname{SignGen}(f, B, \emptyset) = f$. The theorem is a consequence of (70).

(72) $\operatorname{SignGen}(f \restriction n, B, X) = \operatorname{SignGen}(f, B, X) \restriction n$.

(73) Suppose $n + 1 = \operatorname{len} f$ and $n + 1 \in X$. Then $\operatorname{SignGen}(f, B, X) = \operatorname{SignGen}(f \restriction n, B, X) \frown \langle(\text{the inverse operation w.r.t. } B)(f(n+1))\rangle$.
PROOF: Set $n_1 = n+1$. Set $I =$ (the inverse operation w.r.t. $B)(f(n_1))$. $\operatorname{SignGen}(f \restriction n, B, X) = \operatorname{SignGen}(f, B, X) \restriction n$. For every i such that $1 \leqslant i \leqslant \operatorname{len} \operatorname{SignGen}(f, B, X)$ holds $(\operatorname{SignGen}(f, B, X))(i) = (\operatorname{SignGen}(f \restriction n, B, X) \frown \langle I\rangle)(i)$. □

(74) If $n+1 = \operatorname{len} f$ and $n+1 \notin X$, then $\operatorname{SignGen}(f, B, X) = \operatorname{SignGen}(f \restriction n, B, X) \frown \langle f(n+1)\rangle$.
PROOF: Set $n_1 = n+1$. Set $I = f(n_1)$. $\operatorname{SignGen}(f \restriction n, B, X) = \operatorname{SignGen}(f, B, X) \restriction n$. For every i such that $1 \leqslant i \leqslant \operatorname{len} \operatorname{SignGen}(f, B, X)$ holds $(\operatorname{SignGen}(f, B, X))(i) = (\operatorname{SignGen}(f \restriction n, B, X) \frown \langle I\rangle)(i)$. □

(75) If $\operatorname{dom} f \subseteq X$, then $\operatorname{SignGen}(f, B, X) =$ (the inverse operation w.r.t. $B) \cdot f$.
PROOF: For every k such that $k \in \operatorname{dom}(\operatorname{SignGen}(f, B, X))$ holds $(\operatorname{SignGen}(f, B, X))(k) = ((\text{the inverse operation w.r.t. } B) \cdot f)(k)$. □

(76) If B is unital and associative and has inverse operation, then $\operatorname{SignGen}(\operatorname{SignGen}(f, B, X), B, X) = f$.
PROOF: Set $C = \operatorname{SignGen}(f, B, X)$. For every k such that $1 \leqslant k \leqslant \operatorname{len} f$ holds $(\operatorname{SignGen}(C, B, X))(k) = f(k)$. □

Let E be a non empty set, D be a set, p be a D-valued finite sequence, and h be a function from D into E. Let us observe that $h \cdot p$ is $(\operatorname{len} p)$-element and finite sequence-like.

Let D be a non empty set, B be a binary operation on D, f be a finite sequence of elements of D, and F be a finite set. The functor $\operatorname{SignGenOp}(f, B, F)$ yielding a function from F into D^* is defined by

(Def. 12) if $X \in F$, then $it(X) = \operatorname{SignGen}(f, B, X)$.

Now we state the propositions:

(77) Let us consider an enumeration E of $\{x\}$. Then $E = \langle x\rangle$.

(78) Let us consider an enumeration E of $\{X\}$. Then $(\operatorname{SignGenOp}(f, B, \{X\})) \cdot E = \langle \operatorname{SignGen}(f, B, X)\rangle$. The theorem is a consequence of (77).

(79) Let us consider an enumeration E_1 of F_1, and an enumeration E_2 of F_2. Suppose F_1 misses F_2. Then $E_1 \frown E_2$ is an enumeration of $F_1 \cup F_2$.

(80) Let us consider an enumeration E of F. Suppose $i \in \operatorname{dom} E$ or $i \in \operatorname{dom}((\text{SignGenOp}(f, B, F)) \cdot E)$. Then $((\text{SignGenOp}(f, B, F)) \cdot E)(i) = \text{SignGen}(f, B, E(i))$.
PROOF: Set $C = \text{SignGenOp}(f, B, F)$. $i \in \operatorname{dom}(C \cdot E)$. □

(81) Let us consider an enumeration E_1 of F_1, an enumeration E_2 of F_2, and an enumeration E_{12} of $F_1 \cup F_2$. Suppose $E_{12} = E_1 \frown E_2$. Then $(\text{SignGenOp}(f, B, F_1 \cup F_2)) \cdot E_{12} =$
$(\text{SignGenOp}(f, B, F_1)) \cdot E_1 \frown (\text{SignGenOp}(f, B, F_2)) \cdot E_2$.
PROOF: Set $C_1 = \text{SignGenOp}(f, B, F_1)$. Set $C_2 = \text{SignGenOp}(f, B, F_2)$. Set $C_{12} = \text{SignGenOp}(f, B, F_1 \cup F_2)$. For every k such that $1 \leqslant k \leqslant \operatorname{len} C_{12} \cdot E_{12}$ holds $(C_{12} \cdot E_{12})(k) = (C_1 \cdot E_1 \frown C_2 \cdot E_2)(k)$. □

Let us consider an enumeration E of F. Now we state the propositions:

(82) Suppose (B is unital or $\operatorname{len} f \geqslant 1$) and $1 + \operatorname{len} f \notin \bigcup F$. Then $B \odot (\text{SignGenOp}(f \frown \langle d \rangle, B, F)) \cdot E = B^{\circ}(B \odot (\text{SignGenOp}(f, B, F)) \cdot E, d)$.
PROOF: Set $f_{10} = f \frown \langle d \rangle$. Set $C = \text{SignGenOp}(f, B, F)$. Set $C_{23} = \text{SignGenOp}(f_{10}, B, F)$. For every x such that $x \in \operatorname{dom}(C \cdot E)$ holds $(B^{\circ}(B \odot C \cdot E, d))(x) = (B \odot C_{23} \cdot E)(x)$. □

(83) Suppose (B is unital or $\operatorname{len} f \geqslant 1$) and $1 + \operatorname{len} f \in \bigcap F$. Then $B \odot (\text{SignGenOp}(f \frown \langle d \rangle, B, F)) \cdot E =$
$B^{\circ}(B \odot (\text{SignGenOp}(f, B, F)) \cdot E, (\text{the inverse operation w.r.t. } B)(d))$.
PROOF: Set $f_{10} = f \frown \langle d \rangle$. Set $C = \text{SignGenOp}(f, B, F)$. Set $C_{23} = \text{SignGenOp}(f_{10}, B, F)$. Set $I =$ the inverse operation w.r.t. B. For every x such that $x \in \operatorname{dom}(C \cdot E)$ holds $(B^{\circ}(B \odot C \cdot E, I(d)))(x) = (B \odot C_{23} \cdot E)(x)$.
□

(84) Suppose (B is unital or $\operatorname{len} f \geqslant 1$) and B is associative and $1 + \operatorname{len} f \notin \bigcup F$ and $2 + \operatorname{len} f \notin \bigcup F$. Then $B \odot (\text{SignGenOp}((f \frown \langle d_1 \rangle) \frown \langle d_2 \rangle, B, F)) \cdot E = B \odot (\text{SignGenOp}(f \frown \langle B(d_1, d_2) \rangle, B, F)) \cdot E$. The theorem is a consequence of (82).

(85) Suppose (B is unital or $\operatorname{len} f \geqslant 1$) and B is associative and $1 + \operatorname{len} f \notin \bigcup F$ and $2 + \operatorname{len} f \in \bigcap F$. Then $B \odot (\text{SignGenOp}((f \frown \langle d_1 \rangle) \frown \langle d_2 \rangle, B, F)) \cdot E = B \odot (\text{SignGenOp}(f \frown \langle B(d_1, (\text{the inverse operation w.r.t. } B)(d_2)) \rangle, B, F)) \cdot E$. The theorem is a consequence of (83) and (82).

(86) Suppose B is unital, associative, and commutative and has inverse operation and $1 + \operatorname{len} f \in \bigcap F$ and $2 + \operatorname{len} f \notin \bigcup F$. Then $B \odot (\text{SignGenOp}((f \frown \langle d_1 \rangle) \frown \langle d_2 \rangle, B, F)) \cdot E = B \odot (\text{SignGenOp}(f \frown \langle B(d_1, ((\text{the inverse operation w.r.t. } B)(d_2))) \rangle, B, F)) \cdot E$. The theorem is a consequence of (82), (83), and (2).

(87) Suppose B is unital, associative, and commutative and has inverse operation and $1 + \operatorname{len} f$, $2 + \operatorname{len} f \in \bigcap F$. Then $B \odot (\text{SignGenOp}((f \frown \langle d_1 \rangle) \frown$

$\langle d_2\rangle, B, F)) \cdot E = B \odot (\mathrm{SignGenOp}(f \frown \langle B(d_1, d_2)\rangle, B, F)) \cdot E$. The theorem is a consequence of (83) and (2).

(88) If X misses $\bigcup F$, then there exists an enumeration E_{36} of $F \Cup \{X\}$ such that for every i such that $i \in \mathrm{dom}\, E$ holds $E_{36}(i) = X \cup E(i)$.

PROOF: Define $\mathcal{F}(\text{set}) = E(\$_1) \cup X$. Consider f being a function such that $\mathrm{dom}\, f = \mathrm{dom}\, E$ and for every set A such that $A \in \mathrm{dom}\, E$ holds $f(A) = \mathcal{F}(A)$. $\mathrm{rng}\, f \subseteq F \Cup \{X\}$. $F \Cup \{X\} \subseteq \mathrm{rng}\, f$. f is one-to-one. $\square$

(89) $\mathrm{SignGen}(f, B, X) = \mathrm{SignGen}(f, B, X \cap \mathrm{dom}\, f)$.

(90) Let us consider an enumeration E_1 of F_1, and an enumeration E_2 of F_2. Suppose $\overline{\overline{F_1}} = \overline{\overline{F_2}}$ and for every i such that $i \in \mathrm{dom}\, E_1$ holds $\mathrm{dom}\, f \cap E_1(i) = \mathrm{dom}\, f \cap E_2(i)$. Then $(\mathrm{SignGenOp}(f, A, F_1)) \cdot E_1 = (\mathrm{SignGenOp}(f, A, F_2)) \cdot E_2$.

PROOF: Set $C_1 = \mathrm{SignGenOp}(f, A, F_1)$. Set $C_2 = \mathrm{SignGenOp}(f, A, F_2)$. For every i such that $1 \leqslant i \leqslant \mathrm{len}\, E_1$ holds $(C_1 \cdot E_1)(i) = (C_2 \cdot E_2)(i)$. $\square$

(91) Suppose A is unital, associative, and commutative and has inverse operation. Let us consider a finite, non empty set F. Suppose $\bigcup F \subseteq \mathrm{dom}\, f$. Let us consider finite sets F_1, F_2. Suppose $F_1 = F \Cup 2^{\{\mathrm{len}\, f+1\}}$ and $F_2 = F \Cup 2^{\{\mathrm{len}\, f+1, \mathrm{len}\, f+2\}}$. Then there exists an enumeration E_1 of F_1 and there exists an enumeration E_2 of F_2 such that $A \odot (\mathrm{SignGenOp}((f \frown \langle d_1\rangle) \frown \langle d_2\rangle, A, F_2)) \cdot E_2 = (A \odot (\mathrm{SignGenOp}(f \frown \langle A(d_1, d_2)\rangle, A, F_1)) \cdot E_1) \frown (A \odot (\mathrm{SignGenOp}(f \frown \langle A(d_1, (\text{the inverse operation w.r.t. } A)(d_2))\rangle, A, F_1)) \cdot E_1)$.

PROOF: Set $L = \mathrm{len}\, f$. Set $U_1 = F \Cup \{\{L+1\}\}$. Set $U_2 = F \Cup \{\{L+2\}\}$. Set $U_{12} = F \Cup \{\{L+1, L+2\}\}$. Set $E =$ the enumeration of F. Set $I =$ the inverse operation w.r.t. A. Set $f_{12} = (f \frown \langle d_1\rangle) \frown \langle d_2\rangle$. Set $f_3 = f \frown \langle A(d_1, d_2)\rangle$. Set $f_4 = f \frown \langle A(d_1, I(d_2))\rangle$.

Consider E_1 being an enumeration of U_1 such that for every i such that $i \in \mathrm{dom}\, E$ holds $E_1(i) = \{L+1\} \cup E(i)$. $L+2 \notin \bigcup U_1$. $L+1 \notin \bigcup U_2$. If $a \in U_{12}$, then $L+1$, $L+2 \in a$. Consider E_2 being an enumeration of U_2 such that for every i such that $i \in \mathrm{dom}\, E$ holds $E_2(i) = \{L+2\} \cup E(i)$. Consider E_{12} being an enumeration of U_{12} such that for every i such that $i \in \mathrm{dom}\, E$ holds $E_{12}(i) = \{L+1, L+2\} \cup E(i)$. F misses U_1. U_1 misses U_2. Reconsider $E_7 = E_2 \frown E_1$ as an enumeration of $U_2 \cup U_1$. F misses U_{12}. Reconsider $E_{37} = E \frown E_{12}$ as an enumeration of $F \cup U_{12}$. $\overline{\overline{U_{12}}} = \overline{\overline{F}} = \overline{\overline{U_2}}$. $\overline{\overline{U_{12}}} = \overline{\overline{F}} = \overline{\overline{U_1}}$. For every i such that $i \in \mathrm{dom}\, E_1$ holds $\mathrm{dom}\, f_3 \cap E_1(i) = \mathrm{dom}\, f_3 \cap E_{12}(i)$. For every i such that $i \in \mathrm{dom}\, E$ holds $\mathrm{dom}\, f_4 \cap E(i) = \mathrm{dom}\, f_4 \cap E_2(i)$. $F \cup U_{12}$ misses $U_2 \cup U_1$.

Reconsider $E_{16} = E_{37} \frown E_7$ as an enumeration of $(F \cup U_{12}) \cup (U_2 \cup U_1)$. $(\{\emptyset\} \cup \{\{L+1, L+2\}\}) \cup (\{\{L+1\}\} \cup \{\{L+2\}\}) = 2^{\{L+1, L+2\}}$. $F = F \Cup \{\emptyset\}$. $F \cup U_{12} = F \Cup (\{\emptyset\} \cup \{\{L+1, L+2\}\})$ and $U_2 \cup U_1 =$

$F \uplus (\{\{L+1\}\} \cup \{\{L+2\}\})$. Reconsider $e_1 = E_{16}$ as an enumeration of F_2. $F \cup U_1 = F \uplus (\{\emptyset\} \cup \{\{L+1\}\})$. $A \odot (\mathrm{SignGenOp}(f_{12}, A, F \cup U_{12})) \cdot E_{37} = A \odot (\mathrm{SignGenOp}(f_{12}, A, F)) \cdot E \frown (\mathrm{SignGenOp}(f_{12}, A, U_{12})) \cdot E_{12}$. $A \odot (\mathrm{SignGenOp}(f_{12}, A, U_2 \cup U_1)) \cdot E_7 = A \odot (\mathrm{SignGenOp}(f_{12}, A, U_2)) \cdot E_2 \frown (\mathrm{SignGenOp}(f_{12}, A, U_1)) \cdot E_1$. $(\mathrm{SignGenOp}(f_{12}, A, F_2)) \cdot e_1 = (\mathrm{SignGenOp}(f_{12}, A, (F \cup U_{12}) \cup (U_2 \cup U_1))) \cdot E_{16}$. □

7. Product over All Combinations of Sings

Let D be a non empty set, A be a binary operation on D, and M be a binary operation on D. Assume M is commutative and associative. Let f be a finite sequence of elements of D and F be a finite set. The functor $\mathrm{SignGenOp}(f, M, A, F)$ yielding an element of D is defined by

(Def. 13) for every enumeration E of 2^F, $it = M\text{-}\sum_{\Omega_{\mathrm{dom}((\mathrm{SignGenOp}(f,A,2^F))\cdot E)}} (A \odot (\mathrm{SignGenOp}(f, A, 2^F)) \cdot E)$.

Now we state the propositions:

(92) Suppose M is commutative and associative and A is commutative, associative, and unital and has inverse operation and M is distributive w.r.t. A. Let us consider non-empty, non empty finite sequences C_4, C_7, C_5 of elements of D^*. Suppose $C_5 = C_4 \frown C_7$. Let us consider an element S_1 of $\mathrm{Fin}\,\mathrm{dom}(\mathrm{App}(C_4))$, an element s_2 of $\mathrm{dom}(\mathrm{App}(C_7))$, and an element S_{12} of $\mathrm{Fin}\,\mathrm{dom}(\mathrm{App}(C_5))$. Suppose $S_{12} = S_1 \frown \{s_2\}$. Then $M(A\text{-}\sum_{S_1}(M \odot \mathrm{App}(C_4)), (M \odot \mathrm{App}(C_7))(s_2)) = A\text{-}\sum_{S_{12}}(M \odot \mathrm{App}(C_5))$.
Proof: Define $\mathcal{P}[\text{set}] \equiv$ for every element S_1 of $\mathrm{Fin}\,\mathrm{dom}(\mathrm{App}(C_4))$ for every element S_{12} of $\mathrm{Fin}\,\mathrm{dom}(\mathrm{App}(C_5))$ such that $S_1 = \$_1$ and $S_{12} = S_1 \frown \{s_2\}$ holds $M(A\text{-}\sum_{S_1}(M \odot \mathrm{App}(C_4)), A\text{-}\sum_{\{s_2\}_f}(M \odot \mathrm{App}(C_7))) = A\text{-}\sum_{S_{12}}(M \odot \mathrm{App}(C_5))$. $\mathcal{P}[\emptyset_{\mathrm{dom}(\mathrm{App}(C_4))}]$. For every element B' of $\mathrm{Fin}\,\mathrm{dom}(\mathrm{App}(C_4))$ and for every element b of $\mathrm{dom}(\mathrm{App}(C_4))$ such that $\mathcal{P}[B']$ and $b \notin B'$ holds $\mathcal{P}[B' \cup \{b\}]$. For every element B of $\mathrm{Fin}\,\mathrm{dom}(\mathrm{App}(C_4))$, $\mathcal{P}[B]$. □

(93) Suppose M is commutative and associative and A is commutative, associative, and unital and has inverse operation and M is distributive w.r.t. A. Let us consider non-empty, non empty finite sequences C_4, C_7, C_5 of elements of D^*. Suppose $C_5 = C_4 \frown C_7$. Let us consider an element S_1 of $\mathrm{Fin}\,\mathrm{dom}(\mathrm{App}(C_4))$, an element S_2 of $\mathrm{Fin}\,\mathrm{dom}(\mathrm{App}(C_7))$, and an element S_{12} of $\mathrm{Fin}\,\mathrm{dom}(\mathrm{App}(C_5))$. Suppose $S_{12} = S_1 \frown S_2$. Then $M(A\text{-}\sum_{S_1}(M \odot \mathrm{App}(C_4)), A\text{-}\sum_{S_2}(M \odot \mathrm{App}(C_7))) = A\text{-}\sum_{S_{12}}(M \odot \mathrm{App}(C_5))$.
Proof: Set $a_1 = A\text{-}\sum_{S_1}(M \odot \mathrm{App}(C_4))$. Define $\mathcal{P}[\text{natural number}] \equiv$ for every element S_2 of $\mathrm{Fin}\,\mathrm{dom}(\mathrm{App}(C_7))$ for every element S_{12} of Fin dom(A-

pp(C_5)) such that $\overline{\overline{S_2}} = \$_1$ and $S_{12} = S_1 \frown S_2$ holds $M(a_1, A\text{-}\sum_{S_2}(M \odot \mathrm{App}(C_7))) = A\text{-}\sum_{S_{12}}(M \odot \mathrm{App}(C_5))$. $\mathcal{P}[0]$. If $\mathcal{P}[n]$, then $\mathcal{P}[n+1]$ by [6, (55)], [4, (16)]. $\mathcal{P}[n]$. □

(94) Let us consider an enumeration E_1 of F_1. Then $\mathrm{dom}_\kappa(\mathrm{SignGenOp}(f, A, F_1)) \cdot E_1(\kappa) \subseteq \mathrm{dom}_\kappa(\mathrm{SignGenOp}(f \frown g, A, F_1)) \cdot E_1(\kappa)$.
PROOF: $\mathrm{len}\, x = \mathrm{len}\, E_1$. For every i such that $i \in \mathrm{dom}\, x$ holds $x(i) \in \mathrm{dom}(((\mathrm{SignGenOp}(f \frown g, A, F_1)) \cdot E_1)(i))$. □

(95) Suppose A is unital, commutative, and associative. Let us consider an enumeration E_1 of F_1, and non-empty, non empty finite sequences C_4, C_7 of elements of D^*. Suppose $C_4 = (\mathrm{SignGenOp}(f, A, F_1)) \cdot E_1$ and $C_7 = (\mathrm{SignGenOp}(f \frown g, A, F_1)) \cdot E_1$. Let us consider an element S_1 of Fin dom(App(C_4)), and an element S_2 of Fin dom(App(C_7)). Suppose $S_1 = S_2$.
Then $A\text{-}\sum_{S_1}(M \odot \mathrm{App}(C_4)) = A\text{-}\sum_{S_2}(M \odot \mathrm{App}(C_7))$.
PROOF: For every x such that $x \in \mathrm{dom}((M \odot \mathrm{App}(C_4)) \restriction S_1)$ holds $((M \odot \mathrm{App}(C_4)) \restriction S_1)(x) = ((M \odot \mathrm{App}(C_7)) \restriction S_2)(x)$. □

(96) Let us consider an enumeration E of F. Suppose $\mathrm{len}\, E = n + 1$. Then

(i) $E \restriction n$ is an enumeration of $F \setminus \{E(\mathrm{len}\, E)\}$, and

(ii) $\langle E(\mathrm{len}\, E) \rangle$ is an enumeration of $\{E(\mathrm{len}\, E)\}$, and

(iii) $F = F \setminus \{E(\mathrm{len}\, E)\} \cup \{E(\mathrm{len}\, E)\}$.

Let F be a with evenly repeated values-member set. Note that every element of F is finite, function-like, and relation-like and every element of F has evenly repeated values. Now we state the proposition:

(97) Let us consider an enumeration E_1 of F_1, and a function p. Suppose $\bigcup F_1 \subseteq \mathrm{dom}\, p$ and $p \restriction \bigcup F_1$ is one-to-one. Then

(i) $({}^\circ p) \cdot E_1$ is an enumeration of $({}^\circ p)^\circ F_1$, and

(ii) $\overline{\overline{E_1}} = \overline{\overline{({}^\circ p) \cdot E_1}}$.

PROOF: Set $I_3 = {}^\circ f$. Reconsider $f_7 = I_3 \cdot E_1$ as a finite sequence. f_7 is one-to-one. $\mathrm{rng}\, f_7 \subseteq ({}^\circ f)^\circ F_1$. $({}^\circ f)^\circ F_1 \subseteq \mathrm{rng}\, f_7$. □

Let us consider an enumeration E_1 of F_1, a function g, an enumeration g_1 of $({}^\circ g)^\circ F_1$, a finite sequence f_{11} of elements of D, and a finite sequence s. Now we state the propositions:

(98) Suppose $\bigcup F_1 \subseteq \mathrm{dom}\, g$ and $g \restriction \bigcup F_1$ is one-to-one. Then suppose $g_1 = ({}^\circ g) \cdot E_1$. Then suppose $g^\circ\, \mathrm{dom}\, f \subseteq \mathrm{dom}\, f_{11}$. Then suppose $s \in \mathrm{dom}_\kappa(\mathrm{SignGenOp}(f, A, F_1)) \cdot E_1(\kappa)$ and $\mathrm{rng}\, s \subseteq \mathrm{dom}\, g$.
Then $g \cdot s \in \mathrm{dom}_\kappa(\mathrm{SignGenOp}(f_{11}, A, ({}^\circ g)^\circ F_1)) \cdot g_1(\kappa)$.
PROOF: $\mathrm{len}(\mathrm{SignGenOp}(f, A, F_1)) \cdot E_1 = \mathrm{len}\, E_1 = \mathrm{len}\, g_1 = \mathrm{len}(\mathrm{SignGenOp}(f, A, ({}^\circ g)^\circ F_1)) \cdot g_1$. Reconsider $g_3 = g \cdot s$ as a finite sequence. $\mathrm{len}\, s = $ len(Sign-

GenOp$(f, A, F_1)) \cdot E_1$. For every i such that $i \in \operatorname{dom} g_3$ holds $g_3(i) \in \operatorname{dom}(((\mathrm{SignGenOp}(gf, A, (^\circ g)^\circ F_1)) \cdot g_1)(i))$. □

(99) Suppose $\bigcup F_1 \subseteq \operatorname{dom} g$ and g is one-to-one. Then suppose $g_1 = (^\circ g) \cdot E_1$. Then suppose $f_{11} = f \cdot (g^{-1}) \restriction \operatorname{dom} f_{11}$ and $g^\circ \operatorname{dom} f \subseteq \operatorname{dom} f_{11}$. Then suppose $s \in \operatorname{dom}_\kappa(\mathrm{SignGenOp}(f, A, F_1)) \cdot E_1(\kappa)$ and $\operatorname{rng} s \subseteq \operatorname{dom} g$. Then $(\mathrm{App}((\mathrm{SignGenOp}(f, A, F_1)) \cdot E_1))(s) = (\mathrm{App}((\mathrm{SignGenOp}(f_{11}, A, (^\circ g)^\circ F_1)) \cdot g_1))(g \cdot s)$.

PROOF: $\operatorname{len}(\mathrm{SignGenOp}(f, A, F_1)) \cdot E_1 = \operatorname{len} E_1 = \operatorname{len} g_1 = \operatorname{len}(\mathrm{SignGenOp}(f, A, (^\circ g)^\circ F_1)) \cdot g_1$. Reconsider $g_3 = g \cdot s$ as a finite sequence. Reconsider $g_3 = g \cdot s$ as a finite sequence. $\operatorname{len} g_3 = \operatorname{len} s = \operatorname{len}(\mathrm{SignGenOp}(f, A, (^\circ g)^\circ F_1)) \cdot g_1$. $g_3 \in \operatorname{dom}_\kappa(\mathrm{SignGenOp}(gf, A, (^\circ g)^\circ F_1)) \cdot g_1(\kappa)$. $\operatorname{len} s = \operatorname{len}(\mathrm{SignGenOp}(f, A, F_1)) \cdot E_1$. $g_3 = g \cdot s$ and $g_3 \in \operatorname{dom}_\kappa(\mathrm{SignGenOp}(gf, A, (^\circ g)^\circ F_1)) \cdot g_1(\kappa)$. For every i such that $1 \leqslant i \leqslant \operatorname{len} s$ holds $(\mathrm{App}((\mathrm{SignGenOp}(f, A, F_1)) \cdot E_1))(s)(i) = (\mathrm{App}((\mathrm{SignGenOp}(gf, A, (^\circ g)^\circ F_1)) \cdot g_1))(g_3)(i)$. □

(100) Let us consider an enumeration E_1 of F_1. Suppose $\bigcup F_1 \subseteq \operatorname{dom} f$. Let us consider a permutation g of $\operatorname{dom} f$, and an enumeration g_1 of $(^\circ g)^\circ F_1$. Suppose $g_1 = (^\circ g) \cdot E_1$. Let us consider a finite sequence f_{11} of elements of D. Suppose $f_{11} = f \cdot (g^{-1})$. Let us consider an element S_1 of $\operatorname{Fin} \operatorname{dom}(\mathrm{App}((\mathrm{SignGenOp}(f, A, F_1)) \cdot E_1))$. Then $\{g \cdot s$, where s is a finite sequence of elements of $\mathbb{N} : s \in S_1\}$ is an element of $\operatorname{Fin} \operatorname{dom}(\mathrm{App}((\mathrm{SignGenOp}(f_{11}, A, (^\circ g)^\circ F_1)) \cdot g_1))$.

PROOF: $\{g \cdot s$, where s is a finite sequence of elements of $\mathbb{N} : s \in S_1\} \subseteq \operatorname{dom}(\mathrm{App}((\mathrm{SignGenOp}(f_{11}, A, (^\circ g)^\circ F_1)) \cdot g_1))$. □

(101) Suppose A is unital, commutative, and associative. Let us consider an enumeration E_1 of F_1. Suppose $\bigcup F_1 \subseteq \operatorname{dom} f$. Let us consider a permutation g of $\operatorname{dom} f$, and an enumeration g_1 of $(^\circ g)^\circ F_1$. Suppose $g_1 = (^\circ g) \cdot E_1$. Let us consider a finite sequence f_{11} of elements of D. Suppose $f_{11} = f \cdot (g^{-1})$. Let us consider non-empty, non empty finite sequences C_4, C_7 of elements of D^*. Suppose $C_4 = (\mathrm{SignGenOp}(f, A, F_1)) \cdot E_1$ and $C_7 = (\mathrm{SignGenOp}(f_{11}, A, (^\circ g)^\circ F_1)) \cdot g_1$. Let us consider an element S_1 of $\operatorname{Fin} \operatorname{dom}(\mathrm{App}(C_4))$, and an element S_2 of $\operatorname{Fin} \operatorname{dom}(\mathrm{App}(C_7))$. Suppose $S_2 = \{g \cdot s$, where s is a finite sequence of elements of $\mathbb{N} : s \in S_1\}$. Then $A\text{-}\sum_{S_1}(M \odot \mathrm{App}(C_4)) = A\text{-}\sum_{S_2}(M \odot \mathrm{App}(C_7))$.

PROOF: Define $\mathcal{P}[\text{set}] \equiv$ for every element S_1 of $\operatorname{Fin} \operatorname{dom}(\mathrm{App}(C_4))$ for every element S_2 of $\operatorname{Fin} \operatorname{dom}(\mathrm{App}(C_7))$ such that $S_1 = \$_1$ and $S_2 = \{g \cdot s$, where s is a finite sequence of elements of $\mathbb{N} : s \in S_1\}$ holds $A\text{-}\sum_{S_1}(M \odot \mathrm{App}(C_4)) = A\text{-}\sum_{S_2}(M \odot \mathrm{App}(C_7))$. $\mathcal{P}[\emptyset_{\operatorname{dom}(\mathrm{App}(C_4))}]$. For every element B' of $\operatorname{Fin} \operatorname{dom}(\mathrm{App}(C_4))$ and for every element b of $\operatorname{dom}(\mathrm{App}(C_4))$ such that $\mathcal{P}[B']$ and $b \notin B'$ holds $\mathcal{P}[B' \cup \{b\}]$. For every element B of $\operatorname{Fin} \operatorname{dom}(\mathrm{App}(C_4))$, $\mathcal{P}[B]$. □

(102) Let us consider an enumeration E_1 of F_1. Suppose $n \in \operatorname{dom} f$. Then $\operatorname{len} E_1 \mapsto n \in \operatorname{dom}_\kappa(\operatorname{SignGenOp}(f, A, F_1)) \cdot E_1(\kappa)$.
Proof: Set $C_3 = (\operatorname{SignGenOp}(f, A, F_1)) \cdot E_1$. Set $s = \operatorname{len} E_1 \mapsto n$. For every i such that $i \in \operatorname{dom} s$ holds $s(i) \in \operatorname{dom}(C_3(i))$. □

(103) Suppose B is unital, associative, and commutative and has inverse operation. Then (the inverse operation w.r.t. $B)(B(d_1, d_2)) = B(($the inverse operation w.r.t. $B)(d_1)$, (the inverse operation w.r.t. $B)(d_2))$.

Let x be an object and n be an even natural number. One can check that $n \mapsto x$ has evenly repeated values.

Let us consider finite sequences f, g. Now we state the propositions:

(104) If $f \frown g$ has evenly repeated values and f has evenly repeated values, then g has evenly repeated values.

(105) If $f \frown g$ has evenly repeated values and g has evenly repeated values, then f has evenly repeated values.

Let x be an object and n be an even natural number. Let us note that $n \mapsto x$ has evenly repeated values.

Let X, Y be with evenly repeated values-member sets. Note that $X \cup Y$ is with evenly repeated values-member.

Let n, k be natural numbers. The functor $\operatorname{doms}(n, k)$ yielding a finite sequence-membered, finite set is defined by the term

(Def. 14) $(\operatorname{Seg} n)^k$.

Note that every element of $\operatorname{doms}(n, k)$ is $(\operatorname{Seg} n)$-valued.

Let n be a non empty natural number and k be a natural number. Let us note that $\operatorname{doms}(n, k)$ is non empty and every element of $\operatorname{doms}(n, k)$ is k-element.

Now we state the proposition:

(106) Let us consider an enumeration E of F. Then $\operatorname{dom}_\kappa(\operatorname{SignGenOp}(f, A, F)) \cdot E(\kappa) = \operatorname{doms}(\operatorname{len} f, \overline{\overline{F}})$.
Proof: $\operatorname{dom}_\kappa(\operatorname{SignGenOp}(f, A, F)) \cdot E(\kappa) \subseteq \operatorname{doms}(\operatorname{len} f, \overline{\overline{F}})$. Consider s being an element of $(\operatorname{Seg} \operatorname{len} f)^*$ such that $x = s$ and $\operatorname{len} s = \overline{\overline{F}}$. For every i such that $i \in \operatorname{dom} s$ holds $s(i) \in \operatorname{dom}(((\operatorname{SignGenOp}(f, A, F)) \cdot E)(i))$. □

Let us consider an enumeration E_1 of F_1 and an enumeration E_2 of F_2. Now we state the propositions:

(107) Suppose $\overline{\overline{F_1}} = \overline{\overline{F_2}}$ and $\operatorname{len} f \leqslant \operatorname{len} g$. Then $\operatorname{dom}_\kappa(\operatorname{SignGenOp}(f, A, F_1)) \cdot E_1(\kappa) \subseteq \operatorname{dom}_\kappa(\operatorname{SignGenOp}(g, A, F_2)) \cdot E_2(\kappa)$.
Proof: $\operatorname{len} x = \operatorname{len}(\operatorname{SignGenOp}(g, A, F_2)) \cdot E_2$. For every i such that $i \in \operatorname{dom} x$ holds $x(i) \in \operatorname{dom}(((\operatorname{SignGenOp}(g, A, F_2)) \cdot E_2)(i))$. □

(108) Suppose $\overline{\overline{F_1}} = \overline{\overline{F_2}}$. Then $\operatorname{dom}_\kappa(\operatorname{SignGenOp}(f, A, F_1)) \cdot E_1(\kappa) = \operatorname{dom}_\kappa(\operatorname{SignGenOp}(f, A, F_2)) \cdot E_2(\kappa)$.

PROOF: $\mathrm{dom}_\kappa(\mathrm{SignGenOp}(f, A, F_1))\cdot E_1(\kappa) \subseteq \mathrm{dom}_\kappa(\mathrm{SignGenOp}(f, A, F_2))\cdot E_2(\kappa)$. $\mathrm{len}\, x = \mathrm{len}(\mathrm{SignGenOp}(f, A, F_1)) \cdot E_1$. For every i such that $i \in \mathrm{dom}\, x$ holds $x(i) \in \mathrm{dom}(((\mathrm{SignGenOp}(f, A, F_1)) \cdot E_1)(i))$. □

(109) Let us consider an enumeration E of F, and a permutation p of $\mathrm{dom}\, E$. Then $E \cdot p$ is an enumeration of F.

Let us consider an enumeration E of F, a permutation p of $\mathrm{dom}\, E$, and a finite sequence s. Now we state the propositions:

(110) If $s \in \mathrm{dom}_\kappa(\mathrm{SignGenOp}(f, A, F)) \cdot E(\kappa)$, then $s \cdot p \in \mathrm{dom}_\kappa(\mathrm{SignGenOp}(f, A, F)) \cdot (E \cdot p)(\kappa)$.

PROOF: Reconsider $E_{28} = E\cdot p$ as an enumeration of F. $\mathrm{len}\, s = \mathrm{len}(\mathrm{SignGenOp}(f, A, F)) \cdot E = \mathrm{len}\, E = \overline{\overline{F}}$. Reconsider $s_7 = s \cdot p$ as a finite sequence. For every i such that $i \in \mathrm{dom}\, s_7$ holds $s_7(i) \in \mathrm{dom}(((\mathrm{SignGenOp}(f, A, F))\cdot E_{28})(i))$. □

(111) Suppose $s \in \mathrm{dom}_\kappa(\mathrm{SignGenOp}(f, A, F)) \cdot E(\kappa)$. Then $(\mathrm{App}((\mathrm{SignGenOp}(f, A, F)) \cdot E))(s) \cdot p = (\mathrm{App}((\mathrm{SignGenOp}(f, A, F)) \cdot (E \cdot p)))(s \cdot p)$.

PROOF: Set $C = \mathrm{SignGenOp}(f, A, F)$. $s\cdot p \in \mathrm{dom}_\kappa\, C\cdot(E\cdot p)(\kappa)$. Reconsider $s_7 = s \cdot p$ as a finite sequence. $\mathrm{len}\, s = \mathrm{len}\, C \cdot E = \mathrm{len}\, E$. For every i such that $i \in \mathrm{dom}((\mathrm{App}(C \cdot (E \cdot p)))(s_7))$ holds $((\mathrm{App}(C \cdot E))(s) \cdot p)(i) = (\mathrm{App}(C \cdot (E \cdot p)))(s_7)(i)$. □

(112) Suppose M is commutative and associative. Then suppose $s \in \mathrm{dom}_\kappa(\mathrm{SignGenOp}(f, A, F))\cdot E(\kappa)$ and ($\mathrm{len}\, s \geqslant 1$ or M is unital). Then $(M \odot \mathrm{App}((\mathrm{SignGenOp}(f, A, F))\cdot E))(s) = (M \odot \mathrm{App}((\mathrm{SignGenOp}(f, A, F))\cdot(E\cdot p)))(s\cdot p)$. The theorem is a consequence of (110), (47), and (111).

(113) Let us consider an enumeration E of F, a permutation p of $\mathrm{dom}\, E$, and an element S of $\mathrm{Fin}\,\mathrm{dom}(\mathrm{App}((\mathrm{SignGenOp}(f, A, F)) \cdot E))$. Then $\{s \cdot p$, where s is a finite sequence of elements of $\mathbb{N} : s \in S\}$ is an element of $\mathrm{Fin}\,\mathrm{dom}(\mathrm{App}((\mathrm{SignGenOp}(f, A, F)) \cdot (E \cdot p)))$. The theorem is a consequence of (110).

(114) Let us consider an enumeration E of F, a permutation p of $\mathrm{dom}\, E$, and an element S of $\mathrm{Fin}\,\mathrm{doms}(n, \overline{\overline{F}})$. Then $\{s \cdot p$, where s is a finite sequence of elements of $\mathbb{N} : s \in S\}$ is an element of $\mathrm{Fin}\,\mathrm{doms}(n, \overline{\overline{F}})$. The theorem is a consequence of (109), (110), and (106).

(115) Suppose M is commutative and associative and A is unital, commutative, and associative. Let us consider an enumeration E of F, and a permutation p of $\mathrm{dom}\, E$. Suppose M is unital or $\mathrm{len}\, E \geqslant 1$. Let us consider non-empty, non empty finite sequences C_3, C_{11} of elements of D^*. Suppose $C_3 = (\mathrm{SignGenOp}(f, A, F)) \cdot E$ and $C_{11} = (\mathrm{SignGenOp}(f, A, F)) \cdot (E \cdot p)$. Let us consider an element S of $\mathrm{Fin}\,\mathrm{dom}(\mathrm{App}(C_3))$, and an element S_{13} of $\mathrm{Fin}\,\mathrm{dom}(\mathrm{App}(C_{11}))$.

Suppose $S_{13} = \{s \cdot p$, where s is a finite sequence of elements of $\mathbb{N}$: $s \in S\}$. Then $A\text{-}\sum_S(M \odot \mathrm{App}(C_3)) = A\text{-}\sum_{S_{13}}(M \odot \mathrm{App}(C_{11}))$.

PROOF: Define $\mathcal{P}[\text{set}] \equiv$ for every element S of $\mathrm{Fin}\,\mathrm{dom}(\mathrm{App}(C_3))$ for every element S_{13} of $\mathrm{Fin}\,\mathrm{dom}(\mathrm{App}(C_{11}))$ such that $S = \$_1$ and $S_{13} = \{s \cdot p$, where s is a finite sequence of elements of $\mathbb{N} : s \in S\}$ holds $A\text{-}\sum_S(M \odot \mathrm{App}(C_3)) = A\text{-}\sum_{S_{13}}(M \odot \mathrm{App}(C_{11}))$. $\mathcal{P}[\emptyset_{\mathrm{dom}(\mathrm{App}(C_3))}]$. For every element B' of $\mathrm{Fin}\,\mathrm{dom}(\mathrm{App}(C_3))$ and for every element b of $\mathrm{dom}(\mathrm{App}(C_3))$ such that $\mathcal{P}[B']$ and $b \notin B'$ holds $\mathcal{P}[B' \cup \{b\}]$. For every element B of $\mathrm{Fin}\,\mathrm{dom}(\mathrm{App}(C_3))$, $\mathcal{P}[B]$. □

(116) Suppose A is unital and associative and has inverse operation. Let us consider finite sets F, F_9. Suppose $F_9 = F \Cup 2^{\{\mathrm{len}\, f+1\}}$ and $\bigcup F \subseteq \mathrm{dom}\, f$. Let us consider an enumeration E_1 of F_9. Then there exists an enumeration E_2 of F_9 such that $(\mathrm{SignGenOp}(f \frown \langle d_1 \rangle, A, F_9)) \cdot E_1 = (\mathrm{SignGenOp}(f \frown \langle (\text{the inverse operation w.r.t. } A)(d_1) \rangle, A, F_9)) \cdot E_2$.

PROOF: Set $I =$ the inverse operation w.r.t. A. Define $\mathcal{P}[\text{object}, \text{object}] \equiv \$_2 \in \mathrm{dom}\, E_1$ and if $1 + \mathrm{len}\, f \in E_1(\$_1)$, then $E_1(\$_2) = E_1(\$_1) \setminus \{1 + \mathrm{len}\, f\}$ and if $1 + \mathrm{len}\, f \notin E_1(\$_1)$, then $E_1(\$_2) = E_1(\$_1) \cup \{1 + \mathrm{len}\, f\}$. For every x such that $x \in \mathrm{dom}\, E_1$ there exists y such that $\mathcal{P}[x, y]$.

Consider p being a function such that $\mathrm{dom}\, p = \mathrm{dom}\, E_1$ and for every x such that $x \in \mathrm{dom}\, E_1$ holds $\mathcal{P}[x, p(x)]$. $\mathrm{rng}\, p \subseteq \mathrm{dom}\, E_1$. $\mathrm{dom}\, E_1 \subseteq \mathrm{rng}\, p$. Reconsider $E_4 = E_1 \cdot p$ as an enumeration of F_9. For every i such that $1 \leqslant i \leqslant \mathrm{len}(\mathrm{SignGenOp}(f \frown \langle d_1 \rangle, A, F_9)) \cdot E_1$ holds $((\mathrm{SignGenOp}(f \frown \langle d_1 \rangle, A, F_9)) \cdot E_1)(i) = ((\mathrm{SignGenOp}(f \frown \langle I(d_1) \rangle, A, F_9)) \cdot E_4)(i)$. □

(117) Suppose A is unital, associative, and commutative and has inverse operation. Let us consider a finite, non empty set F. Suppose $\bigcup F \subseteq \mathrm{dom}\, f$. Let us consider finite sets F_1, F_2. Suppose $F_1 = F \Cup 2^{\{\mathrm{len}\, f+1\}}$ and $F_2 = F \Cup 2^{\{\mathrm{len}\, f+1, \mathrm{len}\, f+2\}}$. Then there exist enumerations E_1, E_2 of F_1 and there exists an enumeration E of F_2 such that $A \odot (\mathrm{SignGenOp}((f \frown \langle d_1 \rangle) \frown \langle d_2 \rangle, A, F_2)) \cdot E = (A \odot (\mathrm{SignGenOp}(f \frown \langle A(d_1, d_2) \rangle, A, F_1)) \cdot E_1) \frown (A \odot (\mathrm{SignGenOp}(f \frown \langle A((\text{the inverse operation w.r.t. } A)(d_1), d_2) \rangle, A, F_1)) \cdot E_2)$. The theorem is a consequence of (91), (116), and (2).

(118) Suppose A is unital. Let us consider an enumeration E of F, and a finite sequence s. Suppose $F = \emptyset$ and $s \in \mathrm{dom}_\kappa(\mathrm{SignGenOp}(f, B, F)) \cdot E(\kappa)$. Then $(A \odot \mathrm{App}((\mathrm{SignGenOp}(f, B, F)) \cdot E))(s) = \mathbf{1}_A$. The theorem is a consequence of (47) and (59).

(119) Let us consider an enumeration E of F, a permutation p of $\mathrm{dom}\, E$, and a subset S of $\mathrm{doms}(n, \overline{\overline{F}})$. Then $\{s \cdot p$, where s is a finite sequence of elements of $\mathbb{N} : s \in S\}$ is a subset of $\mathrm{doms}(n, \overline{\overline{F}})$. The theorem is a consequence of (109), (110), and (106).

(120) Let us consider finite sequences f, g. Suppose ($\operatorname{len} f = n$ or $\operatorname{len} g = m$) and $f \frown g \in \operatorname{doms}(k, n+m)$. Then

(i) $f \in \operatorname{doms}(k, n)$, and

(ii) $g \in \operatorname{doms}(k, m)$.

(121) Let us consider a finite sequence f. If $f \in \operatorname{doms}(n, k)$, then $\operatorname{len} f = k$.

(122) Let us consider finite sequences f, g. Suppose $f \in \operatorname{doms}(k, n)$ and $g \in \operatorname{doms}(k, m)$. Then $f \frown g \in \operatorname{doms}(k, n+m)$.

(123) $\operatorname{doms}(k, n) \frown \operatorname{doms}(k, m) = \operatorname{doms}(k, n+m)$. The theorem is a consequence of (122) and (120).

(124) Let us consider an enumeration E of F, a permutation p of $\operatorname{dom} E$, and a finite sequence s. Suppose $s \in \operatorname{doms}(m, \overline{\overline{F}})$. Then $s \cdot p \in \operatorname{doms}(m, \overline{\overline{F}})$. The theorem is a consequence of (109) and (121).

(125) If $k \leqslant n$, then $\operatorname{doms}(k, m) \subseteq \operatorname{doms}(n, m)$.

(126) Suppose A is commutative, associative, and unital and has inverse operation and M is associative, commutative, and unital and M is distributive w.r.t. A. Let us consider an enumeration E_1 of F_1, and an enumeration E_2 of F_2. Suppose $\bigcup F_1 \subseteq \operatorname{Seg}(1+m)$ and $\bigcup F_2 \subseteq \operatorname{Seg}(1+m)$. Let us consider an enumeration E_{17} of $\operatorname{ext}(F_1, 1+m, 2+m)$, and an enumeration E_{33} of $\operatorname{swap}(F_2, 1+m, 2+m)$.

Suppose $E_{17} = \operatorname{Ext}(E_1, 1+m, 2+m)$ and $E_{33} = \operatorname{Swap}(E_2, 1+m, 2+m)$. Let us consider an enumeration E_{21} of $\operatorname{ext}(F_1, 1+m, 2+m) \cup \operatorname{swap}(F_2, 1+m, 2+m)$. Suppose $E_{21} = E_{17} \frown E_{33}$. Let us consider finite sequences s_1, s_2. Suppose $s_1 \in \operatorname{doms}(m+1, \overline{\overline{F_1}})$ and $s_2 \in \operatorname{doms}(m+1, \overline{\overline{F_2}})$ and $s_1 \frown s_2$ has evenly repeated values and $\overline{\overline{s_1{}^{-1}(\{1+m\})}} = \overline{\overline{s_2{}^{-1}(\{1+m\})}}$. Then there exists a subset S of $\operatorname{doms}(m+2, \overline{\overline{F_1}} + \overline{\overline{F_2}})$ such that

(i) if $\overline{\overline{s_1{}^{-1}(\{1+m\})}} = 0$, then $s_1 \frown s_2 \in S$, and

(ii) S is with evenly repeated values-member, and

(iii) for every finite sequences C_4, C_7 of elements of D^* and for every f, d_1, and d_2 such that $\operatorname{len} f = m$ and $C_4 = (\operatorname{SignGenOp}(f \frown \langle A(d_1, d_2)\rangle, A, F_1)) \cdot E_1$ and $C_7 = (\operatorname{SignGenOp}(f \frown \langle A((\text{the inverse operation w.r.t. } A)(d_1), d_2)\rangle, A, F_2)) \cdot E_2$ for every non-empty, non empty finite sequence C_{17} of elements of D^* such that $C_{17} = (\operatorname{SignGenOp}((f \frown \langle d_1\rangle) \frown \langle d_2\rangle, A, \operatorname{ext}(F_1, 1 + \operatorname{len} f, 2 + \operatorname{len} f) \cup \operatorname{swap}(F_2, 1 + \operatorname{len} f, 2 + \operatorname{len} f))) \cdot E_{21}$ for every element S_7 of $\operatorname{Fin} \operatorname{dom}(\operatorname{App}(C_{17}))$ such that $S = S_7$ holds $M((M \odot \operatorname{App}(C_4))(s_1), (M \odot \operatorname{App}(C_7))(s_2)) = A\text{-}\sum_{S_7}(M \odot \operatorname{App}(C_{17}))$ and for every finite sequence h and for every i such that $h \in S_7$ and $i \in \operatorname{dom} h$ holds if $(s_1 \frown s_2)(i) = 1 + \operatorname{len} f$, then $h(i) \in \{1 + \operatorname{len} f, 2 + \operatorname{len} f\}$ and if $(s_1 \frown s_2)(i) \neq 1 + \operatorname{len} f$, then $h(i) = (s_1 \frown s_2)(i)$.

Proof: Define $\mathcal{P}$[natural number] $\equiv$ for every F_1 and F_2 for every enumeration E_1 of F_1 for every enumeration E_2 of F_2 such that $\bigcup F_1 \subseteq \mathrm{Seg}(1+m)$ and $\bigcup F_2 \subseteq \mathrm{Seg}(1+m)$ for every enumeration E_{17} of $\mathrm{ext}(F_1, 1+m, 2+m)$ for every enumeration E_{33} of $\mathrm{swap}(F_2, 1+m, 2+m)$ such that $E_{17} = \mathrm{Ext}(E_1, 1+m, 2+m)$ and $E_{33} = \mathrm{Swap}(E_2, 1+m, 2+m)$ for every enumeration E_{21} of $\mathrm{ext}(F_1, 1+m, 2+m) \cup \mathrm{swap}(F_2, 1+m, 2+m)$ such that $E_{21} = E_{17} \frown E_{33}$ for every finite sequences s_1, s_2 such that $s_1 \in \mathrm{doms}(m+1, \overline{\overline{F_1}})$ and $s_2 \in \mathrm{doms}(m+1, \overline{\overline{F_2}})$ and $s_1 \frown s_2$ has evenly repeated values and $\overline{\overline{s_1{}^{-1}(\{1+m\})}} = \$_1 = \overline{\overline{s_2{}^{-1}(\{1+m\})}}$ there exists a subset S of $\mathrm{doms}(m+2, \overline{\overline{F_1}} + \overline{\overline{F_2}})$ such that if $\overline{\overline{s_1{}^{-1}(\{1+m\})}} = 0$, then $s_1 \frown s_2 \in S$.

S is with evenly repeated values-member and for every finite sequences C_4, C_7 of elements of D^* and for every f, d_1, and d_2 such that $\mathrm{len}\, f = m$ and $C_4 = (\mathrm{SignGenOp}(f \frown \langle A(d_1, d_2)\rangle, A, F_1)) \cdot E_1$ and $C_7 = (\mathrm{SignGenOp}(f \frown \langle A((\text{the inverse operation w.r.t. } A)(d_1), d_2)\rangle, A, F_2)) \cdot E_2$ for every non-empty, non empty finite sequence C_{17} of elements of D^* such that $C_{17} = (\mathrm{SignGenOp}((f \frown \langle d_1\rangle) \frown \langle d_2\rangle, A, \mathrm{ext}(F_1, 1+\mathrm{len}\, f, 2+\mathrm{len}\, f) \cup \mathrm{swap}(F_2, 1+\mathrm{len}\, f, 2+\mathrm{len}\, f))) \cdot E_{21}$ for every element S_7 of $\mathrm{Fin}\,\mathrm{dom}(\mathrm{App}(C_{17}))$ such that $S = S_7$ holds $M((M \odot \mathrm{App}(C_4))(s_1), (M \odot \mathrm{App}(C_7))(s_2)) = A\text{-}\sum_{S_7}(M \odot \mathrm{App}(C_{17}))$ and for every finite sequence h and for every i such that $h \in S_7$ and $i \in \mathrm{dom}\, h$ holds if $(s_1 \frown s_2)(i) = 1 + \mathrm{len}\, f$, then $h(i) \in \{1 + \mathrm{len}\, f, 2 + \mathrm{len}\, f\}$ and if $(s_1 \frown s_2)(i) \neq 1 + \mathrm{len}\, f$, then $h(i) = (s_1 \frown s_2)(i)$. If $\mathcal{P}[n]$, then $\mathcal{P}[n+1]$. $\mathcal{P}[0]$. $\mathcal{P}[n]$. □

(127) Suppose A is commutative, associative, and unital and has inverse operation and M is associative, commutative, and unital and M is distributive w.r.t. A. Let us consider an enumeration E_1 of F_1. Suppose $\bigcup F_1 \subseteq \mathrm{Seg}(1+m)$. Let us consider an enumeration E_{17} of $\mathrm{ext}(F_1, 1+m, 2+m)$. Suppose $E_{17} = \mathrm{Ext}(E_1, 1+m, 2+m)$. Then there exists a subset S of $\mathrm{doms}(m+2, \overline{\overline{F_1}})$ such that

(i) $S = \{1+m, 2+m\}^{\mathrm{len}\, E_1}$, and

(ii) for every non-empty, non empty finite sequence C_{16} of elements of D^* and for every f, d_1, and d_2 such that $\mathrm{len}\, f = m$ and $C_{16} = (\mathrm{SignGenOp}((f \frown \langle d_1\rangle) \frown \langle d_2\rangle, A, \mathrm{ext}(F_1, 1+\mathrm{len}\, f, 2+\mathrm{len}\, f))) \cdot E_{17}$ for every element S_7 of $\mathrm{Fin}\,\mathrm{dom}(\mathrm{App}(C_{16}))$ such that $S_7 = S$ holds $(M \odot \mathrm{App}((\mathrm{SignGenOp}(f \frown \langle A(d_1, d_2)\rangle, A, F_1)) \cdot E_1))(\mathrm{len}\, E_1 \mapsto (1 + \mathrm{len}\, f)) = A\text{-}\sum_{S_7}(M \odot \mathrm{App}(C_{16}))$.

Proof: Define $\mathcal{P}$[natural number] $\equiv$ for every F_1 for every enumeration E_1 of F_1 such that $\bigcup F_1 \subseteq \mathrm{Seg}(1+m)$ and $\mathrm{len}\, E_1 = \$_1$ for every enumeration E_{17} of $\mathrm{ext}(F_1, 1+m, 2+m)$ such that $E_{17} = \mathrm{Ext}(E_1, 1+m, 2+m)$ there exists a subset S of $\mathrm{doms}(m+2, \overline{\overline{F_1}})$ such that $S = \{1+m, 2+m\}^{\mathrm{len}\, E_1}$ and for every non-empty, non empty finite sequence C_{16} of elements of D^* and for every f, d_1, and d_2 such that $\operatorname{len} f = m$ and $C_{16} = (\mathrm{SignGenOp}((f \frown \langle d_1 \rangle) \frown \langle d_2 \rangle, A, \mathrm{ext}(F_1, 1 + \operatorname{len} f, 2 + \operatorname{len} f))) \cdot E_{17}$ for every element S_7 of $\mathrm{Fin\,dom}(\mathrm{App}(C_{16}))$ such that $S_7 = S$ holds $(M \odot \mathrm{App}((\mathrm{SignGenOp}(f \frown \langle A(d_1, d_2) \rangle, A, F_1)) \cdot E_1))(\operatorname{len} E_1 \mapsto (1 + \operatorname{len} f)) = A\text{-}\sum_{S_7}(M \odot \mathrm{App}(C_{16}))$. $\mathcal{P}[0]$. If $\mathcal{P}[n]$, then $\mathcal{P}[n+1]$. $\mathcal{P}[n]$. □

(128) Suppose A is commutative, associative, and unital and has inverse operation. Let us consider an enumeration E_1 of F_1. Suppose $\bigcup F_1 \subseteq \mathrm{Seg}(1 + \operatorname{len} f)$. Let us consider an enumeration E_{17} of $\mathrm{ext}(F_1, 1 + \operatorname{len} f, 2 + \operatorname{len} f)$, and an enumeration E_{33} of $\mathrm{swap}(F_1, 1 + \operatorname{len} f, 2 + \operatorname{len} f)$. Suppose $E_{17} = \mathrm{Ext}(E_1, 1 + \operatorname{len} f, 2 + \operatorname{len} f)$ and $E_{33} = \mathrm{Swap}(E_1, 1 + \operatorname{len} f, 2 + \operatorname{len} f)$. Let us consider a non-empty, non empty finite sequence C_{16} of elements of D^*, and a non-empty, non empty finite sequence C_{20} of elements of D^*.

Suppose $C_{16} = (\mathrm{SignGenOp}((f \frown \langle d_1 \rangle) \frown \langle d_2 \rangle, A, \mathrm{ext}(F_1, 1 + \operatorname{len} f, 2 + \operatorname{len} f))) \cdot E_{17}$ and $C_{20} = (\mathrm{SignGenOp}((f \frown \langle (\text{the inverse operation w.r.t. } A)(d_1) \rangle) \frown \langle d_2 \rangle, A, \mathrm{swap}(F_1, 1 + \operatorname{len} f, 2 + \operatorname{len} f))) \cdot E_{33}$. Let us consider an element S_1 of $\mathrm{Fin\,dom}(\mathrm{App}(C_{16}))$, and an element S_2 of $\mathrm{Fin\,dom}(\mathrm{App}(C_{20}))$. Suppose $S_1 = S_2$. Then $A\text{-}\sum_{S_1}(M \odot \mathrm{App}(C_{16})) = A\text{-}\sum_{S_2}(M \odot \mathrm{App}(C_{20}))$.
PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every element S_1 of $\mathrm{Fin\,dom}(\mathrm{App}(C_{16}))$ for every element S_2 of $\mathrm{Fin\,dom}(\mathrm{App}(C_{20}))$ such that $S_1 = S_2$ and $\overline{\overline{S_1}} = \$_1$ holds $A\text{-}\sum_{S_1}(M \odot \mathrm{App}(C_{16})) = A\text{-}\sum_{S_2}(M \odot \mathrm{App}(C_{20}))$. $\mathcal{P}[0]$. If $\mathcal{P}[n]$, then $\mathcal{P}[n+1]$. $\mathcal{P}[n]$. □

(129) Suppose A is commutative, associative, and unital and has inverse operation and M is associative, commutative, and unital and M is distributive w.r.t. A. Let us consider an enumeration E_1 of F_1. Suppose $\bigcup F_1 \subseteq \mathrm{Seg}(1+m)$. Let us consider an enumeration E_{33} of $\mathrm{swap}(F_1, 1+m, 2+m)$. Suppose $E_{33} = \mathrm{Swap}(E_1, 1 + m, 2 + m)$. Then there exists a subset S of $\mathrm{doms}(m + 2, \overline{\overline{F_1}})$ such that

(i) $S = \{1 + m, 2 + m\}^{\operatorname{len} E_1}$, and

(ii) for every non-empty, non empty finite sequence C_{20} of elements of D^* and for every f, d_1, and d_2 such that $\operatorname{len} f = m$ and $C_{20} = (\mathrm{SignGenOp}((f \frown \langle d_1 \rangle) \frown \langle d_2 \rangle, A, \mathrm{swap}(F_1, 1 + \operatorname{len} f, 2 + \operatorname{len} f))) \cdot E_{33}$ for every element S_7 of $\mathrm{Fin\,dom}(\mathrm{App}(C_{20}))$ such that $S_7 = S$ holds $(M \odot \mathrm{App}((\mathrm{SignGenOp}(f \frown \langle A((\text{the inverse operation w.r.t. } A)(d_1), d_2) \rangle, A, F_1)) \cdot E_1))(\operatorname{len} E_1 \mapsto (1 + \operatorname{len} f)) = A\text{-}\sum_{S_7}(M \odot \mathrm{App}(C_{20}))$.

The theorem is a consequence of (28), (127), (80), (10), (11), (107), and (128).

(130) Suppose A is unital, associative, and commutative and has inverse operation and M is commutative and associative and $\operatorname{len} f \neq 0$. Then SignGenOp

$((f \frown \langle d_1\rangle) \frown \langle d_2\rangle, M, A, (\mathrm{Seg}(2+\mathrm{len}\, f)) \setminus \{1\}) = M(\mathrm{SignGenOp}(f \frown \langle A(d_1, d_2)\rangle, M, A, (\mathrm{Seg}(1+\mathrm{len}\, f)) \setminus \{1\}), \mathrm{SignGenOp}(f \frown \langle A($(the inverse operation w.r.t. $A)(d_1), d_2)\rangle, M, A, (\mathrm{Seg}(1+\mathrm{len}\, f)) \setminus \{1\}))$. The theorem is a consequence of (6), (117), and (64).

(131) Let us consider an enumeration E of F. Suppose $\bigcup F \subseteq \mathrm{Seg}(1+\mathrm{len}\, f)$. Let us consider an enumeration E_{17} of $\mathrm{ext}(F, 1+\mathrm{len}\, f, 2+\mathrm{len}\, f)$. Suppose $E_{17} = \mathrm{Ext}(E, 1+\mathrm{len}\, f, 2+\mathrm{len}\, f)$. Let us consider finite sequences C_4, C_9 of elements of D^*. Suppose $C_4 = (\mathrm{SignGenOp}(f \frown \langle d\rangle, A, F)) \cdot E$ and $C_9 = (\mathrm{SignGenOp}((f \frown \langle d_1\rangle) \frown \langle d_2\rangle, A, \mathrm{ext}(F, 1+\mathrm{len}\, f, 2+\mathrm{len}\, f))) \cdot E_{17}$. Let us consider a finite sequence s. Suppose $s \in \mathrm{dom}_\kappa\, C_4(\kappa)$ and $\mathrm{rng}\, s \subseteq \mathrm{dom}\, f$. Then

(i) $s \in \mathrm{dom}_\kappa\, C_9(\kappa)$, and

(ii) $(\mathrm{App}(C_4))(s) = (\mathrm{App}(C_9))(s)$.

Proof: $\mathrm{dom}_\kappa\, C_4(\kappa) \subseteq \mathrm{dom}_\kappa\, C_9(\kappa)$. $\mathrm{len}\, E = \mathrm{len}\, C_4 = \mathrm{len}\, s = \mathrm{len}\, C_9$. For every i such that $1 \leqslant i \leqslant \mathrm{len}\, s$ holds $(\mathrm{App}(C_4))(s)(i) = (\mathrm{App}(C_9))(s)(i)$. □

(132) Let us consider an enumeration E of F. Suppose $\bigcup F \subseteq \mathrm{Seg}(1+\mathrm{len}\, f)$. Let us consider an enumeration E_{33} of $\mathrm{swap}(F, 1+\mathrm{len}\, f, 2+\mathrm{len}\, f)$. Suppose $E_{33} = \mathrm{Swap}(E, 1+\mathrm{len}\, f, 2+\mathrm{len}\, f)$. Let us consider finite sequences C_4, C_{10} of elements of D^*. Suppose $C_4 = (\mathrm{SignGenOp}(f \frown \langle d\rangle, A, F)) \cdot E$ and $C_{10} = (\mathrm{SignGenOp}((f \frown \langle d_1\rangle) \frown \langle d_2\rangle, A, \mathrm{swap}(F, 1+\mathrm{len}\, f, 2+\mathrm{len}\, f))) \cdot E_{33}$. Let us consider a finite sequence s. Suppose $s \in \mathrm{dom}_\kappa\, C_4(\kappa)$ and $\mathrm{rng}\, s \subseteq \mathrm{dom}\, f$. Then

(i) $s \in \mathrm{dom}_\kappa\, C_{10}(\kappa)$, and

(ii) $(\mathrm{App}(C_4))(s) = (\mathrm{App}(C_{10}))(s)$.

Proof: $\mathrm{dom}_\kappa\, C_4(\kappa) \subseteq \mathrm{dom}_\kappa\, C_9(\kappa)$. $\mathrm{len}\, E = \mathrm{len}\, C_4 = \mathrm{len}\, s = \mathrm{len}\, C_9$. For every i such that $1 \leqslant i \leqslant \mathrm{len}\, s$ holds $(\mathrm{App}(C_4))(s)(i) = (\mathrm{App}(C_9))(s)(i)$. □

(133) Let us consider an enumeration E_1 of F_1, and (D^*)-valued finite sequences C_4, C_7. Suppose $C_4 = (\mathrm{SignGenOp}(f \frown \langle d_1\rangle, A, F_1)) \cdot E_1$ and $C_7 = (\mathrm{SignGenOp}(f \frown \langle d_2\rangle, A, F_1)) \cdot E_1$. Let us consider a finite sequence s. Suppose $s \in \mathrm{dom}_\kappa\, C_4(\kappa)$ and $1+\mathrm{len}\, f \notin \mathrm{rng}\, s$. Then

(i) $s \in \mathrm{dom}_\kappa\, C_7(\kappa)$, and

(ii) $(\mathrm{App}(C_4))(s) = (\mathrm{App}(C_7))(s)$.

Proof: $\mathrm{dom}_\kappa\, C_4(\kappa) \subseteq \mathrm{dom}_\kappa\, C_7(\kappa)$. $\mathrm{len}\, C_4 = \mathrm{len}\, s = \mathrm{len}\, C_7$. For every i such that $1 \leqslant i \leqslant \mathrm{len}\, s$ holds $(\mathrm{App}(C_4))(s)(i) = (\mathrm{App}(C_7))(s)(i)$. □

(134) Let us consider a finite sequence s. Suppose $\overline{\overline{s^{-1}(\{y\})}} = k$. Then there exists a permutation p of $\operatorname{dom} s$ and there exists a finite sequence s_1 such that $s \cdot p = s_1 \frown (k \mapsto y)$ and $y \notin \operatorname{rng} s_1$.

(135) Let us consider a finite sequence f of elements of D. Suppose A is commutative, associative, and unital and has inverse operation and M is associative, commutative, and unital and M is distributive w.r.t. A and $n \in \operatorname{dom} f$. Let us consider an enumeration E of F, and a subset D of $\operatorname{dom} E$. Suppose for every i, $i \in D$ iff $n \in E(i)$. Then

(i) if $\overline{\overline{D}}$ is even, then $(M \odot \operatorname{App}((\operatorname{SignGenOp}(f, A, F)) \cdot E))(\operatorname{len} E \mapsto n) = M \odot \operatorname{len} E \mapsto f_{/n}$, and

(ii) if $\overline{\overline{D}}$ is odd, then $(M \odot \operatorname{App}((\operatorname{SignGenOp}(f, A, F)) \cdot E))(\operatorname{len} E \mapsto n) =$ (the inverse operation w.r.t. A)$(M \odot \operatorname{len} E \mapsto f_{/n})$.

PROOF: Set $I_1 =$ the inverse operation w.r.t. A. Define $\mathcal{P}$[natural number] $\equiv$ for every F such that $\overline{\overline{F}} = \$_1$ for every enumeration E of F for every subset I of $\operatorname{dom} E$ such that for every i, $i \in I$ iff $n \in E(i)$ holds if $\overline{\overline{I}}$ is even, then $(M \odot \operatorname{App}((\operatorname{SignGenOp}(f, A, F)) \cdot E))(\operatorname{len} E \mapsto n) = M \odot \operatorname{len} E \mapsto f_{/n}$ and if $\overline{\overline{I}}$ is odd, then $(M \odot \operatorname{App}((\operatorname{SignGenOp}(f, A, F)) \cdot E))(\operatorname{len} E \mapsto n) = I_1(M \odot \operatorname{len} E \mapsto f_{/n})$. $\mathcal{P}[0]$. If $\mathcal{P}[j]$, then $\mathcal{P}[j+1]$. $\mathcal{P}[j]$. □

(136) Suppose M is commutative, associative, and unital and A is commutative, associative, and unital and has inverse operation and M is distributive w.r.t. A. Let us consider a finite sequence f of elements of D, an enumeration E_1 of F_1, an enumeration E_2 of F_2, and finite sequences s_1, s_2. Suppose $s_1 \in \operatorname{dom}_\kappa(\operatorname{SignGenOp}(f \frown \langle d_1 \rangle, A, F_1)) \cdot E_1(\kappa)$ and $s_2 \in \operatorname{dom}_\kappa(\operatorname{SignGenOp}(f \frown \langle d_2 \rangle, A, F_2)) \cdot E_2(\kappa)$ and $\overline{\overline{{s_1}^{-1}(\{1 + \operatorname{len} f\})}} = \overline{\overline{{s_2}^{-1}(\{1 + \operatorname{len} f\})}}$. Then $M((M \odot \operatorname{App}((\operatorname{SignGenOp}(f \frown \langle d_1 \rangle, A, F_1)) \cdot E_1))(s_1), (M \odot \operatorname{App}((\operatorname{SignGenOp}(f \frown \langle d_2 \rangle, A, F_2)) \cdot E_2))(s_2)) = M((M \odot \operatorname{App}((\operatorname{SignGenOp}(f \frown \langle d_2 \rangle, A, F_1)) \cdot E_1))(s_1), (M \odot \operatorname{App}((\operatorname{SignGenOp}(f \frown \langle d_1 \rangle, A, F_2)) \cdot E_2))(s_2))$.

PROOF: Set $L = 1 + \operatorname{len} f$. $\operatorname{dom}_\kappa(\operatorname{SignGenOp}(f \frown \langle d_1 \rangle, A, F_1)) \cdot E_1(\kappa) = \operatorname{dom}_\kappa(\operatorname{SignGenOp}(f \frown \langle d_2 \rangle, A, F_1)) \cdot E_1(\kappa)$ and $\operatorname{dom}_\kappa(\operatorname{SignGenOp}(f \frown \langle d_2 \rangle, A, F_2)) \cdot E_2(\kappa) = \operatorname{dom}_\kappa(\operatorname{SignGenOp}(f \frown \langle d_1 \rangle, A, F_2)) \cdot E_2(\kappa)$. Set $k = \overline{\overline{{s_1}^{-1}(\{L\})}}$. $\operatorname{len} s_1 = \operatorname{len}(\operatorname{SignGenOp}(f \frown \langle d_1 \rangle, A, F_1)) \cdot E_1 = \operatorname{len} E_1$ and $\operatorname{len} s_2 = \operatorname{len}(\operatorname{SignGenOp}(f \frown \langle d_2 \rangle, A, F_2)) \cdot E_2 = \operatorname{len} E_2$. Set $k_1 = k \mapsto L$. Consider p_1 being a permutation of $\operatorname{dom} s_1$, S_1 being a finite sequence such that $s_1 \cdot p_1 = S_1 \frown k_1$ and $L \notin \operatorname{rng} S_1$. Reconsider $E_4 = E_1 \cdot p_1$ as an enumeration of F_1. Set $e_3 = E_4 \restriction \operatorname{len} S_1$.

Consider e_2 being a finite sequence such that $E_4 = e_3 \frown e_2$. Set $F_4 = \operatorname{rng} e_3$. Set $F_3 = \operatorname{rng} e_2$. Reconsider $E_6 = e_3$ as an enumeration of F_4. Reconsider $E_5 = e_2$ as an enumeration of F_3. Consider p_2 being a permutation of $\operatorname{dom} s_2$, S_2 being a finite sequence such that $s_2 \cdot p_2 = S_2 \frown k_1$ and $L \notin \operatorname{rng} S_2$. Reconsider $E_8 = E_2 \cdot p_2$ as an enumeration of F_2. Set $e_5 = E_8 \restriction \operatorname{len} S_2$. Consider e_4 being a finite sequence such that $E_8 = e_5 \frown e_4$. Set $F_6 = \operatorname{rng} e_5$. Set $F_5 = \operatorname{rng} e_4$. Reconsider $E_{10} = e_5$ as an enumeration of F_6. Reconsider $E_9 = e_4$ as an enumeration of F_5. $(\mathrm{SignGenOp}(f \frown \langle d_1 \rangle, A, F_1)) \cdot E_4 = (\mathrm{SignGenOp}(f \frown \langle d_1 \rangle, A, F_4)) \cdot E_6 \frown (\mathrm{SignGenOp}(f \frown \langle d_1 \rangle, A, F_3)) \cdot E_5$ and $(\mathrm{SignGenOp}(f \frown \langle d_2 \rangle, A, F_2)) \cdot E_8 = (\mathrm{SignGenOp}(f \frown \langle d_2 \rangle, A, F_6)) \cdot E_{10} \frown (\mathrm{SignGenOp}(f \frown \langle d_2 \rangle, A, F_5)) \cdot E_9$. □

(137) Suppose M is commutative, associative, and unital and A is commutative, associative, and unital and has inverse operation and M is distributive w.r.t. A. Let us consider an enumeration E_1 of F_1. Suppose $\bigcup F_1 \subseteq \mathrm{Seg}(1+m)$ and $\operatorname{len} E_1$ is even. Let us consider an enumeration E_{17} of $\mathrm{ext}(F_1, 1+m, 2+m)$, and an enumeration E_{33} of $\mathrm{swap}(F_1, 1+m, 2+m)$. Suppose $E_{17} = \mathrm{Ext}(E_1, 1+m, 2+m)$ and $E_{33} = \mathrm{Swap}(E_1, 1+m, 2+m)$. Then there exist subsets s_6, s_8 of $\mathrm{doms}(m+2, \overline{\overline{F_1}})$ such that

(i) $s_6 \subseteq \{1+m, 2+m\}^{\operatorname{len} E_1}$, and

(ii) $s_8 \subseteq \{1+m, 2+m\}^{\operatorname{len} E_1}$, and

(iii) s_6 is with evenly repeated values-member, and

(iv) s_8 is with evenly repeated values-member, and

(v) for every non-empty, non empty finite sequences C_{16}, C_{20} of elements of D^* and for every f, d_1, and d_2 such that $\operatorname{len} f = m$ and $C_{16} = (\mathrm{SignGenOp}((f \frown \langle d_1 \rangle) \frown \langle d_2 \rangle, A, \mathrm{ext}(F_1, 1+\operatorname{len} f, 2+\operatorname{len} f))) \cdot E_{17}$ and $C_{20} = (\mathrm{SignGenOp}((f \frown \langle d_1 \rangle) \frown \langle d_2 \rangle, A, \mathrm{swap}(F_1, 1+\operatorname{len} f, 2+\operatorname{len} f))) \cdot E_{33}$ for every element S_8 of $\mathrm{Fin}\,\mathrm{dom}(\mathrm{App}(C_{16}))$ for every element S_{14} of $\mathrm{Fin}\,\mathrm{dom}(\mathrm{App}(C_{20}))$ such that $S_8 = s_6$ and $S_{14} = s_8$ holds $A((M \odot \mathrm{App}((\mathrm{SignGenOp}(f \frown \langle A(d_1, d_2) \rangle, A, F_1)) \cdot E_1))(\operatorname{len} E_1 \mapsto (1+\operatorname{len} f)), (M \odot \mathrm{App}((\mathrm{SignGenOp}(f \frown \langle A((\text{the inverse operation w.r.t. } A)(d_1), d_2) \rangle, A, F_1)) \cdot E_1))(\operatorname{len} E_1 \mapsto (1+\operatorname{len} f))) = A(A\text{-}\sum_{S_8}(M \odot \mathrm{App}(C_{16})), A\text{-}\sum_{S_{14}}(M \odot \mathrm{App}(C_{20})))$.

Proof: Set $I =$ the inverse operation w.r.t. A. Set $L_3 = \operatorname{len} E_1$. Set $L_1 = 1+m$. Set $L_2 = 2+m$. Consider s_6 being a subset of $\mathrm{doms}(m+2, \overline{\overline{F_1}})$ such that $s_6 = \{1+m, 2+m\}^{\operatorname{len} E_1}$ and for every non-empty, non empty finite sequence C_{16} of elements of D^* and for every f, d_1, and d_2 such that $\operatorname{len} f = m$ and $C_{16} = (\mathrm{SignGenOp}((f \frown \langle d_1 \rangle) \frown \langle d_2 \rangle, A, \mathrm{ext}(F_1, 1+\operatorname{len} f, 2+\operatorname{len} f))) \cdot E_{17}$ for every element S_7 of $\mathrm{Fin}\,\mathrm{dom}(\mathrm{App}(C_{16}))$ such that $S_7 = s_6$ holds $(M \odot \mathrm{App}((\mathrm{SignGenOp}(f \frown \langle A(d_1, d_2) \rangle, A, F_1)) \cdot E_1))(\operatorname{len} E_1 \mapsto (1+\operatorname{len} f)) = A\text{-}\sum_{S_7}(M \odot \mathrm{App}(C_{16}))$.

Consider s_8 being a subset of $\mathrm{doms}(m+2, \overline{\overline{F_1}})$ such that $s_8 = \{1 + m, 2 + m\}^{\mathrm{len}\, E_1}$ and for every non-empty, non empty finite sequence C_{20} of elements of D^* and for every f, d_1, and d_2 such that $\mathrm{len}\, f = m$ and $C_{20} = (\mathrm{SignGenOp}((f \frown \langle d_1 \rangle) \frown \langle d_2 \rangle, A, \mathrm{swap}(F_1, 1 + \mathrm{len}\, f, 2 + \mathrm{len}\, f))) \cdot E_{33}$ for every element S_7 of $\mathrm{Fin}\,\mathrm{dom}(\mathrm{App}(C_{20}))$ such that $S_7 = s_8$ holds $(M \odot \mathrm{App}((\mathrm{SignGenOp}(f \frown \langle A(I(d_1), d_2) \rangle, A, F_1)) \cdot E_1))(\mathrm{len}\, E_1 \mapsto (1 + \mathrm{len}\, f)) = A\text{-}\sum_{S_7}(M \odot \mathrm{App}(C_{20}))$. Set $C = \mathrm{CFS}(\{1+m, 2+m\}^{L_3})$. Define $\mathcal{P}[\text{natural number}] \equiv$ if $\$_1 \leqslant \mathrm{len}\, C$, then there exist subsets S_5, R_4, S_{15}, R_6 of $\mathrm{doms}(m+2, \overline{\overline{F_1}})$ such that $S_5 \subseteq \mathrm{rng}(C \restriction \$_1)$ and $R_4 = \mathrm{rng}(C \restriction \$_1) = R_6$ and $S_{15} \subseteq \mathrm{rng}(C \restriction \$_1)$ and S_5 is with evenly repeated values-member and S_{15} is with evenly repeated values-member and for every non-empty, non empty finite sequences C_{20}, C_{15} of elements of D^* and for every f, d_1, and d_2 such that $\mathrm{len}\, f = m$ and $C_{20} = (\mathrm{SignGenOp}((f \frown \langle d_1 \rangle) \frown \langle d_2 \rangle, A, \mathrm{swap}(F_1, 1 + \mathrm{len}\, f, 2 + \mathrm{len}\, f))) \cdot E_{33}$ and $C_{15} = (\mathrm{SignGenOp}((f \frown \langle I(d_1) \rangle) \frown \langle d_2 \rangle, A, \mathrm{swap}(F_1, 1 + \mathrm{len}\, f, 2 + \mathrm{len}\, f))) \cdot E_{33}$ for every elements S_4, R_3 of $\mathrm{Fin}\,\mathrm{dom}(\mathrm{App}(C_{15}))$.

For every elements S_{14}, R_5 of $\mathrm{Fin}\,\mathrm{dom}(\mathrm{App}(C_{20}))$ such that $S_5 = S_4$ and $R_4 = R_3$ and $S_{15} = S_{14}$ and $R_6 = R_5$ holds $A(A\text{-}\sum_{S_4}(M \odot \mathrm{App}(C_{15})), A\text{-}\sum_{S_{14}}(M \odot \mathrm{App}(C_{20}))) = A(A\text{-}\sum_{R_3}(M \odot \mathrm{App}(C_{15})), A\text{-}\sum_{R_5}(M \odot \mathrm{App}(C_{20})))$. $\mathcal{P}[0]$. If $\mathcal{P}[n]$, then $\mathcal{P}[n+1]$. $\mathcal{P}[n]$. Consider S_5, R_4, S_{15}, R_6 being subsets of $\mathrm{doms}(m+2, \overline{\overline{F_1}})$ such that $S_5 \subseteq \mathrm{rng}(C \restriction \mathrm{len}\, C)$ and $R_4 = \mathrm{rng}(C \restriction \mathrm{len}\, C) = R_6$ and $S_{15} \subseteq \mathrm{rng}(C \restriction \mathrm{len}\, C)$ and S_5 is with evenly repeated values-member and S_{15} is with evenly repeated values-member and for every non-empty, non empty finite sequences C_{20}, C_{15} of elements of D^*.

For every f, d_1, and d_2 such that $\mathrm{len}\, f = m$ and $C_{20} = (\mathrm{SignGenOp}((f \frown \langle d_1 \rangle) \frown \langle d_2 \rangle, A, \mathrm{swap}(F_1, 1 + \mathrm{len}\, f, 2 + \mathrm{len}\, f))) \cdot E_{33}$ and $C_{15} = (\mathrm{SignGenOp}((f \frown \langle I(d_1) \rangle) \frown \langle d_2 \rangle, A, \mathrm{swap}(F_1, 1 + \mathrm{len}\, f, 2 + \mathrm{len}\, f))) \cdot E_{33}$ for every elements S_4, R_3 of $\mathrm{Fin}\,\mathrm{dom}(\mathrm{App}(C_{15}))$ for every elements S_{14}, R_5 of $\mathrm{Fin}\,\mathrm{dom}(\mathrm{App}(C_{20}))$ such that $S_5 = S_4$ and $R_4 = R_3$ and $S_{15} = S_{14}$ and $R_6 = R_5$ holds $A(A\text{-}\sum_{S_4}(M \odot \mathrm{App}(C_{15})), A\text{-}\sum_{S_{14}}(M \odot \mathrm{App}(C_{20}))) = A(A\text{-}\sum_{R_3}(M \odot \mathrm{App}(C_{15})), A\text{-}\sum_{R_5}(M \odot \mathrm{App}(C_{20})))$. Set $C_{15} = (\mathrm{SignGenOp}((f \frown \langle I(d_1) \rangle) \frown \langle d_2 \rangle, A, \mathrm{swap}(F_1, L_1, L_2))) \cdot E_{33}$. For every x such that $x \in \mathrm{dom}\, C_{15}$ holds $C_{15}(x)$ is not empty. $\square$

Let us consider an enumeration E of F, an enumeration E_{17} of $\mathrm{ext}(F, 1 + m, 2 + m)$, an enumeration E_{33} of $\mathrm{swap}(F, 1 + m, 2 + m)$, an enumeration E_{21} of $\mathrm{ext}(F, 1 + m, 2 + m) \cup \mathrm{swap}(F, 1 + m, 2 + m)$, and finite sequences s_1, s_2. Now we state the propositions:

(138) Suppose A is commutative, associative, and unital and has inverse ope-

ration and M is associative, commutative, and unital and M is distributive w.r.t. A. Then suppose $\bigcup F \subseteq \mathrm{Seg}(1+m)$. Then suppose $E_{17} = \mathrm{Ext}(E, 1+m, 2+m)$ and $E_{33} = \mathrm{Swap}(E, 1+m, 2+m)$. Then suppose $E_{21} = E_{17} \frown E_{33}$. Then suppose s_1, $s_2 \in \mathrm{doms}(m+1, \overline{\overline{F}})$ and s_1 has evenly repeated values and s_2 has evenly repeated values and $\overline{\overline{s_1^{-1}(\{1+m\})}} < \overline{\overline{s_2^{-1}(\{1+m\})}}$. Then there exist subsets D_1, D_2 of $\mathrm{doms}(m+2, \overline{\overline{F}} + \overline{\overline{F}})$ such that

(i) D_1 is with evenly repeated values-member, and

(ii) D_2 is with evenly repeated values-member, and

(iii) for every finite sequences C_4, C_7 of elements of D^* and for every f, d_1, and d_2 such that $\mathrm{len}\, f = m$ and $C_4 = (\mathrm{SignGenOp}(f \frown \langle A(d_1, d_2)\rangle, A, F)) \cdot E$ and $C_7 = (\mathrm{SignGenOp}(f \frown \langle A((\text{the inverse operation w.r.t. } A)(d_1), d_2)\rangle, A, F)) \cdot E$ for every non-empty, non empty finite sequence C_{17} of elements of D^* such that $C_{17} = (\mathrm{SignGenOp}((f \frown \langle d_1\rangle) \frown \langle d_2\rangle, A, \mathrm{ext}(F, 1+\mathrm{len}\, f, 2+\mathrm{len}\, f) \cup \mathrm{swap}(F, 1+\mathrm{len}\, f, 2+\mathrm{len}\, f))) \cdot E_{21}$ for every elements S_1, S_2 of $\mathrm{Fin\,dom}(\mathrm{App}(C_{17}))$ such that $S_1 = D_1$ and $S_2 = D_2$ holds S_1 misses S_2 and $A(M((M \odot \mathrm{App}(C_4))(s_1), (M \odot \mathrm{App}(C_7))(s_2)), M((M \odot \mathrm{App}(C_4))(s_2), (M \odot \mathrm{App}(C_7))(s_1))) = A\text{-}\sum_{S_1 \cup S_2}(M \odot \mathrm{App}(C_{17}))$ and for every finite sequence h and for every i such that $h \in S_1$ and $i \in \mathrm{dom}(s_1 \frown s_2)$ holds if $(s_1 \frown s_2)(i) = 1+\mathrm{len}\, f$, then $h(i) \in \{1+\mathrm{len}\, f, 2+\mathrm{len}\, f\}$ and if $(s_1 \frown s_2)(i) \neq 1+\mathrm{len}\, f$, then $h(i) = (s_1 \frown s_2)(i)$ and for every finite sequence h and for every i such that $h \in S_2$ and $i \in \mathrm{dom}(s_2 \frown s_1)$ holds if $(s_2 \frown s_1)(i) = 1+\mathrm{len}\, f$, then $h(i) \in \{1+\mathrm{len}\, f, 2+\mathrm{len}\, f\}$ and if $(s_2 \frown s_1)(i) \neq 1+\mathrm{len}\, f$, then $h(i) = (s_2 \frown s_1)(i)$.

(139) Suppose A is commutative, associative, and unital and has inverse operation and M is associative, commutative, and unital and M is distributive w.r.t. A. Then suppose $\bigcup F \subseteq \mathrm{Seg}(1+m)$. Then suppose $E_{17} = \mathrm{Ext}(E, 1+m, 2+m)$ and $E_{33} = \mathrm{Swap}(E, 1+m, 2+m)$. Then suppose $E_{21} = E_{17} \frown E_{33}$. Then suppose s_1, $s_2 \in \mathrm{doms}(m+1, \overline{\overline{F}})$ and s_1 has evenly repeated values and s_2 has evenly repeated values and $s_1 \neq s_2$. Then there exist subsets D_1, D_2 of $\mathrm{doms}(m+2, \overline{\overline{F}} + \overline{\overline{F}})$ such that

(i) D_1 is with evenly repeated values-member, and

(ii) D_2 is with evenly repeated values-member, and

(iii) for every finite sequences C_4, C_7 of elements of D^* and for every f, d_1, and d_2 such that $\mathrm{len}\, f = m$ and $C_4 = (\mathrm{SignGenOp}(f \frown \langle A(d_1, d_2)\rangle, A, F)) \cdot E$ and $C_7 = (\mathrm{SignGenOp}(f \frown \langle A((\text{the inverse operation w.r.t. } A)(d_1), d_2)\rangle, A, F)) \cdot E$ for every non-empty, non empty finite sequence C_{17} of elements of D^* such that $C_{17} = (\text{SignGenOp}((f ^\frown \langle d_1\rangle)^\frown\langle d_2\rangle, A, \text{ext}(F, 1+\text{len}\, f, 2+\text{len}\, f) \cup \text{swap}(F, 1+\text{len}\, f, 2+\text{len}\, f))) \cdot E_{21}$ for every elements S_1, S_2 of $\text{Fin}\,\text{dom}(\text{App}(C_{17}))$ such that $S_1 = D_1$ and $S_2 = D_2$ holds S_1 misses S_2 and $A(M((M \odot \text{App}(C_4))(s_1), (M \odot \text{App}(C_7))(s_2)), M((M \odot \text{App}(C_4))(s_2), (M \odot \text{App}(C_7))(s_1))) = A\text{-}\sum_{S_1 \cup S_2}(M \odot \text{App}(C_{17}))$ and for every finite sequence h and for every i such that $h \in S_1$ and $i \in \text{dom}(s_1 ^\frown s_2)$ holds if $(s_1 ^\frown s_2)(i) = 1+\text{len}\, f$, then $h(i) \in \{1+\text{len}\, f, 2+\text{len}\, f\}$ and if $(s_1 ^\frown s_2)(i) \neq 1+\text{len}\, f$, then $h(i) = (s_1 ^\frown s_2)(i)$ and for every finite sequence h and for every i such that $h \in S_2$ and $i \in \text{dom}(s_2 ^\frown s_1)$ holds if $(s_2 ^\frown s_1)(i) = 1+\text{len}\, f$, then $h(i) \in \{1 + \text{len}\, f, 2 + \text{len}\, f\}$ and if $(s_2 ^\frown s_1)(i) \neq 1 + \text{len}\, f$, then $h(i) = (s_2 ^\frown s_1)(i)$.

The theorem is a consequence of (126), (40), (106), (47), (80), and (138).

(140) Suppose M is commutative and associative and $\text{len}\, f = 2$. Then $\text{SignGenOp}(f, M, A, \{2\}) = M(A(f(1), f(2)), A(f(1), (\text{the inverse operation w.r.t. } A)(f(2))))$. The theorem is a consequence of (71), (70), and (73).

Let us consider an enumeration E of $2^{\{2\}}$ and a non-empty, non empty finite sequence C_3 of elements of D^*. Now we state the propositions:

(141) Suppose M is commutative and associative and A is commutative, associative, and unital and has inverse operation and M is distributive w.r.t. A. Then suppose $C_3 = (\text{SignGenOp}(f, A, 2^{\{2\}})) \cdot E$ and $\text{len}\, f = 2$. Then there exists an element S of $\text{Fin}\,\text{dom}(\text{App}(C_3))$ such that

(i) $S = \{\langle 1, 1\rangle, \langle 2, 2\rangle\}$, and

(ii) $\text{SignGenOp}(f, M, A, \{2\}) = A\text{-}\sum_S(M \odot \text{App}(C_3))$.

PROOF: Set $I =$ the inverse operation w.r.t. A. Reconsider $f_1 = f(1)$, $f_2 = f(2)$ as an element of D. $\{\langle 1, 1\rangle, \langle 2, 2\rangle\} \subseteq \text{dom}_\kappa\, C_3(\kappa)$. $\text{SignGenOp}(f, M, A, \{2\}) = A(M(f_1, f_1), M(f_2, I(f_2)))$. □

(142) Suppose M is commutative and associative and A is commutative, associative, and unital and has inverse operation and M is distributive w.r.t. A. Then suppose $C_3 = (\text{SignGenOp}(f, A, 2^{\{2\}})) \cdot E$ and $\text{len}\, f = 2$. Then there exists an element S of $\text{Fin}\,\text{dom}(\text{App}(C_3))$ such that

(i) S is with evenly repeated values-member, and

(ii) $\text{SignGenOp}(f, M, A, \{2\}) = A\text{-}\sum_S(M \odot \text{App}(C_3))$.

The theorem is a consequence of (141).

(143) MAIN THEOREM:

Suppose A is commutative, associative, and unital and has inverse operation and M is associative, commutative, and unital and M is distributive w.r.t. A and $m > 1$ and for every d, $M(\mathbf{1}_A, d) = \mathbf{1}_A$.

Then there exists an enumeration E of $2^{(\mathrm{Seg}\, m)\setminus\{1\}}$ and there exists a subset S of $\mathrm{doms}(m, \overline{\overline{2^{(\mathrm{Seg}\, m)\setminus\{1\}}}})$ such that S is with evenly repeated values-member and $\overline{\overline{2^{(\mathrm{Seg}\, m)\setminus\{1\}}}} \mapsto 1 \in S$ and for every non-empty, non empty finite sequence C_3 of elements of D^* and for every f such that $C_3 = (\mathrm{SignGenOp}(f, A, 2^{(\mathrm{Seg}\, m)\setminus\{1\}})) \cdot E$ and $\mathrm{len}\, f = m$ for every element S_6 of $\mathrm{Fin}\,\mathrm{dom}(\mathrm{App}(C_3))$ such that $S_6 = S$ holds $\mathrm{SignGenOp}(f, M, A, (\mathrm{Seg}\, m) \setminus \{1\}) = A\text{-}\sum_{S_6}(M \odot \mathrm{App}(C_3))$.

Proof: Define $\mathcal{P}$[natural number] $\equiv$ there exists an enumeration E of $2^{(\mathrm{Seg}\, \$_1)\setminus\{1\}}$ and there exists a subset S of $\mathrm{doms}(\$_1, \overline{\overline{2^{(\mathrm{Seg}\, \$_1)\setminus\{1\}}}})$ such that S is with evenly repeated values-member and $\overline{\overline{2^{(\mathrm{Seg}\, \$_1)\setminus\{1\}}}} \mapsto 1 \in S$ and for every non-empty, non empty finite sequence C_3 of elements of D^* and for every f such that $C_3 = (\mathrm{SignGenOp}(f, A, 2^{(\mathrm{Seg}\, \$_1)\setminus\{1\}})) \cdot E$ and $\mathrm{len}\, f = \$_1$ for every element S_6 of $\mathrm{Fin}\,\mathrm{dom}(\mathrm{App}(C_3))$ such that $S_6 = S$ holds $\mathrm{SignGenOp}(f, M, A, (\mathrm{Seg}\, \$_1) \setminus \{1\}) = A\text{-}\sum_{S_6}(M \odot \mathrm{App}(C_3))$.

$\mathcal{P}[2]$. For every natural number j such that $2 \leqslant j$ holds if $\mathcal{P}[j]$, then $\mathcal{P}[j+1]$. For every natural number i such that $2 \leqslant i$ holds $\mathcal{P}[i]$. □

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